A DEFECT RELATION FOR MEROMORPHIC MAPS ON PARABOLIC MANIFOLDS INTERSECTING HYPERSURFACES

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ABSTRACT. This paper establishes a defect relation for linearly nondegenerate meromorphic mappings from parabolic manifolds into the projective space intersecting hypersurfaces, extending a result of H. Cartan result and an earlier result of Min Ru.

0. Introduction

The purpose of this paper is to study the value distribution theory of meromorphic maps $f : M \to \mathbb{P}^n(\mathbb{C})$ intersecting hypersurfaces, where M is an m-dimensional parabolic manifold and $\mathbb{P}^n(\mathbb{C})$ is the n-dimensional complex projective space. In 1933, H. Cartan [Ca] established the Second Main Theorem for linearly non-degenerate holomorphic curves $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ intersecting hyperplanes. Later L. Ahlfors [A] gave another ingenious geometric proof. W. Stoll [S] generalized Ahlfors' method and extended Cartan's Second Main Theorem to linearly non-degenerate meromorphic maps $f : M \to \mathbb{P}^n(\mathbb{C})$, where M is an m-dimensional parabolic manifold. In [WS], Wong and Stoll carefully examined the error term appearing in Stoll's inequality.

Recently, the second author [Ru] extended H. Cartan's result to algebraically non-degenerate holomorphic curves $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ intersecting hypersurfaces and solved the following conjecture.

CONJECTURE (SHIFFMAN). *Let* $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ *be an algebraically non-degenerate holomorphic curve (here we say that* f *is algebraically non-degenerate if the image of* f *is not contained in any proper algebraic subvarieties of* $\mathbb{P}^n(\mathbb{C})$*), and let* $D_1, \ldots, D_q$ *be hypersurfaces in* $\mathbb{P}^n(\mathbb{C})$ *in general position. Then* $\sum_{j=1}^q \delta_f(D_j) \leq n + 1$.

The purpose of this paper is to extend this result to meromorphic maps on parabolic manifolds.

Throughout this paper, we shall use the standard notations in the value distribution theory of meromorphic maps on parabolic manifolds (see [WS] or [S]). Some notations and definitions will be recalled in Section 1. To establish the value distribution theory on parabolic manifolds M, similar to [S], we make the following assumptions on M (cf. Section 1):

(i) M is a connected complex manifold of dimension m.
(ii) There exists a parabolic exhaustion function τ on M.
(iii) For every integer n and every linearly non-degenerate map $f : M \to \mathbb{P}^n(\mathbb{C})$, there is a holomorphic differential form B of degree $(m-1, 0)$ on M such that f is general for B (cf. §1.3) and

$$m i_{m-1} B \wedge \bar{B} \le Y(r) v^{m-1}$$

on $M[r]$ for some real positive valued function $Y(r)$ on M, which is independent of f (Y is called a *majorant for* B, see §1.4), where, for any positive integer m,

$$i_m = \left(\frac{\sqrt{-1}}{2\pi} \right)^m (-1)^{\frac{m(m-1)}{2}} m!.$$

A complex manifold M satisfying the assumptions (i)–(iii) is called an *admissible parabolic manifold*. Throughout this paper, we shall work on admissible parabolic manifolds. Hypersurfaces $D_1, \ldots, D_q$, $q > n$, in $\mathbb{P}^n(\mathbb{C})$ are said to be *in general position* if $\bigcap_{k=1}^{n+1} \mathrm{supp}(D_{j_k}) = \emptyset$ for any distinct $j_1, \ldots, j_{n+1}$. In this paper, the following theorem is proved.

MAIN THEOREM. *Let M be an admissible parabolic manifold of dimension m. Let $f : M \to \mathbb{P}^n(\mathbb{C})$ be an algebraically non-degenerate meromorphic map (i.e., the image of f is not contained in any proper algebraic subvarieties of $\mathbb{P}^n(\mathbb{C})$), and let $D_1, \ldots, D_q$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ of degree d_j in general position. Fix $s_0 > 0$. Then for every $\epsilon > 0$, we have*

$$\sum_{j=1}^q d_j^{-1} m_f(r, D_j) \centerdot \le \centerdot (n + 1 + \epsilon) T_f(r, s_0)$$

$$+ c_\epsilon \left(\mathrm{Ric}_\tau(r, s_0) + \log^+ T_f(r, s_0) + \log^+ Y(r) + \log^+ r \right),$$

where $c_\epsilon > 0$ is a constant depends on ϵ, $\mathrm{Ric}_\tau(r, s_0)$ is the Ricci function of M (cf. [S], or [WS]), and "$\centerdot \le \centerdot$" means that the inequality holds for all $r \in [s_0, +\infty)$ outside a union of intervals of finite total length.

We note that $\mathrm{Ric}_\tau(r, s_0)$ is called the Ricci function, which depends only on the geometry (topology) of the manifold M. An important class of admissible parabolic manifolds are affine algebraic manifolds. In this case, there exists a finite branched covering $\pi : M \to \mathbb{C}^m$. If we take $\tau = \|\pi\|^2$, Stoll [S] showed

that there exists a holomorphic differential form B of degree $(m-1, 0)$ on M (cf. [S] or [WS]) such that

$$mi_{m-1}B \wedge \bar{B} \leq (1 + |\tau|)^{m-1}(dd^c\tau)^{m-1}.$$

Hence, we can take $Y(r) = (1 + r^2)^{m-1}$. Further, we can show (cf. [S] or [WS]) that

$$d_\pi = \lim_{r \to +\infty} \frac{\text{Ric}_\tau(r, s_0)}{\log r}$$

exists and equals the degree of the branching divisor of π. The Main Theorem thus implies the following corollary in the case that M is an affine algebraic manifold.

COROLLARY. *Let M be an affine algebraic manifold of complex dimension m. Let $\pi : M \to \mathbb{C}^m$ be a finite branched covering. Let $D_1, \ldots, D_q$ be a finite collection of hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ in general position. Let $f : M \to \mathbb{P}^n(\mathbb{C})$ be an algebraically non-degenerate meromorphic map. Then, for every $\epsilon > 0$, we have*

$$\sum_{j=1}^{q} d_j^{-1}m_f(r, D_j) \centerdot \leq \centerdot (n + 1 + \epsilon)T_f(r, s_0) + c_\epsilon \left(\log^+ T_f(r, s_0) + \log^+ r\right),$$

where $c_\epsilon > 0$ is a constant depends on ϵ, and where "$\centerdot \leq \centerdot$" means that the inequality holds for all $r \in [s_0, +\infty)$ outside a union of intervals of finite total length.

1. The theory of meromorphic maps on parabolic manifolds

In this section, we recall some basic results in the theory of meromorphic maps on parabolic manifolds. For references, see [S] or [WS].

1.1 Parabolic manifolds. Let M be a connected complex manifold of dimension m. Let $\tau \geq 0$ be a non-negative, unbounded function of class C^∞ on M. For $0 \leq r \in \mathbb{R}$ and $A \subseteq M$ define

$$A[r] = \{x \in A \mid \tau(x) \leq r^2\},$$
$$A(r) = \{x \in A \mid \tau(x) < r^2\},$$
$$A\langle r\rangle = \{x \in A \mid \tau(x) = r^2\},$$
$$A_* = \{x \in A \mid \tau(x) > 0\},$$
$$\upsilon = dd^c\tau, \quad \omega = dd^c \log \tau, \quad \sigma = d^c \log \tau \wedge \omega^{m-1}.$$

If $M[r]$ is compact for each $r > 0$, the function τ is then said to be an *exhaustion* of M. The function τ is said to be *parabolic* if

$$\omega \geq 0, \quad \omega^m \equiv 0, \quad \upsilon^m \not\equiv 0$$

on M_*. Note that this also implies that $v \geq 0$ on M. If τ is a parabolic exhaustion, (M, τ) is said to be a *parabolic manifold*. Define

$$\hat{\mathbb{R}}_\tau = \{r \in \mathbb{R}^+ \mid d\tau(x) \neq 0 \quad \text{for all} \quad x \in M\langle r \rangle\}.$$

Then $\mathbb{R}^+ \backslash \hat{\mathbb{R}}_\tau$ has measure zero. If $r \in \hat{\mathbb{R}}_\tau$, then $M\langle r \rangle$ is a compact, real, $(2m-1)$-dimensional submanifold of class C^∞ of M, oriented to the exterior of $M\langle r \rangle$. By Stoll (cf. [S, p. 133]), for all $r \in \hat{\mathbb{R}}_\tau$, $\int_{M\langle r \rangle} \sigma$ is a positive constant, independent of r. Let

$$(1.1.1) \qquad\qquad\qquad \kappa = \int_{M\langle r \rangle} \sigma.$$

For an affine algebraic manifold M, this number represents the sheet number of the projection $\pi : M \to \mathbb{C}^m$ if $\tau = \|\pi\|^2$ (cf. [WS]).

1.2 Meromorphic maps, reduced representation. Let M be a complex manifold with $\dim M = m$. Let $A \neq \emptyset$ be an open subset of M such that $S = M - A$ is analytic. Then A is dense in M. Let $f : A \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map on A. The closure Γ of the graph $\{(x, f(x)) \mid x \in A\}$ in $M \times \mathbb{P}^n(\mathbb{C})$ is called the *closed graph* of f. The map f is said to be *meromorphic* on M if (i) $\Gamma(f)$ is analytic in $M \times \mathbb{P}^n(\mathbb{C})$ and (ii) $\Gamma \cap (K \times \mathbb{P}^n(\mathbb{C}))$ is compact for each compact subset $K \subseteq M$, i.e., the projection $\rho : \Gamma(f) \to M$ is proper. If f is meromorphic, then the *set of indeterminacy* $I_f = \{x \in M \mid \#\rho^{-1}(x) > 1\}$ is analytic with $\dim I_f \leq m - 2$ and is contained in S. The holomorphic map $f : A \to \mathbb{P}^n(\mathbb{C})$ continues to a holomorphic map $f : M - I_f \to \mathbb{P}^n(\mathbb{C})$ such that we can assume, a posteriori, that $S = I_f$. If $m = 1$, I_f is necessarily empty and $f : M \to \mathbb{P}^n(\mathbb{C})$ is holomorphic.

Given M, A, S and a holomorphic map $f : A \to \mathbb{P}^n(\mathbb{C})$ as above, a holomorphic map $\mathbf{f}(\not\equiv 0) : U \to \mathbb{C}^{n+1}$ on an open and connected subset U of M is said to be a *representation* of f if $f(x) = \mathbb{P}(\mathbf{f}(x))$ for all $x \in A \cap U$ with $\mathbf{f}(x) \neq 0$. A representation $\mathbf{f}$ is said to be *reduced* if $\dim \mathbf{f}^{-1}(0) \leq m - 2$. The map f is meromorphic if and only if for every point $p \in M$, there is a representation $\mathbf{f} : U \to \mathbb{C}^{n+1}$ of f with $p \in U$. If so, a representation $\mathbf{f}$ is reduced if and only if $U \cap I_f = \mathbf{f}^{-1}(0)$. For every meromorphic map $f : M \to \mathbb{P}^n(\mathbb{C})$, there is a reduced representation of f at every point of M.

1.3 The associated map. A distinguished differential operator, in general, might be missing on a connected, complex manifold M of dimension $m > 1$. Therefore, Stoll [S] assumed that a holomorphic form B of bidegree $(m-1, 0)$ is given on M. Let $\mathbf{f}$ be a holomorphic vector valued function on an open subset U of M. If $z = (z_1, \ldots, z_m)$ is a chart with $U_z \cap U \neq \emptyset$, then the B-derivative $\mathbf{f}'_{B,z} = \mathbf{f}'$ on $U \cap U_z$ for z is defined by $d\mathbf{f} \wedge B = \mathbf{f}' dz_1 \wedge \cdots \wedge dz_m$. The operation can be iterated so that the kth $B-$derivative $\mathbf{f}^{(k)}$ is defined:

$\mathbf{f}^{(k)} = (\mathbf{f}^{(k-1)})'$. Put $\mathbf{f}^{(0)} = \mathbf{f}$. Abbreviate

$$\mathbf{f}_k = \mathbf{f} \wedge \mathbf{f}' \wedge \cdots \wedge \mathbf{f}^{(k)} : U \to \wedge^{k+1} \mathbb{C}^{n+1}.$$

Let $f : M \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic map. If $\mathbf{f}_k \not\equiv 0$ for one choice of a reduced representation $\mathbf{f} : U \to \mathbb{C}^{n+1}$ on a chart U_z, then $\mathbf{f}_k \not\equiv 0$ for all possible choices and f is said to be *general of order k for B*. In this case, the kth associated map $f_k : M \to \mathbb{P}(\bigwedge^{k+1} \mathbb{C}^{n+1})$ is well-defined as a meromorphic map by $f_k|U = \mathbb{P}(\mathbf{f}_k)$ for all possible choices of $\mathbf{f}$ and charts z. We say that f *is general for B* if f is general of order k for B for all k, $1 \leq k \leq n$.

1.4 The majorant function. A positive real valued function $Y(r)$ is said to be a *majorant for B* if

$$mi_{m-1} B \wedge \bar{B} \leq Y(r) v^{m-1}$$

on $M[r]$. In the assumption (iii) for M, we assumed that such a function exists. The majorant function $Y(r)$ introduces an extra term in the Second Main Theorem.

1.5 Projective distance. Denote by $\mathbb{C}^{*n+1}$ the dual space of $\mathbb{C}^{n+1}$. For $0 \leq k \leq n$, let $\lfloor : (\bigwedge^{k+1} \mathbb{C}^{n+1}) \times \mathbb{C}^{*n+1} \to \bigwedge^k \mathbb{C}^{n+1}$ be the interior product defined in the usual way. Let $x \in \mathbb{P}(\bigwedge^{k+1} \mathbb{C}^{n+1})$ with representative $\xi \in \mathbb{C}^{n+1} - \{0\}$ and let $a \in \mathbb{P}(\mathbb{C}^{*n+1})$ with representative $\alpha \in \mathbb{C}^{*n+1} - \{0\}$, the *projective distance* between x and a is defined by

$$(1.5.1) \qquad\qquad 0 \leq \|x; a\| = \frac{\|\xi \lfloor \alpha\|}{\|\xi\| \|\alpha\|} \leq 1,$$

where the norm on $\bigwedge^k \mathbb{C}^{n+1}$ is induced by the standard norm on $\mathbb{C}^{n+1}$. Note that the above definition is independent of the choice of the representatives α and ξ. Note that a hyperplane H in $\mathbb{P}^n(\mathbb{C})$ can also be regarded as a point in $\mathbb{P}^n(\mathbb{C}^*)$. Hence, for every meromorphic map $f : M \to \mathbb{P}^n(\mathbb{C})$, $\|f_k(z); H\|$ is defined for $z \in M$. This defines a distance function (from $f_k(z)$ to H) on M.

1.6 The First Main Theorem. Let M be an admissible parabolic manifold. Let $f : M \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic map which is linearly non-degenerate; hence f is general for B. Let f_k be the kth associated map of f. Let Ω_k be the Fubini-Study form on $\mathbb{P}^n(\bigwedge^{k+1} \mathbb{C}^{n+1})$. Define the kth characteristic function for $0 < s_0 < r$ by

$$T_{f_k}(r, s_0) = \int_{s_0}^r \frac{dt}{t^{2m-1}} \int_{M[t]} f_k^*(\Omega_k) v^{m-1}.$$

It is known that $T_{f_n}(r, s_0) \equiv 0$. Set $T_{f_{-1}}(r, s_0) \equiv 0$.

Let ν be a divisor on M with $S = \operatorname{supp}\nu$. The counting function of ν is defined to be

$$N_\nu(r, s_0) = \int_{s_0}^r n_\nu(t)\frac{dt}{t},$$

where

$$n_\nu(t) = t^{2-2m}\int_{S[t]} \nu\upsilon^{m-1} = \int_{S_*[t]} \nu\omega^{m-1} + n_\nu(0), \quad \text{if} \quad m > 1,$$

$$n_\nu(t) = \sum_{z \in S[t]} \nu(z), \quad \text{if} \quad m = 1.$$

For a hyperplane H in $\mathbb{P}^n(\mathbb{C})$, define an $H-$divisor $\nu = \mu_{f_k}^H$ as in Stoll [S]. Fix $s_0 > 0$, let $N_{f_k}(r, H) = N_\nu(r, s_0)$ and let

$$m_{f_k}(r, H) = \int_{M\langle r\rangle} \log \frac{1}{\|f_k; H\|}\sigma,$$

where $\|f_k; H\|$ is defined as in (1.5.1) (here we regard H as a point in $\mathbb{P}^n(\mathbb{C}^*)$, the dual space of $\mathbb{P}^n(\mathbb{C})$).

THEOREM 1.6 (FIRST MAIN THEOREM) ([S, (8.21), p. 153]). *Let $f : M \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic map which is general for B. Then, for every hyperplane $H \in \mathbb{P}^n(\mathbb{C})$ and for every $0 \le k \le n, s_0, r \in \hat{R}_\tau, 0 < s_0 < r$, we have*

$$T_{f_k}(r, s_0) \ge N_{f_k}(r, H) + m_{f_k}(r, H) - m_{f_k}(s_0, H).$$

1.7 The Calculus Lemma. Let T be a nonnegative function defined on an interval $[s_0, +\infty)$ with $s_0 \ge 0$. Define the error functions $E(T, r)$ and $\tilde{E}(T, r)$ by

(1.7.1)
$$\tilde{E}(T, r) = T(r)\log^{1+\epsilon}(1 + T(r))\log^{1+\epsilon}[1 + r^{2m-1}T(r)\log^{1+t}(1 + T(r))]$$

and

(1.7.2) $$E(T, r) = \log^+ \tilde{E}(T, r).$$

THEOREM 1.7 (CALCULUS LEMMA). *Let h be a nonnegative measurable function on M such that $h\upsilon^m$ is locally integrable. Let s_0 be a positive real number and let T be the function defined by*

$$T(r) = \int_{s_0}^r \frac{dt}{t^{2m-1}}\int_{M[t]} h\upsilon^m.$$

Then $h\sigma$ is integrable over $M\langle r\rangle$ for almost all $r > 0$ and

$$2m\int_{M\langle r\rangle} h\sigma = r^{-(2m-1)}\frac{d}{dr}\left(r^{2m-1}\frac{dT}{dr}\right) \centerdot \le \centerdot \tilde{E}(T, r),$$

where $. \leq .$ *means that the inequality holds for all* $r \in [s_0, +\infty)$ *outside a union of intervals of finite total length.*

Proof. By Corollary 2.4 in [WS] with $g(t) = \log^{1+\epsilon}(1 + t)$. $\qquad\square$

1.8 The Plücker formula. Let d_k be the zero divisor of f_k. When $k = n$, we obtain the Wronskian divisor d_n. The divisor $l_k = d_{k-1} - 2d_k + d_{k+1} \geq 0$ is called the *kth stationary index*; here we assume that $d_{-1} = 0$. Let I_k be the indeterminacy of f_k. On $M - I_k$ define

$$\hat{h}_k = m i_{m-1} f_k^*(\Omega_k) \wedge B \wedge \bar{B}, \tag{1.8.1}$$

where Ω_k is the Fubini-Study form on $\mathbb{P}^n(\bigwedge^{k+1} \mathbb{C}^{n+1})$. Since $i_{m-1} B \wedge \bar{B} \geq 0$, we have $\hat{h}_k \geq 0$ on $M - I_k$. Define

$$h_k = \hat{h}_k / \upsilon^m. \tag{1.8.2}$$

For all $r \in \hat{R}_\tau$, define

$$S_k(r) = \frac{1}{2} \int_{M\langle r \rangle} \log \ h_k \ \sigma. \tag{1.8.3}$$

THEOREM 1.8.1 (PLÜCKER FORMULA) ([S, Theorem 7.6]). *For almost all* $s_0, r \in \hat{R}_\tau, 0 < s_0 < r,$

$$N_{l_k}(r, s_0) + T_{f_{k-1}}(r, s_0) - 2T_{f_k}(r, s_0) + T_{f_{k+1}}(r, s_0)$$
$$= S_k(r) - S_k(s_0) + \mathrm{Ric}_\tau(r, s_0).$$

The *Plücker formula* implies the following result (see [S, (10.24), p. 164]).

THEOREM 1.8.2. *For* $0 \leq k \leq n - 1,$

$$T_{f_k}(r, s_0) \leq 3^k T_f(r, s_0) + \frac{1}{2}(3^k - 1)(\kappa \log Y(r) + \mathrm{Ric}_\tau(r, s_0) + \epsilon\kappa \log r)$$

holds for $r \in [s_0, \infty)$ *outside a union of intervals of finite total length.*

1.9 The Ahlfors Estimate.

THEOREM 1.9.1 (AHLFORS ESTIMATE) ([S, Theorem 10.3]). *Let* H *be a hyperplane in* $\mathbb{P}^n(\mathbb{C})$. *Then for any* $0 < \lambda < 1$, $0 < s_0 < r$, *and any integer* $0 \leq k \leq n - 1$, *we have*

$$\int_{s_0}^r \frac{dt}{t^{2m-1}} \int_{M[t]} \frac{\|f_{k+1}; H\|^2}{\|f_k; H\|^{2-2\lambda}} h_k \upsilon^m \leq Y(r)\left(\frac{4 + 4\lambda}{\lambda} T_{f_k}(r, s_0) + \frac{2\kappa}{\lambda^2} \log 2\right),$$

where h_k *is defined in (1.8.2), and* κ *is the constant defined in (1.1.1).*

THEOREM 1.9.2.   *Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$. Let $\Lambda(r) = \min_k\{1/(1+ T_k(r, s_0))\}$. Then, for every $0 \le k \le n-1$,*

$$\log^+ \int_{M\langle r\rangle} \frac{\|f_{k+1}; H\|^2}{\|f_k; H\|^{2-2\Lambda(r)}} h_k \sigma$$

$$\centerdot \le \centerdot\, 2\log^+ T_f(r, s_0) + 2(2+\epsilon)\log^+ \log^+ T_f(r, s_0)$$

$$+ 2\log^+ Y(r) + 3\log^+ \mathrm{Ric}_\tau(r, s_0) + 5\log^+ \log^+ r + O(1),$$

where "$\centerdot \le \centerdot$" means that the inequality holds for all $r \in [s_0, +\infty)$ outside a union of intervals of finite total length, and where the constant $O(1)$ is independent of r.

Proof. Let $0 < \Lambda(r) < 1$ be a decreasing function of $r \ge 0$. Define

$$K_k(r, s_0) = \int_{s_0}^r \frac{dt}{t^{2m-1}} \int_{M[t]} \frac{\|f_{k+1}; H\|^2}{\|f_k; H\|^{2-2\Lambda^*}} h_k \upsilon^m,$$

where $\Lambda^* = \Lambda \circ \tau^{1/2}$. By Theorem 1.7 (Calculus Lemma), we have

$$(1.9.1) \qquad \int_{M\langle r\rangle} \frac{\|f_{k+1}; H\|^2}{\|f_k; H\|^{2-2\Lambda(r)}} h_k \sigma \centerdot \le \centerdot \tilde{E}(K_k, r).$$

On the other hand, noticing that Λ is a decreasing function, we have $\|f_k; H\|^{\Lambda^*} \le \|f_k; H\|^{\Lambda(r)}$. Hence, by Theorem 1.9.1 (Ahlfors Estimate) with $\lambda = \Lambda(r)$, we have

$$K_k(r, s_0) = \int_{s_0}^r \frac{dt}{t^{2m-1}} \int_{M[t]} \frac{\|f_{k+1}; H\|^2}{\|f_k; H\|^{2-2\Lambda^*}} h_k \upsilon^m$$

$$\le Y(r)\left(\frac{8}{\Lambda(r)} T_{f_k}(r, s_0) + \frac{2\kappa \log 2}{\Lambda(r)^2}\right).$$

Since $\Lambda(r) = \min_k\{1/(1 + T_{f_k}(r, s_0))\}$,

$$K_k(r, s_0) \le Y(r)(b_1 T_{f_k}^2(r, s_0) + b_2),$$

where b_1 and b_2 are constants depending only on κ. By choosing a larger constant b_3, we have

$$(1.9.2) \qquad \tilde{E}(K_k, r) \le \tilde{E}(b_3 Y(r) T_{f_k}^2(r, s_0), r).$$

Combining (1.9.1) and (1.9.2) yields

$$\int_{M\langle r\rangle} \frac{\|f_{k+1}; H\|^2}{\|f_k; H\|^{2-2\Lambda(r)}} h_k \sigma \centerdot \le \centerdot \tilde{E}(b_3 Y(r) T_{f_k}^2(r, s_0), r).$$

Hence

$$(1.9.3) \qquad \log^+ \int_{M\langle r\rangle} \frac{\|f_{k+1}; H\|^2}{\|f_k; H\|^{2-2\Lambda(r)}} h_k \sigma \centerdot \le \centerdot E(b_3 Y(r) T_{f_k}^2(r, s_0), r).$$

By the definition, we have (see [WS, (2.8), p. 1046])

$$E(b_3 Y(r) T_{f_k}^2(r, s_0), r)$$
$$\leq \log^+[b_3 Y(r) T_{f_k}^2(r, s_0)] + 2(1 + \epsilon) \log^+ \log^+[b_3 Y(r) T_k^2(r, s_0)]$$
$$+ (1 + \epsilon) \log^+ \log^+ \log^+ (b_3 Y(r) T_{f_k}^2(r, s_0)) + (1 + \epsilon) \log^+ \log^+ r + O(1)$$
$$\leq 2 \log^+ T_{f_k}(r, s_0) + 2(1 + \epsilon) \log^+ \log^+ T_{f_k}(r, s_0)$$
$$+ (1 + \epsilon) \log^+ \log^+ \log^+ T_{f_k}(r, s_0) + 2 \log^+ Y(r) + 2 \log^+ \log^+ r + O(1).$$

By Theorem 1.8.2,

$$T_{f_k}(r, s_0) \centerdot \leq \centerdot 3^k T_f(r, s_0) + \frac{1}{2}(3^k - 1)(\kappa \log Y(r) + \mathrm{Ric}_\tau(r, s_0) + \epsilon \kappa \log r).$$

Hence

$$E(b_3 Y(r) T_{f_k}^2(r, s_0), r)$$
$$\centerdot \leq \centerdot 2 \log^+ T_f(r, s_0) + 2(2 + \epsilon) \log^+ \log^+ T_f(r, s_0)$$
$$+ 2 \log^+ Y(r) + 3 \log^+ \mathrm{Ric}_\tau(r, s_0) + 5 \log^+ \log^+ r + O(1).$$

This, together with (1.9.3), concludes the proof. □

2 A slight generalization of Wong-Stoll's theorem

In this section, we extend the Second Main Theorem of Wong-Stoll (cf. [WS]) with good error term to the case when the given hyperplanes $H_1, \ldots, H_q$ in $\mathbb{P}^n(\mathbb{C})$ are not necessarily in general position. This formulation is crucial in the proof of our Main Theorem.

THEOREM 2.1. *Let M be an admissible parabolic manifold. Let $H_1, \ldots, H_q$ be arbitrary hyperplanes in $\mathbb{P}^n(\mathbb{C})$ (also regarded as linear forms). Let $f : M \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic map which is linearly non-degenerate. Fix $s_0 > 0$. Then, for $\epsilon > 0$,*

$$\int_{M\langle r \rangle} \max_K \sum_{j \in K} \log \frac{1}{\|f(z); H_j\|} \sigma$$

$$\centerdot \leq \centerdot (n + 1) T_f(r, s_0) + \frac{n(n + 1)}{2} \mathrm{Ric}_\tau(r, s_0)$$

$$+ \kappa \frac{n(n + 1)}{2} \Big[\log^+ T_f(r, s_0) + (2 + \epsilon) \log^+ \log^+ T_f(r, s_0) + \log^+ Y(r)$$

$$+ 2 \log^+ \mathrm{Ric}_\tau(r, s_0) + 3 \log^+ \log^+ r + O(1) \Big],$$

where "$\centerdot \leq \centerdot$" means that the inequality holds for all $r \in [s_0, +\infty)$ outside a union of intervals of finite total length, and the max is taken over all subsets K of $\{1, \ldots, q\}$ such that the linear forms $H_i, i \in K$, are linearly independent.

Proof. Let $K \subset \{1, \ldots, q\}$ be such that the linear forms $\{H_k, k \in K\}$, are linearly independent. Without loss of generality, we may assume $q \geq n + 1$ and that $\#K = n + 1$. Let T be the set of all the injective maps $\mu : \{0, 1, \ldots, n\} \to \{1, \ldots, q\}$ such that $H_{\mu(0)}, \ldots, H_{\mu(n)}$ are linearly independent. Set $\Gamma = \max_{1 \leq j \leq q} \{\sum_{k=0}^{n-1} m_{f_k}(s_0, H_j)\}$ and $\Lambda(r) = \min_k \{1/(1 + T_{f_k}(r, s_0))\}$. For any $\mu \in T, z \notin I_f$, the Product to Sum Estimate (cf. [WS, Lemma 1.12]), with $\lambda = \Lambda(r)$, reads

$$\prod_{j=0}^{n} \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} \leq c_k \left(\sum_{j=0}^{n} \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}}\right)^{n-k},$$

where $c_k > 1$ is a constant. Since $\|f_n; H_{\mu(j)}\|$ is a constant for any $0 \leq j \leq n$, we have

$$\prod_{j=0}^{n} \frac{1}{\|f(z); H_{\mu(j)}\|^2}$$

$$= \prod_{k=0}^{n-1} \prod_{j=0}^{n} \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} \cdot \prod_{k=0}^{n-1} \prod_{j=0}^{n} \frac{1}{\|f_k(z); H_{\mu(j)}\|^{2\Lambda(r)}}$$

$$\leq c \prod_{k=0}^{n-1} \left(\sum_{j=0}^{n} \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}}\right)^{n-k} \cdot \prod_{k=0}^{n-1} \prod_{j=0}^{n} \frac{1}{\|f_k(z); H_{\mu(j)}\|^{2\Lambda(r)}},$$

where $c > 1$ is a constant. Therefore, for $r > s_0$, we have

$$(2.1) \quad \int_{M\langle r \rangle} \max_K \sum_{j \in K} \log \frac{1}{\|f(z); H_j\|^2} \ \sigma$$

$$= \int_{M\langle r \rangle} \max_{\mu \in T} \log \left(\prod_{j=0}^{n} \frac{1}{\|f(z); H_{\mu(j)}\|^2}\right) \ \sigma$$

$$\leq \sum_{k=0}^{n-1} \int_{M\langle r \rangle} \max_{\mu \in T} \log \left(\sum_{j=0}^{n} \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}}\right)^{n-k} \ \sigma$$

$$+ \sum_{k=0}^{n-1} \sum_{j=0}^{n} \int_{M\langle r \rangle} \max_{\mu \in T} \log \frac{1}{\|f_k(z); H_{\mu(j)}\|^{2\Lambda(r)}} \sigma + O(1).$$

We now estimate each term appearing in the above inequality. First,

$$(2.2) \qquad \sum_{k=0}^{n-1} \int_{M\langle r\rangle} \max_{\mu\in T} \log \left(\sum_{j=0}^{n} \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} \right)^{n-k} \sigma$$

$$= \sum_{k=0}^{n-1} \int_{M\langle r\rangle} \log \max_{\mu\in T} \left(\sum_{j=0}^{n} \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} h_k \right)^{n-k} \sigma$$

$$- 2 \sum_{k=0}^{n-1} (n-k) S_k(r),$$

where h_k is defined by (1.8.2), and $S_k(r)$ is defined by (1.8.3). However, we have

$$(2.3) \qquad \int_{M\langle r\rangle} \log \max_{\mu\in T} \left(\sum_{j=0}^{n} \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} h_k \right)^{n-k} \sigma$$

$$= \kappa(n-k) \int_{M\langle r\rangle} \log \max_{\mu\in T} \left(\sum_{j=0}^{n} \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} h_k \right) \frac{\sigma}{\kappa}$$

$$\leq \kappa(n-k) \log \int_{M\langle r\rangle} \max_{\mu\in T} \left(\sum_{j=0}^{n} \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} h_k \right) \frac{\sigma}{\kappa}$$

$$\leq (n-k)\kappa \max_{0\leq j\leq n} \log^+ \int_{M\langle r\rangle} \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} h_k \sigma$$

$$+ (n-k)\kappa \log q + C',$$

where C' is a constant. By Theorem 1.9.2,

$$\max_{0\leq j\leq n} \log^+ \int_{M\langle r\rangle} \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} h_k \sigma$$

$$\centerdot \leq \centerdot 2 \Big[\log^+ T_f(r, s_0) + (2+\epsilon) \log^+ \log^+ T_f(r, s_0)$$

$$+ \log^+ Y(r) + 2\log^+ \mathrm{Ric}_\tau(r, s_0) + 3\log^+ \log^+ r + O(1) \Big].$$

Hence,

$$(2.4) \qquad \int_{M\langle r\rangle} \log \max_{\mu \in T} \left(\sum_{j=0}^{n} \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} h_k \right)^{n-k} \sigma$$

$$. \leq . n(n+1)\kappa \Big[\log^+ T_f(r)$$
$$+ (2+\epsilon) \log^+ \log^+ T_f(r) + \log^+ Y(r)$$
$$+ 2 \log^+ \mathrm{Ric}_\tau(r, s_0) + 3 \log^+ \log^+ r + O(1) \Big].$$

Next, using Theorem 1.8.1 (the Plücker formula), we have

$$N_{l_k}(r, s_0) + T_{f_{k-1}}(r, s_0) - 2T_{f_k}(r, s_0) + T_{f_{k+1}}(r, s_0)$$
$$= S_k(r) - S_k(s_0) + \mathrm{Ric}_\tau(r, s_0).$$

Noticing that $T_{f_n}(r, s_0) = 0$, we get

$$(2.5) \qquad \sum_{k=0}^{n-1} (n-k) S_k(r) = N_{d_n}(r, s_0) - (n+1) T_f(r, s_0)$$
$$- \frac{n(n+1)}{2} \mathrm{Ric}_\tau(r, s_0) + O(1).$$

Combining (2.2), (2.3), (2.4) and (2.5), we obtain

$$(2.6) \qquad \sum_{k=0}^{n-1} \int_{M\langle r\rangle} \max_{\mu \in T} \log \left(\sum_{j=0}^{n} \frac{\|f_{k+1}(z); H_{\mu(j)}\|^2}{\|f_k(z); H_{\mu(j)}\|^{2-2\Lambda(r)}} \right)^{n-k} \sigma$$

$$. \leq . (n+1) T_f(r) + \frac{n(n+1)}{2} \mathrm{Ric}_\tau(r, s_0) - N_{d_n}(r, s_0)$$
$$+ \kappa \frac{n(n+1)}{2} [\log^+ T_f(r) + (2+\epsilon) \log^+ \log^+ T_f(r)$$
$$+ \log^+ Y(r) + 2 \log^+ \mathrm{Ric}_\tau(r, s_0) + 3 \log^+ \log^+ r + O(1)].$$

Finally, by the First Main Theorem,

$$(2.7) \qquad \sum_{k=0}^{n-1} \sum_{j=0}^{n} \int_{M\langle r\rangle} \max_{\mu \in T} \log \frac{1}{\|f_k(z); H_{\mu(j)}\|^{2\Lambda(r)}} \sigma$$

$$\leq \sum_{\mu \in T} \sum_{k=0}^{n-1} \sum_{j=0}^{n} \int_{M\langle r\rangle} 2\Lambda(r) \log \frac{1}{\|f_k(z); H_{\mu(j)}\|} \sigma + O(1)$$

$$= \sum_{\mu \in T} \sum_{k=0}^{n-1} \sum_{j=0}^{n} 2\Lambda(r) m_{f_k}(r, H_{\mu(j)}) + O(1)$$

$$\leq \sum_{k=0}^{n-1} \sum_{j=0}^{n} 2q! \Lambda(r) (T_{f_k}(r, s_0) + m_{f_k}(s_0, H_{\mu(j)})) + O(1)$$

$$\leq O(1).$$

Combining (2.1), (2.6), and (2.7) yields

$$\int_{M\langle r\rangle} \max_K \sum_{j\in K} \log \frac{1}{\|f(z); H_j\|} \sigma$$

$$\cdot \leq \cdot (n+1)T_f(r) + \frac{n(n+1)}{2} \operatorname{Ric}_\tau(r, s_0) - N_{d_n}(r, s_0)$$

$$+ \kappa \frac{n(n+1)}{2} [\log^+ T_f(r) + (2+\epsilon) \log^+ \log^+ T_f(r)$$

$$+ \log^+ Y(r) + 2\log^+ \operatorname{Ric}_\tau(r, s_0) + 3\log^+ \log^+ r + O(1)]. \quad \square$$

3. Proof of the Main Theorem

Let M be an admissible parabolic manifold of dimension m. Let $f : M \to \mathbb{P}^n(\mathbb{C})$ be a meromorphic map. Choose a reduced representation $\mathbf{f} : U \to \mathbb{C}^{n+1}$ on a chart U_z. Let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d. Assume that $f(M) \not\subset D$. Let Q be the homogeneous polynomial (form) of degree d defining D. The proximity function $m_f(r, D)$ is defined as

$$m_f(r, D) = \int_{M\langle r\rangle} \log \frac{\|\mathbf{f}(z)\|^d \|Q\|}{|Q(\mathbf{f})(z)|} \sigma,$$

where $\|Q\|$ is the maximum norm of the coefficients appearing in Q. We note that, although $\mathbf{f}$ depends on the choice of representations, the function $\|\mathbf{f}(z)\|^d \|Q\|/|Q(\mathbf{f})(z)|$ in fact is a (globally defined) function on M. Also, the zeros of $Q(\mathbf{f})$ are independent of the choice of representations. We define the divisor μ_f^D on M by $\mu_f^D|_U = \mu_{Q(\mathbf{f})}^0$, where $\mu_{Q(\mathbf{f})}^0$ is the zero divisor of $Q(\mathbf{f})$. Let $S = \operatorname{supp} \mu_f^D$, and define

$$n_f(r, D) = \frac{1}{r^{2m-2}} \int_{S[r]} \mu_f^D v^{m-1}, \quad \text{if} \quad m > 1,$$

$$n_f(r, D) = \sum_{z \in S[r]} \mu_f^D(z), \quad \text{if} \quad m = 1.$$

Fix $s_0 > 0$. The counting function is defined by

$$N_f(r, D) = \int_{s_0}^r \frac{n_f(t, D)}{t} dt.$$

Green's formula (see [S]) implies the following First Main Theorem:

THEOREM 3.1 (FIRST MAIN THEOREM). *Let $f : M \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map, and let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d. Fix $s_0 > 0$. If $f(M) \not\subset D$, then for every real number r with $s_0 < r < \infty$*

$$m_f(r, D) + N_f(r, D) = dT_f(r, s_0) + O(1),$$

where $O(1)$ is a constant independent of r.

We now prove the Main Theorem:

Proof of the Main Theorem. Let $D_1, \ldots, D_q$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$, located in general position. Let $Q_j, 1 \le j \le q$, be the homogeneous polynomials in $\mathbb{C}[X_0, \ldots, X_n]$ of degree d_j defining D_j. Replacing Q_j by Q_j^{d/d_j} if necessary, where d is the l.c.m of the $d_j's$, we can assume that $Q_1, \ldots, Q_q$ have the same degree of d. Choose a reduced representation $\mathbf{f} = (f_0, \ldots, f_n) : U \to \mathbb{C}^{n+1}$ on a chart U_z. Given $z \in U$, there exists a renumbering $\{i_1, \ldots, i_q\}$ of the indices $\{1, \ldots, q\}$ such that

$$(3.1) \qquad |Q_{i_1} \circ \mathbf{f}(z)| \le |Q_{i_2} \circ \mathbf{f}(z)| \le \cdots \le |Q_{i_q} \circ \mathbf{f}(z)|.$$

Since $Q_1, \ldots, Q_q$ are in general position, by Hilbert's Nullstellensatz (cf. [W]), for any integer $k, 0 \le k \le n$, there is an integer $m_k \ge d$ such that

$$x_k^{m_k} = \sum_{j=1}^{n+1} b_{jk}(x_0, \ldots, x_n) Q_{i_j}(x_0, \ldots, x_n),$$

where $b_{jk}, 1 \le j \le n+1, 0 \le k \le n$, are homogeneous forms with coefficients in $\mathbb{C}$ of degree $m_k - d$. So

$$|f_k(z)|^{m_k} \le c_1 \|f(z)\|^{m_k - d} \max\{|Q_{i_1}(f)(z)|, \ldots, |Q_{i_{n+1}}(f)(z)|\},$$

where c_1 is a positive constant that depends only on the coefficients of $b_{ik}, 1 \le i \le n+1, 0 \le k \le n$, and thus depends only on the coefficients of $Q_i, 1 \le i \le n+1$. Therefore,

$$(3.2) \qquad \|\mathbf{f}(z)\|^d \le c_1 \max\{|Q_{i_1}(\mathbf{f})(z)|, \ldots, |Q_{i_{n+1}}(\mathbf{f})(z)|\}.$$

By (3.1) and (3.2),

$$\prod_{j=1}^{q} \frac{\|\mathbf{f}(z)\|^d}{|Q_j(\mathbf{f})(z)|} \le c_1^{q-n} \prod_{k=1}^{n} \frac{\|\mathbf{f}(z)\|^d}{|Q_{i_k}(\mathbf{f})(z)|}.$$

Hence, by the definition,

$$(3.3) \qquad \sum_{j=1}^{q} m_f(r, D_j)$$

$$\le \int_{M\langle r \rangle} \max_{\{i_1, \ldots, i_n\}} \left\{ \log \prod_{k=1}^{n} \frac{\|\mathbf{f}(z)\|^d}{|Q_{i_k}(\mathbf{f})(z)|} \right\} \sigma + (q - n) \log c_1.$$

Now pick n distinct polynomials $\gamma_1, \ldots, \gamma_n \in \{Q_1, \ldots, Q_q\}$. By the "in general position" assumption, they define a subvariety of $\mathbb{P}^n(\mathbb{C})$ of dimension 0. For a fixed large integer N, which will be chosen later, denote by V_N the space of homogeneous polynomials in $\mathbb{C}[X_0, \ldots, X_n]$ of degree N. Arrange,

in lexicographic order, the n-tuples $(\mathbf{i}) = (i_1, \ldots, i_n)$ of non-negative integers such that $\sigma(\mathbf{i}) := \sum_j i_j \leq N/d$. Define the spaces $W_{(\mathbf{i})} = W_{N,(\mathbf{i})}$ by

$$W_{(\mathbf{i})} = \sum_{(\mathbf{e}) \geq (\mathbf{i})} \gamma_1^{e_1} \cdots \gamma_n^{e_n} V_{N-d\sigma(\mathbf{e})}.$$

Plainly $W_{(0,\ldots,0)} = V_N$ and $W_{(\mathbf{i})} \supset W_{(\mathbf{i}')}$ if $(\mathbf{i}') \geq (\mathbf{i})$, so the $W_{(\mathbf{i})}$ in fact define a filtration of V_N.

Our next step is to investigate quotients between consecutive spaces in the filtration. Suppose that $(\mathbf{i}')$ follows $(\mathbf{i})$ in the ordering. We recall the following lemma in [Ru]:

LEMMA 3.2. *There exists an integer N_0 dependent only on $\gamma_1, \ldots, \gamma_n$ such that, in the above notation,*

$$\Delta_{(\mathbf{i})} := \dim \frac{W_{(\mathbf{i})}}{W_{(\mathbf{i}')}} = d^n,$$

provided $d\sigma(\mathbf{i}) < N - N_0$. Also, for the remaining n-tuples $(\mathbf{i})$, $\dim W_{(\mathbf{i})}/W_{(\mathbf{i}')}$ is bounded (by $\dim V_{N_0}$).

For the proof see Lemma 3.3 in [Ru].

Set $u = u_N := \dim V_N$. We choose a suitable basis $\{\psi_1, \ldots, \psi_u\}$ for V_N in the following way. We start with the last nonzero $W_{(\mathbf{i})}$ and pick any basis of it. Then we continue inductively as follows: Suppose $(\mathbf{i}') > (\mathbf{i})$ are consecutive n-tuples such that $d\sigma(\mathbf{i}), d\sigma(\mathbf{i}') \leq N$ and assume that we have chosen a basis of $W_{(\mathbf{i}')}$. It follows directly from the definition that we may pick representatives in $W_{(\mathbf{i})}$ for the quotient space $W_{(\mathbf{i})}/W_{(\mathbf{i}')}$, of the form $\gamma_1^{i_1} \cdots \gamma_n^{i_n} \gamma$, where $\gamma \in V_{N-d\sigma(\mathbf{i})}$. We extend the previously constructed basis in $W_{(\mathbf{i}')}$ by adding these representatives. In particular, we obtain a basis for $W_{(\mathbf{i})}$ and our inductive procedure can be continued until $W_{(\mathbf{i})} = V_N$, in which case we stop. In this way, we obtain a basis $\{\psi_1, \ldots, \psi_u\}$ for V_N.

Let $\{U_\lambda, \lambda \in \Lambda\}$ be an open covering of M, and let $\mathbf{f}_\lambda : U_\lambda \to \mathbb{C}^{n+1}$ be a reduced representation of f on U_λ. We now estimate $\log \prod_{t=1}^u |\psi_t(\mathbf{f}_\lambda)(z)|$ on U_λ. Let ψ be an element of the basis, constructed with respect to $W_{(\mathbf{i})}/W_{(\mathbf{i}')}$. Thus we may write $\psi = \gamma_1^{i_1} \cdots \gamma_n^{i_n} \gamma$, where $\gamma \in V_{N-d\sigma(\mathbf{i})}$. Then we have a bound

$$|\psi(\mathbf{f}_\lambda)(z)| \leq |\gamma_1(\mathbf{f}_\lambda)(z)|^{i_1} \cdots |\gamma_n(\mathbf{f}_\lambda)(z)|^{i_n} |\gamma(\mathbf{f}_\lambda)(z)|$$

$$\leq c_2 |\gamma_1(\mathbf{f}_\lambda)(z)|^{i_1} \cdots |\gamma_n(\mathbf{f}_\lambda)(z)|^{i_n} \|\mathbf{f}_\lambda(z)\|^{N-d\sigma(\mathbf{i})}$$

$$= c_2 \left(\frac{|\gamma_1(\mathbf{f}_\lambda)(z)|}{\|\mathbf{f}_\lambda(z)\|^d} \right)^{i_1} \cdots \left(\frac{|\gamma_n(\mathbf{f}_\lambda)(z)|}{\|\mathbf{f}_\lambda(z)\|^d} \right)^{i_n} \cdot \|\mathbf{f}_\lambda(z)\|^N,$$

where c_2 is a positive constant that depends only on ψ, but not on f and z. Observe that there are precisely $\Delta_{(\mathbf{i})}$ such functions ψ in our basis. Hence,

taking the product over all functions in the basis, we get, after taking logarithms,

(3.4)
$$\log \prod_{t=1}^{u} |\psi_t(\mathbf{f}_\lambda)(z)| \leq \sum_{(\mathbf{i})} \Delta_{(\mathbf{i})} \left(i_1 \log \frac{|\gamma_1(\mathbf{f}_\lambda)(z)|}{\|\mathbf{f}_\lambda(z)\|^d} + \cdots + i_n \log \frac{|\gamma_n(\mathbf{f}_\lambda)(z)|}{\|\mathbf{f}_\lambda(z)\|^d} \right)$$
$$+ uN \log \|\mathbf{f}_\lambda(z)\| + c_3,$$

where c_3 depends only on the ψ's, but not on f and z. Here the summation is taken over the n-tuples with $\sigma(\mathbf{i}) \leq N/d$. We now estimate the sums. First,

(3.5)
$$u = \frac{(N+n)!}{N!n!} = \frac{N^n}{n!} + O(N^{n-1}).$$

Second, since the number of non-negative integer m-tuples with sum $\leq T$ is equal to the number of non-negative integer $(m+1)$-tuples with sum exactly $T \in \mathbb{Z}$, which is $\binom{T+m}{m}$, and since the sum below is independent of j, we have that, for N divisible by d and for every j,

(3.6)
$$\sum_{(\mathbf{i})} i_j = \frac{1}{n+1} \sum_{(\hat{\mathbf{i}})} \sum_{j=1}^{n+1} i_j = \frac{1}{n+1} \sum_{(\hat{\mathbf{i}})} \frac{N}{d}$$
$$= \frac{1}{n+1} \binom{N/d+n}{n} \frac{N}{d} = \frac{N^{n+1}}{d^{n+1}(n+1)!} + O(N^n),$$

where the sum $\sum_{(\hat{\mathbf{i}})}$ is taken over the non-negative integer $(n+1)$-tuples with sum exactly N/d. Combining (3.6) and Lemma 3.2, we have, for every $1 \leq j \leq n$,

(3.7)
$$\sum_{(\mathbf{i})} i_j \Delta_{(\mathbf{i})} = \frac{N^{n+1}}{d(n+1)!} + O(N^n),$$

where again the summations are taken over the n-tuples with sum $\leq N/d$ and the various constants in the "O" terms depend only on the original data, $\gamma_1, \ldots, \gamma_n$, and hence only on $Q_1, \ldots, Q_q$, but not on f and z. (3.4) and (3.7) yield

(3.8)
$$\log \prod_{t=1}^{u} |\psi_t(\mathbf{f}_\lambda)(z)| \leq \left(\log \prod_{j=1}^{n} \frac{|\gamma_j(\mathbf{f}_\lambda)(z)|}{\|\mathbf{f}_\lambda\|^d} \right) \frac{N^{n+1}}{d(n+1)!} (1 + O(N^{-1}))$$
$$+ uN \log \|\mathbf{f}_\lambda(z)\| + c_3,$$

where c_3 and the various constants in the "O" terms depend only on $Q_1, \ldots, Q_q$, but not on f and z.

Now let $\phi_1, \ldots, \phi_u$ be a fixed basis of V_N. Let $\mathbf{F}_\lambda = (\phi_1(\mathbf{f}_\lambda), \ldots, \phi_u(\mathbf{f}_\lambda))$. Then $F = \mathbb{P}(\mathbf{F}_\lambda)$ is independent of λ. Hence it defines a meromorphic map

$F : M \to \mathbb{P}^{u-1}(\mathbb{C})$, where $\mathbb{P}$ is the projection. Write $\{\psi_1, \ldots, \psi_u\}$ as linear forms $L_1, \ldots, L_u$ in $\phi_1, \ldots, \phi_u$, so that $\psi_t(\mathbf{f}_\lambda) = L_t(\mathbf{F}_\lambda), t = 1, \ldots, u$. The linear forms $L_1, \ldots, L_u$ are linearly independent, and we know, by the assumption of algebraical non-degeneracy of f, that $F : M \to \mathbb{P}^{u-1}(\mathbb{C})$ is linearly nondegenerate. From (3.8),

$$(3.9) \quad \log \prod_{t=1}^{u} |L_t(\mathbf{F}_\lambda)(z)| \leq \left(\log \prod_{j=1}^{n} \frac{|\gamma_j(\mathbf{f}_\lambda)(z)|}{\|\mathbf{f}_\lambda\|^d} \right) \frac{N^{n+1}}{d(n+1)!} (1 + O(N^{-1}))$$
$$+ uN \log \|\mathbf{f}_\lambda(z)\| + c_3.$$

This implies that

$$\log \prod_{j=1}^{n} \frac{\|\mathbf{f}_\lambda(z)\|^d}{|\gamma_j(\mathbf{f}_\lambda)(z)|}$$
$$\leq \frac{d(n+1)!}{N^{n+1}(1 + O(N^{-1}))} \left[\log \prod_{t=1}^{u} \frac{\|\mathbf{F}_\lambda(z)\|}{|L_t(\mathbf{F}_\lambda)(z)|} \right.$$
$$\left. - u \log \|\mathbf{F}_\lambda(z)\| + uN \log \|\mathbf{f}_\lambda(z)\| + c_3 \right].$$

By the definition of F, we have $C_1 \|\mathbf{f}_\lambda(z)\|^N \leq \|\mathbf{F}_\lambda(z)\| \leq C_2 \|\mathbf{f}_\lambda(z)\|^N$, where C_1 and C_2 are positive constants independent of λ. Hence

$$(3.10) \quad \log \prod_{j=1}^{n} \frac{\|\mathbf{f}_\lambda(z)\|^d}{|\gamma_j(\mathbf{f}_\lambda)(z)|}$$
$$\leq \frac{d(n+1)!}{N^{n+1}(1 + O(N^{-1}))} \left[\log \prod_{t=1}^{u} \frac{\|\mathbf{F}_\lambda(z)\|}{|L_t(\mathbf{F}_\lambda)(z)|} + c_3 + uc_4 \right],$$

where c_3, c_4 and the various constants in the "O" terms depend only on $Q_1, \ldots, Q_q$, but not on f and z. Note that the expressions $\|\mathbf{f}_\lambda(z)\|^d / |\gamma_j(\mathbf{f}_\lambda)(z)|$ and $\|\mathbf{F}_\lambda(z)\| / |L_t(\mathbf{F}_\lambda)(z)|$ are independent of λ. Hence they are globally defined functions on M. Note that the linear forms $L_1, \ldots, L_u$ depend on the set $\{\gamma_1, \ldots, \gamma_n\}$. Setting $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$, we write $L_{\Gamma,1}, \ldots, L_{\Gamma,u}$ for the linear forms $L_1, \ldots, L_u$ to indicate this dependency. (3.10) thus implies that

(3.11)

$$\int_{M\langle r \rangle} \max_{\{i_1, \ldots, i_n\}} \left\{ \log \prod_{k=1}^{n} \frac{\|f(z)\|^d}{|Q_{i_k}(f)(z)|} \right\} \sigma$$
$$\leq \frac{d(n+1)!}{N^{n+1}(1 + O(N^{-1}))} \left[\int_{M\langle r \rangle} \max_{\Gamma} \log \prod_{j=1}^{u} \frac{\|\mathbf{F}_\lambda(z))\| \|L_{\Gamma,j}\|}{|L_{\Gamma,j}(\mathbf{F}_\lambda)(z)|} \sigma + \kappa(c_3 + uc_4) \right].$$

Applying Theorem 2.1 with $\epsilon = 1$ to the holomorphic map F gives

$$(3.12) \qquad \int_{M\langle r \rangle} \max_{\Gamma} \log \prod_{j=1}^{u} \frac{\|\mathbf{F}_\lambda(z))\|\|L_{\Gamma,j}\|}{|L_{\Gamma,j}(\mathbf{F}_\lambda)(z)|} \sigma$$

$$\leq (u+1)T_F(r, s_0) + c_u[\log^+ T_F(r, s_0)$$

$$+ \operatorname{Ric}_\tau(r, s_0) + \log^+ Y(r) + \log^+ r]$$

for all r outside of a set E with finite Lebesgue measure, where $c_u > 0$ is a constant depending on u. Combining (3.3), (3.11) and (3.12) yields

$$(3.13) \qquad \sum_{j=1}^{q} m_f(r, D_j) \leq \frac{d(n+1)!}{N^{n+1}(1 + O(N^{-1}))}\Big[(u+1)T_F(r, s_0)$$

$$+ c_u(\log^+ T_F(r, s_0) + \operatorname{Ric}_\tau(r, s_0)$$

$$+ \log^+ Y(r) + \log^+ r) + \kappa(c_3 + uc_4)\Big]$$

We now compare $T_F(r, s_0)$ and $T_f(r, s_0)$. To do so, we need to introduce the concept of a *reduced representation section of f* (see [S]) to overcome the difficulty that there is, in general, no goal reduced representation of f on M. Let $\{U_\lambda, \lambda \in \Lambda\}$ be an open covering of M, and let $\mathbf{f}_\lambda : U_\lambda \to \mathbb{C}^{n+1}$ be a reduced representation of f on U_λ. Then there is a holomorphic function $g_{\lambda\mu} : U_\lambda \cap U_\mu \to \mathbb{C}_*$ such that

$$(3.14) \qquad\qquad \mathbf{f}_\lambda = g_{\lambda\mu}\mathbf{f}_\mu \quad \text{on} \quad U_\lambda \cap U_\mu.$$

It is easy to check that $\{g_{\lambda\mu}\}$ is a basic cocycle (cf. [S]). Therefore there exists a holomorphic line bundle H_f on M with a holomorphic frame atlas $\{U_\lambda, s_\lambda\}_{\lambda \in \Lambda}$ such that

$$(3.15) \qquad\qquad s_\lambda = g_{\mu\lambda}v_\mu \quad \text{on} \quad U_\lambda \cap U_\mu.$$

The line bundle H_f is called the *hyperplane section bundle of f*. Also define $\tilde{\mathbf{f}}_\lambda \in \Gamma(U_\lambda, M \times \mathbb{C}^{n+1})$ by $\tilde{\mathbf{f}}_\lambda(z) = (z, \mathbf{f}_\lambda(z))$ for $z \in U_\lambda$. Notice that $\tilde{\mathbf{f}}_\lambda \otimes s_\lambda = g_{\lambda\mu}\tilde{\mathbf{f}}_\mu \otimes s_\lambda = \tilde{\mathbf{f}}_\mu \otimes g_{\lambda\mu}s_\lambda = \tilde{\mathbf{f}}_\mu \otimes s_\mu$ on $U_\lambda \cap U_\mu$. Therefore there exists a holomorphic section Γ_f of $(M \times \mathbb{C}^{n+1}) \otimes H_f$ such that $\Gamma_f|_{U_\lambda} = \tilde{\mathbf{f}}_\lambda \otimes s_\lambda$. Γ_f is called the *standard reduced representation section of f*. Let ℓ be the standard hermitian metric along the fibers of the trivial bundle $M \times \mathbb{C}^{n+1}$ and let κ be a hermitian metric along the fibers of H_f. Then

$$dd^c \log \|\Gamma_f\|_{\ell \otimes \kappa}^2 = dd^c \log \|\mathbf{f}_\lambda\|^2 + dd^c \log \|s_\lambda\|_\kappa^2 = f^*\Omega_{FS} - c_1(H_f, \kappa),$$

where Ω_{FS} is the Fubini-Study form on $\mathbb{P}^n(\mathbb{C})$. Hence, by Green's formula (cf. [S]), we have

$$(3.16) \quad T_f(r, s_0) = \int_{s_0}^r \frac{dt}{t^{2m-1}} \int_{M[t]} f^* c_1(H_f, \kappa) \wedge v^{m-1} + \int_{M\langle r \rangle} \log \|\Gamma_f\|_{\ell \otimes \kappa} \sigma$$
$$- \int_{M<s_0>} \log \|\Gamma_f\|_{\ell \otimes \kappa} \sigma.$$

Let

$$(3.17) \qquad T(H_f, r) = \int_{s_0}^r \frac{dt}{t^{2m-1}} \int_{M[t]} f^* c_1(H_f, \kappa) \wedge v^{m-1}.$$

Then

$$(3.18) \qquad T_f(r, s_0) = \int_{M\langle r \rangle} \log \|\Gamma_f\|_{\ell \otimes \kappa} \sigma + T(H_f, r) + O(1).$$

Similarly, we have

$$(3.19) \qquad T_F(r, s_0) = \int_{M\langle r \rangle} \log \|\Gamma_F\|_{\ell \otimes \kappa} \sigma + T(H_F, r) + O(1).$$

By comparing the transition functions of H_f and H_F, it is clear that $H_F = H_f^N$, so we have $T(H_F, r) = NT(H_f, r)$ and

$$\frac{\|\Gamma_F\|_{\ell \otimes \kappa}}{\|\Gamma_f\|_{\ell \otimes \kappa}^N} = \frac{\|\mathbf{F}_\lambda(z)\|}{\|\mathbf{f}_\lambda(z)\|^N} \leq O(1).$$

These estimates, together with (3.18) and (3.19), imply that $T_F(r, s_0) \leq NT_f(r, s_0) + O(1)$. Therefore, (3.13) becomes

$$(3.20) \quad \sum_{j=1}^q m_f(r, D_j) \leq \frac{d(n+1)!}{N^{n+1}(1 + O(N^{-1}))} \Big[(u+1)NT_f(r, s_0)$$
$$+ c_u(\log^+ T_f(r, s_0) + \mathrm{Ric}_\tau(r, s_0)$$
$$+ \log^+ Y(r) + \log^+ r + O(1)) + \kappa(c_3 + uc_4) \Big].$$

Since, by (3.5),

$$(3.5) \qquad u = \frac{(N+n)!}{N! n!} = \frac{N^n}{n!} + O(N^{n-1}),$$

and the various constants in the "O" term above depend only on $Q_1, \ldots, Q_q$, but not on f and z, we can take N large enough such that

$$\frac{d(n+1)! Nu}{N^{n+1}(1 + O(N^{-1}))} = \frac{d(n+1)(1 + O(N^{-1}))}{(1 + O(N^{-1}))} < d(n+1) + \epsilon/2.$$

Then we have, for N large enough,

$$\sum_{j=1}^{q} m_f(r, D_j) \leq (d(n+1) + \epsilon)T_f(r, s_0) + c_\epsilon(\log^+ T_f(r, s_0)$$
$$+ \operatorname{Ric}_\tau(r, s_0) + \log^+ Y(r) + \log^+ r),$$

where the inequality holds for all r outside of a set E with finite Lebesgue measure and $c_\epsilon > 0$ is a constant depends on ϵ. We note that the exceptional set E appearing here may be different each time, but still has finite Lebesgue measure. This finishes the proof of the Main Theorem. $\square$

For a hypersurface D of degree d, define the defect

$$\delta_f(D) = \liminf_{r \to +\infty} \frac{m_f(r, D)}{dT_f(r)}.$$

Then we have the following defect relation.

COROLLARY (DEFECT RELATION).   *Let M be an admissible parabolic manifold. Let $f : M \to \mathbb{P}^n(\mathbb{C})$ be an algebraically non-degenerate meromorphic map, and let $D_1, \ldots, D_q$ be hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ in general position. Assume that*

$$\limsup_{r \to +\infty} \frac{\operatorname{Ric}_\tau(r, s_0)}{T_f(r)} = 0, \quad \limsup_{r \to +\infty} \frac{\log Y(r)}{T_f(r)} = 0.$$

Then we have

$$\sum_{j=1}^{q} \delta_f(D_j) \leq n + 1.$$

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