

BERGMAN AND REINHARDT WEIGHTED SPACES OF HOLOMORPHIC FUNCTIONS

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ABSTRACT. We study isometries between spaces of weighted holomorphic functions defined on bounded domains in \mathbf{C}^n . Using the Bergman kernel we see that it is possible to define a ‘natural’ weight on bounded domains in \mathbf{C}^n . We calculate the isometries of weighted spaces of holomorphic functions on the unit ball, the Thullen domains, the generalised Thullen domains and the domain with minimal complex norm.

In a recent paper Bonet and Wolf [3] showed that the space of weighted holomorphic functions $\mathcal{H}_{v_o}(U)$, v a continuous strictly positive weight on an open subset U of \mathbf{C}^n , is almost isometrically isomorphic to a subspace of c_o . This paper extends the work of Lusky [17], [18], [19], [20], [21] and [22]. In [18] he showed that there are weights on the unit disc Δ which are isomorphic to a subspace of c_o but which are not isomorphic to c_o . In [6] the authors looked at the isometric theory of weighted spaces of holomorphic functions. This work is based on the geometric theory of spaces of holomorphic functions described in [4] and [5]. There, it was observed that the geometric structure of weighted spaces of holomorphic functions was determined by a distinguished subset of U . This subset is denoted by $\mathcal{B}_v(U)$ and is called the v -boundary of U . A weight where $\mathcal{B}_v(U) = U$ is said to be complete. In [6] it is shown that if T is an isometric isomorphism of $\mathcal{H}_{v_o}(U)$ then there is a homeomorphism ϕ of $\mathcal{B}_v(U)$ onto $\mathcal{B}_v(U)$ and h_ϕ in $\mathcal{H}_{v_o}(U)$ so that

$$T(f)(z) = h_\phi(z)f \circ \phi(z)$$

for all z in $\mathcal{B}_v(U)$. When v is complete ϕ is an automorphism of U . The set of all ϕ which give us an isometric isomorphism of $\mathcal{H}_{v_o}(U)$ is called the isometry group of v . In [6] we calculate the isometry group of several families of weights on the unit disc in \mathbf{C} , Δ . In addition we prove that the only weights on the unit disc with isometry group equal to $\text{Aut}(\Delta)$ are those of the form $v(z) = e^{k(z)}(1 - |z|^2)^\alpha$ where $\alpha > 0$ and k is a bounded real-valued harmonic function

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on Δ . In this paper we will give other examples of weights on a domain D which have isometry group equal to $\text{Aut}(D)$. The Riemann Mapping Theorem limits our choices of proper open simply connected subsets of \mathbf{C} and therefore to obtain examples of such weights we must look in higher dimensions. The Laplacian which we used in [6] is replaced in higher dimensions by the Monge-Ampère operator. The reason we have restricted ourselves to two dimensions, in general, is the difficulty in calculations in even higher dimensions. We show that there is a ‘natural’ weight which may be defined on any domain using the Bergman metric. In the case of bounded symmetric domains these give us weights with large isometry groups.

Let us give some definitions. Let U be a bounded open subset of \mathbf{C}^n . A continuous weight v on U is a continuous, bounded, strictly positive real valued function on U . We will use $\mathcal{H}_v(U)$ to denote the space of all holomorphic functions f on U which have the property that $\|f\|_v := \sup_{z \in U} v(z)|f(z)| < \infty$. Endowed with the norm $\|\cdot\|_v$ $\mathcal{H}_v(U)$ becomes a Banach space. Consider all f in $\mathcal{H}_v(U)$ with the property that $v(z)|f(z)|$ converges to 0 as z converges to the boundary of U , i.e., given $\epsilon > 0$ there is a compact subset K of U such that $v(z)|f(z)| < \epsilon$ for z in $U \setminus K$. The set of all such functions is a subspace of $\mathcal{H}_v(U)$ denoted by $\mathcal{H}_{v_o}(U)$. We say that the weight v on a balanced domain is radial if $v(\lambda z) = v(z)$ for all λ in $\Gamma := \{\lambda \in \mathbf{C} : |\lambda| = 1\}$ and a weight v on the unit ball of \mathbf{C}^n is unitary if it is invariant under all unitary matrices. In [4] we showed that the set of extreme points of the unit ball of $\mathcal{H}_{v_o}(U)'$ is contained in $\{\lambda v(z)\delta_z : z \in U, \lambda \in \Gamma\}$. To avoid the degenerate case we shall assume that $v(z)$ converges to 0 as z converges to the boundary of U . The v -boundary of U is defined as the set of all $z \in U$ such that $v(z)\delta_z$ is an extreme point of the unit ball of $\mathcal{H}_{v_o}(U)'$. Note that $v(z)\delta_z$ is an extreme point of the unit ball of $\mathcal{H}_{v_o}(U)'$ if and only if $\lambda v(z)\delta_z$ is an extreme point for every λ in Γ . We use $\mathcal{B}_v(U)$ to denote the v -boundary. It is shown in [4] that $\mathcal{B}_v(U)$ is radial when v is radial and unitary when v is unitary. Furthermore, the mapping $\mu: U \rightarrow (\mathcal{H}_{v_o}(U)', \sigma(\mathcal{H}_{v_o}(U)', \mathcal{H}_{v_o}(U)))$, $\mu(z) = v(z)\delta_z$, is a homeomorphism onto its range that allows to show that $\mathcal{B}_v(U)$ is a G_δ subset of U . We say that a weight v on U is complete if $\mathcal{B}_v(U) = U$. A criterion for a unitary weight v on the unit ball of \mathbf{C}^n to be complete is given in [5, Proposition 8].

In [6] we proved the following

THEOREM 0.1. *Let U be a bounded open subset of \mathbf{C}^n and v be a continuous strictly positive weight converging to 0 on the boundary of U . If $T: \mathcal{H}_{v_o}(U) \rightarrow \mathcal{H}_{v_o}(U)$ is an isometric isomorphism there is a homeomorphism $\phi: \mathcal{B}_v(U) \rightarrow \mathcal{B}_v(U)$ and $h_\phi \in \mathcal{H}_{v_o}(U)$ such that*

$$T(f)(z) = h_\phi(z)f \circ \phi(z),$$

for all $f \in \mathcal{H}_{v_o}(U)$, $z \in \mathcal{B}_v(U)$. Moreover

$$|h_\phi| = \frac{v \circ \phi}{v}$$

on $\mathcal{B}_v(U)$.

Further, if $\mathring{\mathcal{B}}_v(U)$ is non-empty then $\phi : \mathring{\mathcal{B}}_v(U) \rightarrow \mathring{\mathcal{B}}_v(U)$ is a biholomorphic mapping. In the case where v is a radial weight on Δ , ϕ extends to an automorphism of Δ . Further it is shown in [6, Theorem 2] that when U is balanced and v is radial then the isometries of $\mathcal{H}_v(U)$ also have the form

$$T(f)(z) = h_\phi(z)f \circ \phi(z),$$

for all $f \in \mathcal{H}_v(U)$, $z \in \mathcal{B}_v(U)$, where ϕ and h_ϕ are as in the case of $\mathcal{H}_{v_o}(U)$. When v is radial the isometries of vector-valued weighted spaces of holomorphic functions can also be described. The set of homeomorphisms ϕ of $\mathcal{B}_v(U)$ which induce an isometry of $\mathcal{H}_{v_o}(U)$ is denoted by $\Lambda_v(U)$ and is called the isometry group of v . To calculate the isometry group we will make use of the following theorem found in [6].

THEOREM 0.2. *Let v be a complete, continuous, strictly positive weight on a bounded open simply connected set U which converges to 0 on the boundary of U . Then $\phi \in \text{Aut}(U)$ belongs to $\Lambda_v(U)$ if and only if $\log(v \circ \phi/v)$ is pluriharmonic.*

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1. Bergman weights

Let D be a domain in \mathbf{C}^n and λ denote Lebesgue measure on \mathbf{C}^n . Let $\mathcal{H}^2(D) = \mathcal{H}(D) \cap \mathcal{L}^2(D)$. Given $w \in D$ there is a holomorphic function k_w in $\mathcal{H}^2(D)$ such that

$$f(w) = \int_D f(z) \overline{k_w(z)} d\lambda(z),$$

for all f in $\mathcal{H}^2(D)$ (see [16] and [14]). The Bergman kernel on D is defined by

$$k(z, w) = k_{\bar{w}}(z).$$

It follows from [14] (see also [16]) that $k(z, z) \geq 0$. Furthermore, if D is bounded then $k(z, z) > 0$ for $z \in D$.

Let us define the *Bergman weight*, v_B , on a bounded domain D by $v_B(z) = k(z, z)^{-1}$. Then v_B is a continuous weight on D . It follows from [14, Proposition 1.4.12] that if D is balanced then v_B is a radial weight.

We let $\mathcal{A}(D)$ denote the space of all f which are analytic on D and which extend continuously to the boundary of D . As noted by Krantz in [14, p. 53] if D is strictly convex (or each point of the boundary of D has a peaking function in $\mathcal{A}(D)$) then $k(z, z) \rightarrow \infty$ as z converges to ∂D . Thus if each point

of the boundary of D has a peaking function in $\mathcal{A}(D)$ then v_B converges to 0 as z converges to the boundary of D .

PROPOSITION 1.1. *Let D be a bounded domain in \mathbf{C}^n such that each point of the boundary of D has a peaking function in $\mathcal{A}(D)$. Then $\Lambda_{v_B}(D) = \text{Aut}(D)$.*

Proof. Given $\phi \in \text{Aut}(D)$ we define $T_\phi: \mathcal{H}_{(v_B)_o}(D) \rightarrow \mathcal{H}_{(v_B)_o}(D)$ by

$$T_\phi(f)(z) = (\det \partial_z \phi)^{-2} f \circ \phi(z).$$

Then using [14, Proposition 1.4.12] we get

$$\begin{aligned} \|T_\phi(f)\|_v &= \sup_{z \in D} v_B(z) |(\det \partial_z \phi)^{-2} f \circ \phi(z)| \\ &= \sup_{z \in D} k(z, z)^{-1} |(\det \partial_z \phi)^{-2} f \circ \phi(z)| \\ &= \sup_{z \in D} k(\phi(z), \phi(z))^{-1} |f \circ \phi(z)| \\ &= \|f\|_v, \end{aligned}$$

proving that T_ϕ is an isometry. □

Applying [2, Theorem 3.1] we obtain the following result.

COROLLARY 1.2. *Let D be a bounded pseudoconvex domain in \mathbf{C}^2 with smooth real analytic boundary. Then $\Lambda_{v_B}(D) = \text{Aut}(D)$.*

According to E. Cartan a bounded domain D is classical if $\text{Aut}(D)$ is a classical Lie group which acts transitively on D . A domain is irreducible if it cannot be written as the non-trivial Cartesian products of two domains. Cartan classified the irreducible domains into four classical families and two exceptional domains in dimensions 16 and 27 ($M_{1,2}(\mathbf{O})$ and $H_3(\mathbf{O})$). Each of these domains corresponds to the unit ball of an irreducible JB*-triple; see [7].

The *classical domains of the first type* are denoted by $\Omega_{p,q}^1$ and consist of all $p \times q$ matrices Z , $p \geq q \geq 1$, such that $I_q - Z^*Z$ is positive definite, where I_q is the $q \times q$ identity matrix and $Z^* = \bar{Z}'$. This domain has dimension $p \times q$. The domain $\Omega_{n,1}^1$ is the unit ball of \mathbf{C}^n and v_B in this case is the weight $(1 - \|z\|^2)^{n+1}$.

The *classical domains of the second type* are denoted by $\Omega_{p,p}^2$ and consist of all symmetric $p \times p$ matrices, Z , such that $I_p - Z^*Z$ is positive definite. The domain $\Omega_{p,p}^2$ has dimension $p(p+1)/2$.

The *classical domains of the third type* are denoted by $\Omega_{p,p}^3$ and consist of all skew-symmetric $p \times p$ matrices, Z , such that $I_p + Z^*Z$ is positive definite. The domain $\Omega_{p,p}^3$ has dimension $p(p-1)/2$.

The *classical domains of the fourth type* are denoted by Ω_n^4 and consist of all $z \in \mathbf{C}^n$, such that $|\langle z, z \rangle|^2 + 1 - 2\langle \bar{z}, z \rangle > 0$ and $|\langle z, z \rangle| < 1$. Domains in Ω_n^4 are called *spin factors*.

The *first of Cartan's two exceptional domains* Ω_{16}^5 is defined as

$$\Omega_{16}^5 = \left\{ (Z, U) \in \mathbf{C}^8 \times \mathbf{C}^8 : \frac{1}{2i}(Z - Z^*) - \frac{1}{2}(UU^* + \bar{U}U^t) > 0 \right\},$$

where

$$Z = \begin{pmatrix} z_1, & z_2 & \dots & \dots & z_7 \\ z_2, & z_8 & O, & \dots, & O \\ z_3, & O & \ddots & \dots & O \\ \vdots & \vdots & \vdots & \ddots & O \\ z_7, & O & \dots & O & z_8 \end{pmatrix}, \quad U = \begin{pmatrix} t \\ uQ_1 \\ \vdots \\ uQ_6 \end{pmatrix},$$

where $t, u \in \mathbf{C}^4$ and Q_1, Q_2, \dots, Q_6 are six 4×4 -matrices (see [26] for a complete list).

The *second exceptional domain* is

$$\Omega_{27}^6 = \left\{ (z_{11}, z_{12}, z_{13}, z_{22}, z_{23}, z_{33}) \in \mathbf{C} \times \mathbf{C}^8 \times \mathbf{C}^8 \times \mathbf{C} \times \mathbf{C}^8 \times \mathbf{C} : \frac{1}{2i}(Z - Z^*) > 0 \right\},$$

where

$$Z = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z'_{12} & z_{22}I_8 & z_{23} \\ z'_{13} & z'_{23} & z_{33}I_8 \end{pmatrix}, \quad z_{23} = \begin{pmatrix} zT_1 \\ \vdots \\ zT_8 \end{pmatrix},$$

$z = (z_1, \dots, z_8) \in \mathbf{C}^8$ and T_1, \dots, T_8 are a family of 8×8 -matrices (see [26] for the complete list).

When $p = n = 1$ the first four domains are equal to the unit disc in \mathbf{C} . We observe that $\Omega_{2,1}^1$ is equal to the unit ball in \mathbf{C}^2 . Domains of type two and three cannot arise as subspaces of \mathbf{C}^2 . In \mathbf{C}^3 , $\Omega_{3,1}^1$ is the unit ball while $\Omega_{2,2}^2$, $\Omega_{3,3}^3$ and Ω_3^4 all coincide. In \mathbf{C}^4 we get distinct domains of types one, two, three and four.

Given a domain D in \mathbf{C}^n we let $V(D)$ denote the volume of D with respect to the Bergman metric.

The Bergman kernel for the four classical domains was calculated in 1958 by Hua Loo-Keng [10] as

$$\begin{aligned} \Omega_{p,q}^1 : \quad & k(W, Z) = V(\Omega_{p,q}^1)^{-1}(\det(I - W^*Z))^{-(p+q)}, \\ \Omega_{p,p}^2 : \quad & k(W, Z) = V(\Omega_{p,p}^2)^{-1}(\det(I - W^*Z))^{-(p+1)}, \\ \Omega_{p,p}^3 : \quad & k(W, Z) = V(\Omega_{p,p}^3)^{-1}(\det(I + W^*Z))^{-(p-1)}, \\ \Omega_n^4 : \quad & k(w, z) = V(\Omega_n^4)^{-1}(1 + |\langle w, z \rangle|^2 - 2\langle \bar{w}, z \rangle)^{-n}. \end{aligned}$$

The Bergman kernel, restricted to the diagonal, for the two exceptional domains was calculated by Yin Weiping [26] as

$$\begin{aligned} \Omega_{16}^5 : \quad & k((Z, U), (Z, U)) = \frac{\left(\frac{1}{2i}(z_8 - \bar{z}_8 - uu^*)\right)^{60}}{\det\left(\frac{1}{2i}(Z - Z^*) - \frac{1}{2}(UU^* + \bar{U}U')\right)^{12}}, \\ \Omega_{27}^6 : \quad & k(Z, Z) = \frac{\left(\frac{1}{2i}(z_{22} - \bar{z}_{22})\frac{1}{2i}(z_{33} - \bar{z}_{33}) - \frac{1}{2i}(z - \bar{z})\frac{1}{2i}(z - \bar{z})'\right)^{126}}{\det\left(\frac{1}{2i}(Z - Z^*)\right)^{18}}. \end{aligned}$$

Further on each of these domains the extreme points coincide with the Bergman-Shilov boundary of D which may be strictly smaller than the boundary of D . See [11, pages 31–34].

THEOREM 1.3. *Let D be the unit ball of a finite dimensional JB^* -triple. Then $\Lambda_{v_B}(D) = \text{Aut}(D)$.*

Proof. For each of the domains $\Omega_{p,q}^1, \Omega_{p,p}^2, \Omega_{p,p}^3, \Omega_n^4, \Omega_{16}^5$ and Ω_{27}^6 we see that $v_B(z)$ converges to 0 as z converges to the boundary of D . Since every finite dimensional JB^* -triple is a subtriple of a product of irreducible JB^* -triples and the unit ball of an irreducible JB^* -triple has one of the above forms we see that $v_B(z)$ tends to 0 as z tends to the boundary of D . Now notice that the Bergman kernel of $D_1 \times D_2$ is the product of the Bergman kernel of D_1 and the Bergman kernel of D_2 and apply the argument of Proposition 1.1. \square

We note that the automorphism group D acts transitively on D when D is the unit ball of a JB^* -triple. We recall that if U is a bounded open subset of \mathbf{C}^n and v is a continuous strictly positive weight which converges to 0 on the boundary of U then z in U is said to be a v -peak point if there is f in the unit ball of $\mathcal{H}_{v_o}(U)$ with $v(z)f(z) = 1$ and $v(w)|f(w)| < 1$ for all w in $U \setminus \{z\}$. In [5, Theorem 3] we proved that z is a v -peak point if and only if $v(z)\delta_z$ is a weak*-exposed point of $B_{\mathcal{H}_{v_o}(U)'}.$ Furthermore, the sets of weak*-strongly exposed points and of weak*-exposed points of $B_{\mathcal{H}_{v_o}(U)'}.$ coincide. In particular, if each point of U is a v -peak point then v is complete.

COROLLARY 1.4. *Let D be the unit ball of a finite dimensional JB^* -triple. Then v_B is complete on D .*

Proof. We first observe that as $\mathcal{H}_{(v_B)_o}(U)$ is a separable dual space its unit ball has a weak*-exposed point and therefore the set of v -peak points is non-empty. Therefore we can find z_o in D and $f \in \mathcal{H}_{(v_B)_o}(D)$ so that $v_B(z_o)f(z_o) = 1$ and $v_B(z)|f(z)| < 1$ for all $z \in D \setminus \{z_o\}$. Given $w_o \in D$ we can find an automorphism g of D so that $w_o = g(z_o)$. Let $h(z) = (\det \partial_z g)^2 f \circ g^{-1}(z)$. Then

$$v_B(w_o)h(w_o) = v_B(w_o)(\det \partial_{w_o} g)^2 f \circ g^{-1}(w_o) = v_B(z_o)f(z_o) = 1$$

and

$$v_B(w)|h(w)| = v_B(w)|(\det \partial_w g)^2| |f \circ g^{-1}(w)| = v_B(z)|f(z)| < 1$$

for $w \in D \setminus \{w_o\}$. Therefore each point of D is a v -peak point and so v_B is complete. \square

2. Reinhardt weights

In [5, Proposition 8] we showed that if $v: B_{\mathbf{C}^n} \rightarrow \mathbf{R}$ is a continuous strictly positive strictly decreasing unitary weight on the unit ball of \mathbf{C}^n which converges to 0 on the boundary of $B_{\mathbf{C}^n}$ such that $v(x)$ is twice differentiable and

$$\left(\frac{\partial v(x)}{\partial x_1}\right)^2 - v(x)\frac{\partial^2 v(x)}{\partial x_1^2} > 0$$

for x of the form $(x_1, 0, \dots, 0)$ with x_1 in $(0, 1)$, then the set of weak*-exposed points of the unit ball of $\mathcal{H}_{v_o}(B_{\mathbf{C}^n})'$ is the set $\{v(z)\lambda\delta_z : \lambda \in \Gamma, z \in B_{\mathbf{C}^n}\}$ and hence v is complete. In this section we see how like unitary invariance another form of symmetry can lead to a criterion for completeness.

DEFINITION 2.1. A domain D is said to be balanced (resp. radial) if z in D , $\lambda \in \overline{\Delta}$ (resp. $\lambda \in \Gamma$) implies $\lambda z \in D$.

DEFINITION 2.2. A domain D in \mathbf{C}^2 is said to be *Reinhardt* if given $(z_1, z_2) \in D$, $\lambda_1, \lambda_2 \in \Gamma$ we have that $(\lambda_1 z_1, \lambda_2 z_2) \in D$.

We note that every Reinhardt domain is radial.

We shall say that a weight v on a balanced open subset U of \mathbf{C}^n is strictly decreasing if for each z on the boundary of U , v is strictly decreasing on the ray $\{\lambda z : \lambda \in [0, 1]\}$.

DEFINITION 2.3. Let U be a balanced Reinhardt domain in \mathbf{C}^2 and v be a continuous, strictly decreasing, strictly positive weight which converges to 0 on the boundary of U . We say that v is Reinhardt if $v(z_1, z_2) < v(z_1, 0)$, $v(z_1, z_2) < v(0, z_2)$ when $z_1 \neq 0$, $z_2 \neq 0$ and $v(\lambda_1 z_1, \lambda_2 z_2) = v(z_1, z_2)$ for all $(z_1, z_2) \in U$, $\lambda_1, \lambda_2 \in \Gamma$.

Strictly decreasing unitary weights are Reinhardt and Reinhardt weights are radial.

Given $p, q \geq 1/2$ and $p, q \neq 1$ we define the generalised Thullen domain D_{pq} by

$$D_{pq} := \{(z_1, z_2) \in \mathbf{C}^2 : |z_1|^{2p} + |z_2|^{2q} < 1\}.$$

The space D_{pq} is a Reinhardt domain.

The weights

$$v_{p,q}(z) = (1 - |z_1|^{2p} - |z_2|^{2q})$$

and

$$w_{p,q}(z) = e^{\frac{-1}{(1-|z_1|^{2p}-|z_2|^{2q})}}$$

on the generalised Thullen domain are examples of Reinhardt weights.

THEOREM 2.4. *Let U be a bounded balanced Reinhardt domain in \mathbf{C}^2 and v be a continuous strictly positive strictly decreasing Reinhardt weight decreasing to 0 on the boundary of U which has continuous second order partial derivatives on $U \setminus \{0\}$. Suppose:*

- (a) $vv_{xx} - v_x^2 < 0$ on $\{(x_1, 0) : x_1 > 0\} \cap U$.
- (b) $vv_{yy} - v_y^2 < 0$ on $\{(0, x_2) : x_2 > 0\} \cap U$.
- (c) *The function $(x, y) \rightarrow ((v_x/v)(x, y), (v_y/v)(x, y))$ is injective on $\Sigma := \{(x, y) \in U : x, y \geq 0\}$.*
- (d) $vv_{xx} - v_x^2 < 0$ and

$$(vv_{xy} - v_x v_y)^2 < (vv_{xx} - v_x^2)(vv_{yy} - v_y^2)$$

$$\text{on } \{(x_1, x_2) : (x_1, x_2) \in U, x_1, x_2 > 0\}.$$

Then v is complete.

Proof. By [5, Theorem 3] it suffices to show that each point of U is a v -peak point. First observe that a suitably normalised constant function will weak*-expose the unit ball of $H_v(U)'$ at $v(0)\delta_0$. Consider $(x_1, 0)$ where $0 < x_1$ is real and let

$$\alpha = \frac{-1}{v(x_1, 0)} \frac{\partial v}{\partial x_1}(x_1, 0).$$

Define f on U by $f(z) = e^{\alpha z_1}$. It follows as in unitary case [5, Theorem 3] that the restriction of vf to $\{z \in U : z_2 = 0\}$ peaks at $(x_1, 0)$. For $z_2 \neq 0$ we have that $v(z_1, z_2)|f(z_1, z_2)| < v(z_1, 0)|f(z_1, 0)|$ and therefore $v|f|$ peaks on U at $(x_1, 0)$. Next suppose that $(z_1, 0) \in U$, $z_1 \neq 0$. If f is constructed in the above way so that $v|f|$ peaks at $(|z_1|, 0)$ and $\lambda = z_1/|z_1|$ then $v|f \circ \bar{\lambda}|$ will peak at $(z_1, 0)$.

An analogous argument shows that we can find a function $\tilde{f} \in \mathcal{H}_{v_o}(U)$ so that $v|\tilde{f}|$ peaks at $(0, z_2)$ for any $z_2 \neq 0$.

Let us now consider the case where $(x_1, x_2) \in U$ with $x_1, x_2 > 0$. Let us define α and β by

$$\alpha = \frac{-1}{v(x_1, x_2)} \frac{\partial v}{\partial x_1}(x_1, x_2)$$

and

$$\beta = \frac{-1}{v(x_1, x_2)} \frac{\partial v}{\partial x_2}(x_1, x_2).$$

Define $g: U \rightarrow \mathbf{C}$ by

$$g(z_1, z_2) = v(z_1, z_2) |e^{\alpha z_1 + \beta z_2}|.$$

Consider $(z_1, z_2) \in U$ with $(|z_1|, |z_2|) = (x_1, x_2)$. Then

$$g(z_1, z_2) = v(x_1, x_2) e^{\alpha x_1 \cos \theta_1 + \beta x_2 \cos \theta_2}$$

where θ_1 and θ_2 are the arguments of z_1 and z_2 respectively. Thus we see that g attains its maximum on $A := \{(z_1, z_2) : |z_1| = x_1, |z_2| = x_2\}$ at (x_1, x_2) and $g(z_1, z_2) < g(x_1, x_2)$ for $(z_1, z_2) \in A, (z_1, z_2) \neq (x_1, x_2)$.

We now consider g restricted to the set $\Sigma := \{(x, y) \in U : x, y \geq 0\}$. Then g has a critical point on Σ at (x_1, x_2) . Since the function $(x, y) \rightarrow ((v_x/v)(x, y), (v_y/v)(x, y))$ is injective on Σ it follows that (x_1, x_2) is the only critical point of the restriction of g to Σ . Doing a little calculation we obtain

$$\begin{aligned} g_{xx}(x_1, x_2) &= e^{\alpha x_1 + \beta x_2} (v_{xx}(x_1, x_2) - v_x^2(x_1, x_2)/v(x_1, x_2)), \\ g_{yy}(x_1, x_2) &= e^{\alpha x_1 + \beta x_2} (v_{yy}(x_1, x_2) - v_y^2(x_1, x_2)/v(x_1, x_2)) \end{aligned}$$

and

$$g_{xy}(x_1, x_2) = e^{\alpha x_1 + \beta x_2} (v_{xy}(x_1, x_2) - v_x(x_1, x_2)v_y(x_1, x_2)/v(x_1, x_2)).$$

The second derivative test tells us that g restricted to Σ has a local maximum at (x_1, x_2) if

$$g_{xy}(x_1, x_2)^2 < g_{xx}(x_1, x_2)g_{yy}(x_1, x_2)$$

and

$$g_{xx}(x_1, x_2) < 0.$$

This now reduces to the conditions

$$(vv_{xy} - v_x v_y)^2 < (vv_{xx} - v_x^2)(vv_{yy} - v_y^2)$$

and $vv_{xx} - v_x^2 < 0$.

Finally, we consider a point (z_1, z_2) in U with $z_1, z_2 \neq 0$. By the previous paragraph we can find a $f \in \mathcal{H}_{v_o}(U)$ so that $v|f|$ peaks at $(|z_1|, |z_2|)$. If $\lambda_1 = z_1/|z_1|$ and $\lambda_2 = z_2/|z_2|$, using the fact that v is Reinhardt it follows that $v(z_1, z_2)|f(\bar{\lambda}_1 z_1, \bar{\lambda}_2 z_2)|$ peaks at (z_1, z_2) . Thus v is complete. \square

We can rewrite the above condition as follows:

THEOREM 2.5. *Let U be a bounded balanced Reinhardt domain in \mathbf{C}^2 and v be a continuous strictly positive strictly decreasing Reinhardt weight decreasing to 0 on the boundary of U which has continuous second order partial derivatives on $U \setminus \{0\}$. Suppose:*

- (a) $(\log v)_{xx} < 0$ on $\{(x_1, 0) : x_1 > 0\} \cap U$.
- (b) $(\log v)_{yy} < 0$ on $\{(0, x_2) : x_2 > 0\} \cap U$.
- (c) *The function $(x, y) \rightarrow ((\log v)_x(x, y), (\log v)_y(x, y))$ is injective on $\Sigma := \{(x, y) \in U : x, y \geq 0\}$.*
- (d) $(\log v)_{xx} < 0$ and

$$((\log v)_{xy})^2 < (\log v)_{xx}(\log v)_{yy}$$

$$\text{on } \{(x_1, x_2) : x_1, x_2 > 0, (x_1, x_2) \in U\}.$$

Then v is complete.

COROLLARY 2.6. *The weight*

$$v_{p,q}(z) = (1 - |z_1|^{2p} - |z_2|^{2q})$$

on the generalised Thullen domain D_{pq} is complete.

Proof. For simplicity of notation let $v = v_{pq}$. Since $v_{xx} < 0$ we have that $vv_{xx} - v_x^2 < 0$. Further, as $v_{xy} = 0$, $0 > -v_x^2 > vv_{xx} - v_x^2$ and $0 > -v_y^2 > vv_{yy} - v_y^2$ we see that

$$(vv_{xy} - v_x v_y)^2 = v_x^2 v_y^2 = (-v_x)^2 (-v_y)^2 < (vv_{xx} - v_x^2)(vv_{yy} - v_y^2).$$

Suppose we have distinct points (x_1, y_1) and (x_2, y_2) in Σ with

$$((v_x/v)(x_1, y_1), (v_y/v)(x_1, y_1)) = ((v_x/v)(x_2, y_2), (v_y/v)(x_2, y_2)).$$

Since $(v_x/v_y)(x_1, y_1) = (v_x/v_y)(x_2, y_2)$ we see that

$$(x_2, y_2) = (\lambda x_1, \lambda^{2p-1/2q-1} y_1).$$

This means that we may assume without loss of generality that $x_1 < x_2$ and $y_1 < y_2$. But as $vv_{xx} - v_x^2 < 0$, $vv_{xy} - v_x v_y < 0$, this means that $(v_x/v)(x_1, y_1) < (v_x/v)(x_2, y_2)$, a contradiction. Theorem 2.4 implies that v_{pq} is complete. \square

COROLLARY 2.7. *The weight w_{pq} on the generalised Thullen domain D_{pq} given by*

$$w_{pq}(z) = e^{\frac{-1}{1-|z_1|^{2p}-|z_2|^{2q}}}$$

is complete.

Proof. Let $w = w_{pq}$. We have

$$\begin{aligned}
(\log w)_{xy}^2 &= (-2(2p)(2q)x_1^{2p-1}x_2^{2q-1}(1-x_1^{2p}-x_2^{2q})^{-3})^2 \\
&= (-2(2p)^2x_1^{4p-2}(1-x_1^{2p}-x_2^{2q})^{-3})(-2(2q)^2x_2^{4q-2}(1-x_1^{2p}-x_2^{2q})^{-3}) \\
&< (-2(2p)^2x_1^{4p-2}(1-x_1^{2p}-x_2^{2q})^{-3} - 2p(2p-1)x_1^{2p-2} \\
&\quad (1-x_1^{2p}-x_2^{2q})^{-2})(-2(2q)^2x_2^{4q-2}(1-x_1^{2p}-x_2^{2q})^{-3} \\
&\quad - 2q(2q-1)x_2^{2q-2}(1-x_1^{2p}-x_2^{2q})^{-2}) \\
&= (\log w)_{xx}(\log w)_{yy}, \\
(\log w)_{xx} &= -2(2p)^2x_1^{4p-2}(1-x_1^{2p}-x_2^{2q})^{-3} - 2p(2p-1)x_1^{2p-2} \\
&\quad (1-x_1^{2p}-x_2^{2q})^{-2} < 0, \\
(\log w)_{yy} &= -2(2q)^2x_2^{4q-2}(1-x_1^{2p}-x_2^{2q})^{-3} - 2q(2q-1)x_2^{2q-2} \\
&\quad (1-x_1^{2p}-x_2^{2q})^{-2} < 0, \\
(\log w)_{xy} &= -2(2p)(2q)x_1^{2p-1}x_2^{2q-1}(1-x_1^{2p}-x_2^{2q})^{-3} < 0.
\end{aligned}$$

As w_x/w and w_y/w give the same values as v_x/v and v_y/v in the previous corollary we see that the function $(x, y) \rightarrow ((\log w)_x(x, y), (\log w)_y(x, y))$ is injective on Σ . Applying Theorem 2.5 we see that w_{pq} is complete. \square

3. Isometries between spaces of weighted holomorphic functions in higher dimensions

Let Ω be an open set in \mathbf{C}^n , $a \in \Omega$ and $f: \Omega \rightarrow \mathbf{C}$ be real differentiable. We let

$$\partial_a f = \sum_{j=1}^n \frac{\partial f}{\partial z_j}(a) dz_j,$$

and

$$\bar{\partial}_a f = \sum_{j=1}^n \frac{\partial f}{\partial \bar{z}_j}(a) d\bar{z}_j.$$

This allows us to define the differential forms ∂f and $\bar{\partial} f$ by

$$\partial f: a \rightarrow \partial_a f$$

and

$$\bar{\partial} f: a \rightarrow \bar{\partial}_a f.$$

More generally if

$$w = \sum_{\#\alpha=p} \sum_{\#\beta=q} w_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta$$

is a differential form of type (p, q) we let

$$\partial w = \sum_{\alpha, \beta} \partial w_{\alpha\beta} \wedge dz^\alpha \wedge d\bar{z}^\beta$$

and

$$\bar{\partial} w = \sum_{\alpha, \beta} \bar{\partial} w_{\alpha\beta} \wedge dz^\alpha \wedge d\bar{z}^\beta.$$

We now define the operators d and d^c by $d = \partial + \bar{\partial}$ and $d^c = i(\bar{\partial} - \partial)$. Then $dd^c = 2i\partial\bar{\partial}$ and when $u \in \mathcal{C}^2(\Omega)$

$$dd^c u = 2i \sum_{j,k=1}^n \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k.$$

The Monge-Ampère operator, $(dd^c)^n$, in \mathbf{C}^n is defined as the n^{th} exterior power of dd^c . That is,

$$(dd^c)^n = \underbrace{dd^c \wedge \dots \wedge dd^c}_{n\text{-times}}.$$

If

$$dV := \left(\frac{i}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \wedge \dots \wedge dz_n \wedge d\bar{z}_n$$

is the volume form on \mathbf{C}^n then

$$(dd^c u)^n = 4^n n! \det \left[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right] dV.$$

We note that in the case $n = 1$ the Monge-Ampère operator can be identified with the Laplacian.

If f is holomorphic on Ω and does not attain the value zero then $\log |f|$ is pluriharmonic on Ω and therefore $(dd^c \log |f|)^n(z) = 0$ on Ω .

It is shown in [13, Exercise 1.5.4] that if Ω and Ω' are open in \mathbf{C}^n , $u \in \mathcal{C}^2(\Omega')$ and $f: \Omega \rightarrow \Omega'$ is holomorphic then

$$(dd^c(u \circ f))^n(a) = |\det \partial_a f|^2 (dd^c u)^n(f(a)).$$

In 1931 Thullen [25] proved that the only bounded Reinhardt domains D in \mathbf{C}^2 with the identity component of $\text{Aut}(D)$ being non-linear are the ball, $B_{\mathbf{C}^2}$, the polydisc, $\Delta \times \Delta$, and domains of the form $E_m = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1|^2 + |z_2|^{2m} < 1\}$. Domains of the form E_m are called Thullen domains. In [6] we showed that automorphisms of the weighted spaces of holomorphic functions on the polydisc, $\mathcal{H}_{(v \times w)_o}(\Delta \times \Delta)$, were either of the form

$$(T(f))(z_1, z_2) = h_\phi(z_1) h_\psi(z_2) f(\phi(z_1), \psi(z_2))$$

for all $(z_1, z_2) \in \mathring{B}_{v \times w}(\Delta \times \Delta)$, where $\phi \in \Lambda_v(\Delta)$, $\psi \in \Lambda_w(\Delta)$, $h_\phi \in \mathcal{H}_{v_o}(\Delta)$ and $h_\psi \in \mathcal{H}_{w_o}(\Delta)$, or the form

$$(T(f))(z_1, z_2) = h_\phi(\eta(z_2))h_\psi(\eta^{-1}(z_1))f(\phi(\eta(z_2)), \psi(\eta^{-1}(z_1)))$$

for all $(z_1, z_2) \in \mathring{B}_{v \times w}(\Delta \times \Delta)$, for some biholomorphic mapping $\eta: \mathring{B}_w(\Delta) \rightarrow \mathring{B}_v(\Delta)$, where $\phi \in \Lambda_v(\Delta)$, $\psi \in \Lambda_w(\Delta)$, $h_\phi \in \mathcal{H}_{v_o}(\Delta)$ and $h_\psi \in \mathcal{H}_{w_o}(\Delta)$.

The following proposition covers the case of weighted spaces of holomorphic functions on the unit ball in \mathbf{C}^n .

PROPOSITION 3.1. *Let $v_{\alpha,2}$ be the weight on $B_{\mathbf{C}^n}$ defined by $v_{\alpha,2}(z) = (1 - \|z\|^2)^\alpha$. Then $\Lambda_{v_{\alpha,2}}(B_{\mathbf{C}^n}) = \text{Aut}(B_{\mathbf{C}^n})$ and the isometries of $\mathcal{H}_{(v_{\alpha,2})_o}(B_{\mathbf{C}^n})$ are all of the form*

$$S_\phi(f)(z) = \lambda(\det(\partial_z \phi))^{2\alpha/(n+1)} f \circ \phi(z),$$

for $\phi \in \text{Aut}(B_{\mathbf{C}^n})$, $\lambda \in \Gamma$.

Proof. It follows from Theorem 1.3 that $\Lambda_{v_{\alpha,2}}(B_{\mathbf{C}^n}) = \text{Aut}(B_{\mathbf{C}^n})$.

Next let us suppose that $T: \mathcal{H}_{(v_{\alpha,2})_o}(B_{\mathbf{C}^n}) \rightarrow \mathcal{H}_{(v_{\alpha,2})_o}(B_{\mathbf{C}^n})$ is an isometry. It follows from [5, Example 10] that $v_{\alpha,2}$ is complete. By Theorem 0.1 we can find $\phi \in \text{Aut}(B_{\mathbf{C}^n})$ and $h_\phi \in \mathcal{H}_{(v_{\alpha,2})_o}(B_{\mathbf{C}^n})$ so that

$$T(f)(z) = h_\phi(z)f \circ \phi(z)$$

for all $f \in \mathcal{H}_{(v_{\alpha,2})_o}(B_{\mathbf{C}^n})$ and all $z \in B_{\mathbf{C}^n}$. Then [6, Theorem 1] and [1, Corollary 2.2.3] tell us that

$$|h_\phi(z)| = \frac{v_{\alpha,2}(\phi(z))}{v_{\alpha,2}(z)} = \left(\frac{(1 - \|\phi(z)\|^2)}{(1 - \|z\|^2)} \right)^\alpha = |\det(\partial_z \phi)|^{2\alpha/(n+1)}.$$

The Open Mapping Theorem tells us that

$$h_\phi(z) = \lambda(\det \partial_z \phi)^{2\alpha/(n+1)}$$

for some $\lambda \in \Gamma$, and this completes the proof. □

PROPOSITION 3.2. *For $\alpha > 0$ and $\beta \geq 1$ let $v_{\alpha,\beta}$ be the weight on $B_{\mathbf{C}^2}$ defined by $v_{\alpha,\beta}(z) = (1 - \|z\|^\beta)^\alpha$. Then:*

- (1) $\mathcal{H}_{(v_{\alpha,\beta})_o}(B_{\mathbf{C}^2})$ is isometrically isomorphic to $\mathcal{H}_{(v_{\alpha',\beta'})_o}(B_{\mathbf{C}^2})$ if and only if $\alpha = \alpha'$ and $\beta = \beta'$.
- (2) When $\beta \neq 2$ each isometry T of $\mathcal{H}_{(v_{\alpha,\beta})_o}(B_{\mathbf{C}^2})$ has the form

$$T(f)(z) = \lambda f \circ U(z),$$

where $\lambda \in \Gamma$ and U is unitary.

Proof. We will first suppose that $\mathcal{H}_{(v_{\alpha,\beta})_o}(B_{\mathbf{C}^2})$ is isometrically isomorphic to $\mathcal{H}_{(v_{\alpha',\beta'})_o}(B_{\mathbf{C}^2})$ via an isometry T . Then by [6, Theorem 1] we can find $\phi \in \text{Aut}(B_{\mathbf{C}^2})$ and $h_\phi \in \mathcal{H}_{(v_{\alpha',\beta'})_o}(B_{\mathbf{C}^2})$ so that

$$T(f)(z) = h_\phi(z)f \circ \phi(z)$$

for all $f \in \mathcal{H}_{(v_{\alpha',\beta'})_o}(B_{\mathbf{C}^2})$ and all $z \in B_{\mathbf{C}^2}$. Furthermore, we have that

$$|h_\phi(z)| = \frac{v_{\alpha',\beta'} \circ \phi(z)}{v_{\alpha,\beta}(z)}.$$

Since h_ϕ is a non-zero holomorphic function, $\log |h_\phi|$ is pluriharmonic on $B_{\mathbf{C}^2}$. In particular, we have that

$$\left(\frac{\partial^2 \log(v_{\alpha',\beta'}(\phi(z)))}{\partial z_j \partial \bar{z}_k} \right)_{i,j=1,2} = \left(\frac{\partial^2 \log(v_{\alpha,\beta}(z))}{\partial z_j \partial \bar{z}_k} \right)_{i,j=1,2}.$$

Taking determinants this gives us that

$$((dd^c)^2 \log \circ v_{\alpha',\beta'} \circ \phi(z))^2 = ((dd^c)^2 \log \circ v_{\alpha,\beta}(z))^2$$

from which it follows that

$$((dd^c)^2 \log \circ v_{\alpha',\beta'}(\phi(z)))^2 |\det(\partial_z \phi)|^2 = ((dd^c)^2 \log \circ v_{\alpha,\beta}(z))^2.$$

In this particular case we get that

$$(*) \quad (\alpha')^2 \frac{(\beta')^3 \|\phi(z)\|^{2\beta'-4} |\det(\partial_z \phi)|^2}{(1 - \|\phi(z)\|^{\beta'})^3} = \alpha^2 \frac{\beta^3 \|z\|^{2\beta-4}}{(1 - \|z\|^\beta)^3}.$$

When $\beta, \beta' \neq 2$ this gives us that $\phi(0) = 0$ and so ϕ is unitary.

Let $z = re^{i\theta}$ be in \mathbf{C}^2 . Then $\|\phi(z)\| = r$ and putting this into (*) gives us

$$(\alpha')^2 \frac{(\beta')^3 r^{2\beta'-4}}{(1 - r^{\beta'})^3} = \alpha^2 \frac{\beta^3 r^{2\beta-4}}{(1 - r^\beta)^3}$$

for all $r \in (0, 1)$. Equating powers of r we see that $\beta = \beta'$. Equating coefficients of leading powers of r now gives us that $\alpha = \alpha'$.

If $\beta' \neq \beta = 2$, as $\Lambda_{v_{\alpha',\beta'}} = U(2)$, the group of 2 by 2 unitary matrices, while $\Lambda_{v_{\alpha,\beta}} = \text{Aut}(B_{\mathbf{C}^2})$, we see by [6, Proposition 11] that $\mathcal{H}_{(v_{\alpha,\beta})_o}(B_{\mathbf{C}^2})$ cannot be isometrically isomorphic to $\mathcal{H}_{(v_{\alpha',\beta'})_o}(B_{\mathbf{C}^2})$.

Finally let us suppose that $\mathcal{H}_{(v_{\alpha,2})_o}(B_{\mathbf{C}^2})$ is isometrically isomorphic to $\mathcal{H}_{(v_{\alpha',2})_o}(B_{\mathbf{C}^2})$. Then we get that

$$(\alpha')^2 \frac{|\det(\partial_z \phi)|^2}{(1 - \|\phi(z)\|^2)^3} = \alpha^2 \frac{1}{(1 - \|z\|^2)^3}$$

for all $z \in B_{\mathbf{C}^2}$. From [1, Corollary 2.2.1] it follows that $\alpha = \alpha'$. \square

PROPOSITION 3.3. For $\beta \geq 1$ let w_β be the weight on $B_{\mathbf{C}^2}$ defined by

$$w_\beta(z) = e^{\frac{-1}{1-\|z\|^\beta}}.$$

Then:

- (1) $\mathcal{H}_{(w_\beta)_o}(B_{\mathbf{C}^2})$ is isometrically isomorphic to $\mathcal{H}_{(w_{\beta'})_o}(B_{\mathbf{C}^2})$ if and only if $\beta = \beta'$.
- (2) Each isometry T of $\mathcal{H}_{(w_\beta)_o}(B_{\mathbf{C}^2})$ has the form

$$T(f)(z) = \lambda f \circ U(z),$$

where $\lambda \in \Gamma$ and U is unitary.

Proof. First we observe that [5, Example 10] implies that w_β is complete. If $\mathcal{H}_{(w_\beta)_o}(B_{\mathbf{C}^2})$ is isometrically isomorphic to $\mathcal{H}_{(w_{\beta'})_o}(B_{\mathbf{C}^2})$ via an isometry T it follows as in Proposition 3.2 that we can find $\phi \in \text{Aut}(B_{\mathbf{C}^2})$ and $h_\phi \in \mathcal{H}_{(w_\beta)_o}(B_{\mathbf{C}^2})$ so that

$$T(f)(z) = h_\phi(z) f \circ \phi(z)$$

for all $z \in B_{\mathbf{C}^2}$. Further ϕ satisfies the equation

$$(dd^c \log \circ w_{\beta'}(\phi(z)))^2 |\det(\partial_z \phi)|^2 = (dd^c \log \circ w_\beta(z))^2,$$

which becomes

$$\frac{(\beta')^3 \|\phi(z)\|^{2\beta'-4}(1 + \|\phi(z)\|^{\beta'})}{8(1 - \|\phi(z)\|^{\beta'})^5} |\det \partial_z \phi|^2 = \frac{\beta^3 \|z\|^{2\beta-4}(1 + \|z\|^\beta)}{8(1 - \|z\|^\beta)^5}.$$

If $\beta', \beta > 2$, putting $z = 0$ we see that $\phi(0) = 0$ and therefore ϕ is unitary. If $\|z\| = r$ we see that

$$\beta'^3(1 - r^\beta)^5 r^{2\beta'-4}(1 + r^{\beta'}) = \beta^3(1 - r^{\beta'})^5 r^{2\beta-4}(1 + r^\beta)$$

and equating lowest powers of r gives us that $\beta = \beta'$. An analogous argument works for $\beta, \beta' < 2$ and $\beta < 2 < \beta'$.

Next suppose that $\beta' > \beta = 2$. In this case we get that

$$\frac{(\beta')^3 \|\phi(z)\|^{2\beta'-4}(1 + \|\phi(z)\|^{\beta'})}{8(1 - \|\phi(z)\|^{\beta'})^5} |\det \partial_z \phi|^2 = \frac{2(1 + \|z\|^2)}{8(1 - \|z\|^2)^5}.$$

As there is $z_o \in B_{\mathbf{C}^2}$ such that $\phi(z_o) = 0$ but the right-hand-side is never 0, we see that the above equation cannot be solved and hence there is no isometry from $\mathcal{H}_{(w_\beta)_o}(B_{\mathbf{C}^2})$ onto $\mathcal{H}_{(w_{\beta'})_o}(B_{\mathbf{C}^2})$, $\beta > 2$. An analogous argument works for $\beta' < \beta = 2$.

Finally let us suppose that $\beta' = \beta = 2$. This gives us that

$$\frac{(1 + \|\phi(z)\|^2)}{(1 - \|\phi(z)\|^2)^5} |\det \partial_z \phi|^2 = \frac{(1 + \|z\|^2)}{(1 - \|z\|^2)^5}.$$

Using [1, Corollary 2.2.3] this becomes

$$\frac{(1 + \|\phi(z)\|^2)}{(1 - \|\phi(z)\|^2)^2} = \frac{(1 + \|z\|^2)}{(1 - \|z\|^2)^2}$$

and gives us

$$\|z\|^2 \|\phi(z)\|^2 (\|z\|^2 - \|\phi(z)\|^2) = (\|z\|^2 - \|\phi(z)\|^2) (3 - (\|\phi(z)\|^2 + \|z\|^2))$$

As $\|z\|$ and $\|\phi(z)\|$ are strictly less than 1 we get that $\|\phi(z)\| = \|z\|$ for all $z \in B_{\mathbf{C}^2}$ and therefore ϕ is unitary. \square

For $n \in \mathbf{N}$ and $z \in \mathbf{C}^n$ we define $N^*(z)$ by

$$N^*(z) = \frac{1}{\sqrt{2}} \left(\sum_{j=1}^n |z_j|^2 + \left| \sum_{j=1}^n z_j^2 \right| \right)^{1/2}.$$

Let

$$B_n^* = \{z \in \mathbf{C}^n : N^*(z) < 1\}.$$

It is shown in [8] that N^* is the minimal complex norm on \mathbf{C}^n which extends the real Euclidean norm on \mathbf{R}^n . Further N^* is singular at points on the boundary of B_n^* where $\sum_{j=1}^n z_j^2 = 0$. K.T. Kim proves in [12] that

$$\text{Aut}(B_n^*) = \{e^{i\theta}.A : \theta \in \mathbf{R}, A \in O_n\},$$

where O_n is the group of $n \times n$ real orthogonal matrices. Define a weight v_n on B_n^* by

$$v_n(z) = 1 - N^*(z).$$

Since $v_n(e^{i\theta}Az) = v_n(z)$ for all $z \in B_n^*$, all $\theta \in \mathbf{R}$ and all $A \in O_n$ we get:

PROPOSITION 3.4. For $n \in \mathbf{N}$ we have $\Lambda_{v_n}(B_n^*) = \text{Aut}(B_n^*)$.

We recall that D_{pq} denotes the generalised Thullen domain $\{(z_1, z_2) \in \mathbf{C}^2 : |z_1|^{2p} + |z_2|^{2q} < 1\}$ and that v_{pq} is the weight on D_{pq} defined by $v_{pq}(z) = 1 - |z_1|^{2p} - |z_2|^{2q}$.

PROPOSITION 3.5.

- (1) Let $p, q, p', q' \geq 1/2$, $q, q' \neq 1$. Then $\mathcal{H}_{(v_{pq})_o}(D_{pq})$ is isometrically isomorphic to $\mathcal{H}_{(v_{p'q'})_o}(D_{p'q'})$ if and only if either $p = p'$ and $q = q'$ or $p = q'$ and $p' = q$.
- (2) Let $p, q \geq 1/2$, $p, q \neq 1$ and $p \neq q$. Then every isometric isomorphism of $\mathcal{H}_{(v_{pq})_o}(D_{pq})$ has the form

$$(Tf)(z) = \lambda f(e^{i\theta_1} z_1, e^{i\theta_2} z_2)$$

for $\lambda \in \Gamma$ and $\theta_1, \theta_2 \in \mathbf{R}$.

- (3) Let $p \geq 1/2, p \neq 1$. Then every isometric isomorphism of $\mathcal{H}_{(v_{pp})_o}(D_{pp})$ has the form

$$(Tf)(z) = \lambda f(e^{i\theta_1} z_{\sigma(1)}, e^{i\theta_2} z_{\sigma(2)})$$

for $\lambda \in \Gamma$ and $\theta_1, \theta_2 \in \mathbf{R}$ and $\sigma \in S_2$.

Proof. Let us first suppose that $\mathcal{H}_{(v_{pq})_o}(D_{pq})$ is isometrically isomorphic to $\mathcal{H}_{(v_{p'q'})_o}(D_{p'q'})$ via an isometry T . Then by [6, Theorem 1] and using Corollary 2.6 there is a biholomorphic mapping $\phi: D_{p'q'} \rightarrow D_{pq}$ such that

$$T(f)(z) = h_\phi(z) f \circ \phi(z)$$

for all $z \in D_{p'q'}$. Proceeding as in Proposition 3.2 we see that ϕ satisfies the equation

$$((dd^c)^2 \log \circ v_{pq}(\phi(z)))^2 |\det(\partial_z \phi)|^2 = ((dd^c)^2 \log \circ v_{p'q'}(z))^2,$$

which gives us that

$$\frac{p^2 q^2 |\phi_1(z)|^{2p-2} |\phi_2(z)|^{2q-2} |\det \partial_z \phi|^2}{(1 - |\phi_1(z)|^{2p} - |\phi_2(z)|^{2q})^3} = \frac{(p')^2 (q')^2 |z_1|^{2p'-2} |z_2|^{2q'-2}}{(1 - |z_1|^{2p'} - |z_2|^{2q'})^3}.$$

Therefore $\phi_1(z)\phi_2(z)$ contains a term of the form $z_1 z_2$. As ϕ is biholomorphic, $\det \partial_0 \phi \neq 0$, and therefore one of the terms z_1 and z_2 is a factor of ϕ_1 , while the other is a factor of ϕ_2 . Setting each of z_1 and z_2 equal to 0 in turn we see that either $p = p'$ and $q = q'$ or $p = q'$ and $q = p'$.

Next suppose that T is an isometry of $\mathcal{H}_{(v_{pq})_o}(D_{pq})$ where $p, q > 1/2, p, q \neq 2$ and $p \neq q$. Repeating the above argument we see that

$$\frac{p^2 q^2 |\phi_1(z)|^{2p-2} |\phi_2(z)|^{2q-2} |\det \partial_z \phi|^2}{(1 - |\phi_1(z)|^{2p} - |\phi_2(z)|^{2q})^3} = \frac{p^2 q^2 |z_1|^{2p-2} |z_2|^{2q-2}}{(1 - |z_1|^{2p} - |z_2|^{2q})^3}.$$

As $p \neq q$ we see that $\phi_1(0, z_2) = 0$ and $\phi_2(z_1, 0) = 0$ for all $(z_1, z_2) \in D_{pq}$. As $\phi(0) = 0$ a theorem of Cartan [23, Proposition 5.3] implies that ϕ is unitary (i.e., ϕ is linear.). The equations $\phi_1(0, z_2) = 0$ and $\phi_2(z_1, 0) = 0$ imply that $\phi_1(z_1, z_2) = e^{i\theta_1} z_1$ and $\phi_2(z_1, z_2) = e^{i\theta_2} z_2$. As $|h_\phi| = 1$ it follows that $h_\phi \equiv \lambda$ for some $\lambda \in \Gamma$.

If $p \geq 1/2, p \neq 2$, the automorphisms of $\mathcal{H}_{(v_{pp})_o}(D_{pp})$ follow as in the above allowing for the fact that there are now automorphisms of D_{pp} of the form $\phi(z_1, z_2) = (\phi_1(z_2), \phi_2(z_1))$. □

Given $m > 0$ we consider the Thullen domain E_m as

$$E_m := \{(z_1, z_2) \in \mathbf{C}^2 : |z_1|^2 + |z_2|^{2m} < 1\}.$$

The domain E_m is convex if and only if $m \geq 1/2$ while E_1 is the unit ball in \mathbf{C}^2 . We have $E_m = D_{1m}$. The Thullen domains were considered by Thullen in

[25] where he proved that if $m \neq 1$ then $\text{Aut}(E_m)$ is the real four dimensional group given by

$$\left\{ \phi(z_1, z_2) = \left(e^{i\theta_1} \frac{z_1 - a}{1 - \bar{a}z_1}, e^{i\theta_2} \frac{(1 - |a|^2)^{1/2m}}{(1 - \bar{a}z_1)^{1/m}} z_2 \right) \right\},$$

where θ_1, θ_2 are real numbers, $a \in \Delta$, and we take any branch of the m -th root.

On E_m we shall consider the weight $v_{\alpha,m}$ defined by

$$v_{\alpha,m}(z) = (1 - |z_1|^2 - |z_2|^{2m})^\alpha.$$

THEOREM 3.6. *Let $m > 1/2$. Then $\Lambda_{v_{\alpha,m}}(E_m) = \text{Aut}(E_m)$.*

Proof. Let $\phi \in \text{Aut}(E_m)$. By [25] ϕ has the form

$$\phi(z_1, z_2) = \left(e^{i\theta_1} \frac{z_1 - a}{1 - \bar{a}z_1}, e^{i\theta_2} \frac{(1 - |a|^2)^{1/2m}}{(1 - \bar{a}z_1)^{1/m}} z_2 \right),$$

where θ_1, θ_2 are real numbers, $a \in \Delta$, and we take any branch of the m -th root. Define $T_\phi: \mathcal{H}_{(v_{\alpha,m})_o}(E_m) \rightarrow \mathcal{H}_{(v_{\alpha,m})_o}(E_m)$ by

$$T_\phi(f)(z) = (\det \partial_z \phi)^{\alpha/(1+\frac{1}{2m})} f \circ \phi(z)$$

for $f \in \mathcal{H}_{(v_{\alpha,m})_o}(E_m)$ and $z \in E_m$. We observe that T_ϕ is an isomorphism of $\mathcal{H}_{(v_{\alpha,m})_o}(E_m)$. Further, for $f \in \mathcal{H}_{(v_{\alpha,m})_o}(E_m)$ we have

$$\begin{aligned} \|T_\phi(f)\|_{v_{\alpha,m}} &= \sup_{z \in E_m} |(1 - |z_1|^2 - |z_2|^{2m})^\alpha (\det \partial_z \phi)^{\alpha/(1+\frac{1}{2m})} f \circ \phi(z)| \\ &= \sup_{z \in E_m} |(1 - |z_1|^2 - |z_2|^{2m})^\alpha (e^{i\theta_1} e^{i\theta_2})^{\alpha/(1+\frac{1}{2m})} \\ &\quad \left(\frac{1 - |a|^2}{|1 - z_1 \bar{a}|^2} \right)^\alpha f \circ \phi(z)| \\ &= \sup_{z \in E_m} |(1 - |\phi_1(z)|^2 - |\phi_2(z)|^{2m})^\alpha f \circ \phi(z)| \\ &= \|f\|_{v_{\alpha,m}}, \end{aligned}$$

proving that T_ϕ is an isometry of $\mathcal{H}_{(v_{\alpha,m})_o}(E_m)$. □

The Bergman kernel on E_m is given in [9, page 210]. When $m \neq 1$ we see that v_B and $v_{1,m}$ differ on E_m . Thus we have two distinct weights on E_m with isometry group equal to $\text{Aut}(E_m)$. Therefore [6, Corollary 33] cannot be extended to the Thullen domain E_m .

We have new examples where $\Lambda_v(U) = \text{Aut}(U)$. This is also true for the generalised Thullen domains, see [25]. These domains have relatively small automorphism groups however. Prior to Thullen's result, in 1928 Kritikos [15] showed that every automorphism ϕ of $D_{\frac{1}{2}\frac{1}{2}}$ had the form

$$\phi(z_1, z_2) = (e^{i\theta_1} z_{\sigma(1)}, e^{i\theta_2} z_{\sigma(2)})$$

for $\theta_1, \theta_2 \in \mathbf{R}$ and $\sigma \in S_2$. It is interesting to note that in [9] Hahn and Pflug prove that $D_{\frac{1}{2}, \frac{1}{2}}$ and B_2^* are biholomorphically equivalent.

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