COVERS OF GROUPS DEFINABLE IN O-MINIMAL STRUCTURES

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ABSTRACT. In this paper we develop the theory of covers for locally definable groups in o-minimal structures.

1. Introduction

Throughout this paper, $\mathcal{N}$ will be an $\aleph_1$-saturated o-minimal structure and definable will mean $\mathcal{N}$-definable (possibly with parameters). We will assume the reader's familiarity with the basic theory of o-minimal structures and definable groups in such structures. See, for example, [3] and [11], respectively.

When studying definable groups one often makes use of certain groups which are not definable and are called in the literature $\bigvee$-definable groups (see [9]). Roughly, these are groups whose underlying sets are unions of definable sets and the graphs of the group operations are unions of definable sets. In a real closed field these sets, when equipped with a natural topology, are called in [2] locally semi-algebraic spaces. For this reason, we prefer to call $\bigvee$-definable groups locally definable groups since the groups that we will study here will be equipped with a topology such that in the semi-algebraic case they are locally semi-algebraic spaces. Furthermore, as we shall see in Section 2 when we introduce the exact definitions, what we call here a locally definable group is a small modification of what is called in [9] a $\bigvee$-definable group. In [9] $\bigvee$-definable groups are defined with a restriction on the size of the parameter set and with no restriction on the size of the cover by definable subsets. Here we require that locally definable groups have a countable subcover by definable subsets. This is not a big restriction since all the important examples are of this form and this constraint allows us to prove many results which otherwise would be impossible to verify.

Let us mention a few examples where locally definable groups have occurred in connection with the theory of definable groups. In [4], we prove the Lie-Kolchin-Mal'cev theorem for solvable definable groups. This theorem says that given a solvable definable group G, the commutator subgroup $G^{(1)}$ of G and the smallest definable subgroup $d(G^{(1)})$ of G containing $G^{(1)}$ are nilpotent. The commutator subgroup $G^{(1)}$ is a locally definable subgroup of G. In [9], Peterzil and Starchenko show that if G is a solvable definable group which is definably compact (the o-minimal analogue of semi-algebraically complete), then G is abelian by finite. The proof of this result given in [9] uses the groups of definable homomorphisms between definable abelian groups. The group of definable homomorphisms between two definable abelian groups is a locally definable group. In [8], Peterzil, Pillay and Starchenko use locally definable groups to show that if a definable group is not nilpotent by finite, then the group structure interprets a field. In [10] (see also [12]), Peterzil and Steinhorn construct certain definably compact, abelian definable groups which are not the direct product of one-dimensional definably compact, abelian definable groups. In a sense, these definable groups are constructed by first giving their o-minimal universal covers and their o-minimal fundamental groups. These o-minimal universal covers and these o-minimal fundamental groups are locally definable groups.

In the paper [5] we developed the general theory of locally definable groups. The goal of this paper was to generalise the last example and show that the o-minimal universal cover of a definable group or a locally definable group is a locally definable group. In this way, in Section 3 the theory of locally definable covering homomorphisms is developed. The o-minimal universal covering homomorphism $\widetilde{p} : \widetilde{G} \longrightarrow G$ of a definably connected locally definable group G is defined as the inverse limit of the locally definable covering homomorphisms $h : H \longrightarrow G$ of G with H definably connected. The o-minimal fundamental group $\pi(G)$ of G is defined as the kernel of $\widetilde{p}$. Thus we have a short exact sequence

$$1 \to \pi(G) \to \widetilde{G} \ \overset{\widetilde{p}}{\to} \ G \to 1.$$

It is not immediately clear that this short exact sequence is in the category of locally definable groups. Note also that in arbitrary o-minimal structures the o-minimal fundamental group as defined above is a completely new object even for definable groups. Nevertheless we show that $\pi(G)$ is always abelian and the following theorem holds (see Theorem 3.15).

THEOREM 1.1.   *Let G be a definably connected, abelian locally definable group. Suppose that, for each $k > 0$, G is k-divisible and the subgroup of k-torsion points of G has dimension zero. Then the following hold:*

(i) *the o-minimal universal covering group $\widetilde{G}$ of G is divisible and torsion free;*

(ii) *the o-minimal fundamental group $\pi(G)$ of G is torsion-free abelian group;*

(iii) *the k-torsion subgroup of G is isomorphic to $\pi(G)/k\pi(G)$, for each $k > 0$.*

In Section 4, we assume that $\mathcal{N}$ is an o-minimal expansion of a real closed field and show that, for a definably connected locally definable group G, the o-minimal fundamental group $\pi(G)$ is isomorphic to the o-minimal fundamental group $\pi_1(G)$ defined using definable paths and definable homotopies as in [1].

2. Locally definable groups

In this section we define locally definable groups and present some of the properties of these groups that we will need in the paper.

DEFINITION 2.1.   A set Z is a *locally definable set over A*, where $A \subseteq N$ and $|A| < \aleph_1$, if there is a collection $\{Z_i : i \in I\}$ of definable subsets of N^n, all definable over A, such that: (i) $Z = \cup\{Z_i : i \in I\}$; (ii) there is $I_0 \subseteq I$ with $|I_0| < \aleph_1$ and $Z = \cup\{Z_i : i \in I_0\}$; (iii) for every $i, j \in I$ there is $k \in I$ such that $Z_i \cup Z_j \subseteq Z_k$.

Given two locally definable sets X and Z over A, we say that X is a *locally definable subset of Z over A* if X is a subset of Z.

A map $\alpha : Z \longrightarrow X$ between locally definable sets over A is called a *locally definable map over A* if for every definable subset $V \subseteq Z$ defined over A the restriction $\alpha_{|V}$ is a definable map over A.

By saturation, the set Z does not depend on the choice of the collection $\{Z_i : i \in I\}$ in the sense that if $Z = \cup\{Y_j : j \in J\}$ with each Y_j definable over B, $|B| < \aleph_1$, then the following hold: (i) every Y_j is contained in some Z_i; (ii) there is $J_0 \subseteq J$ with $|J_0| < \aleph_1$ and $Z = \cup\{Y_j : j \in J_0\}$. *For this reason we will always assume from now on that $|I| < \aleph_1$.*

Let $\alpha : Z \longrightarrow X$ be a locally definable map over A between locally definable sets over A and let Y be a locally definable subset of X over A. Then $\alpha(Z)$ is a locally definable set over A and $\alpha^{-1}(Y)$ is a locally definable subset of Z over A.

If $Z = \cup\{Z_i : i \in I\}$ is a locally definable set over A, we define the *dimension* of Z by $\dim Z = \max\{\dim Z_i : i \in I\}$.

Recall that, by [9], a group $G = (G, \cdot)$ is a $\bigvee$-*definable group* over $A \subseteq N$, where $|A| < \aleph_1$, if there is a collection $\{Z_i : i \in I\}$ of definable subsets of N^n, all definable over A, such that: (i) $G = \cup\{Z_i : i \in I\}$; (ii) for every $i, j \in I$ there is $k \in I$ such that $Z_i \cup Z_j \subseteq Z_k$; (iii) the restriction of the group multiplication to $Z_i \times Z_j$ is a definable map into N^n. We modify this definition slightly in the following way.

DEFINITION 2.2.   A group $(G, \cdot)$ is a *locally definable group over A*, with $A \subseteq N$ and $|A| < \aleph_1$, if there is a collection $\{Z_i : i \in I\}$ of definable subsets of N^n, all definable over A, such that: (i) $G = \cup\{Z_i : i \in I\}$; (ii) there is $I_0 \subseteq I$ with $|I_0| < \aleph_1$ and $G = \cup\{Z_i : i \in I_0\}$; (iii) for every $i, j \in I$ there is $k \in I$ such that $Z_i \cup Z_j \subseteq Z_k$; (iv) the restriction of the group multiplication to $Z_i \times Z_j$ is a definable map into N^n.

Given two locally definable groups H and G over A, we say that H is a *locally definable subgroup of G over A* if H is a subgroup of G.

A homomorphism $\alpha : G \longrightarrow H$ between locally definable groups over A is called a *locally definable homomorphism over A* if for every definable subset $Z \subseteq G$ defined over A the restriction $\alpha_{|Z}$ is a definable map over A.

As in the $\bigvee$-definable case, the locally definable group G over A does not depend on the choice of the collection $\{Z_i : i \in I\}$ in the sense that if $G = \cup\{Y_j : j \in J\}$ with each Y_j definable over B, $|B| < \aleph_1$, then by saturation and the definition the following hold: (i) every Y_j is contained in some Z_i; (ii) there is $J_0 \subseteq J$ with $|J_0| < \aleph_1$ and $G = \cup\{Y_j : j \in J_0\}$. *For this reason we will always assume from now on that $|I| < \aleph_1$.* This condition will be used in the proof of one of our main results, namely Theorem 3.6.

Also note that if $\alpha : H \longrightarrow G$ is a locally definable homomorphism over A between locally definable groups over A and if K is a locally definable subgroup of G over A, then $\alpha(H)$ is a locally definable group over A and $\alpha^{-1}(K)$ is a locally definable subgroup of H over A.

Before we proceed any further we recall the two main examples of locally definable groups.

EXAMPLE 2.3.   The following are the two main examples of locally definable groups over A, with $A \subseteq N$ and $|A| < \aleph_1$.

(1) The locally definable groups over A of dimension zero: Let $\{Z_i : i \in I\}$ be a collection of finite subsets of N^k all of which are defined over A such that for all $i, j \in I$ there is $k \in I$ with $Z_i \cup Z_j \subseteq Z_k$ and $(\mathcal{Z}, \cdot)$ is an abstract group, where $\mathcal{Z} = \cup\{Z_i : i \in I\}$, and there is $I_0 \subseteq I$ with $|I_0| < \aleph_1$ and $\mathcal{Z} = \cup\{Z_i : i \in I_0\}$. Then $(\mathcal{Z}, \cdot)$ is a locally definable group over A of dimension zero.

(2) The locally definable groups over A which are the subgroups of (type) definable groups: Let $(G, \cdot)$ be a (type) definable group over $B \subseteq A$; let $\{Z_i : i \in I\}$ be a collection of definable subsets of G all of which are defined over A such that for all $i, j \in I$ there is $k \in I$ with $Z_i \cup Z_j \subseteq Z_k$, $(\mathcal{Z}, \cdot)$ is a subgroup of $(G, \cdot)$, where $\mathcal{Z} = \cup\{Z_i : i \in I\}$, and there is $I_0 \subseteq I$ with $|I_0| < \aleph_1$ and $\mathcal{Z} = \cup\{Z_i : i \in I_0\}$. Then $(\mathcal{Z}, \cdot)$ is a locally definable group over A.

The proof of [9, Proposition 2.2] also shows the following theorem.

THEOREM 2.4. *Let $G \subseteq N^k$ be a locally definable group over A. Then there is a uniformly definable family $\{V_s : s \in S\}$ of definable subsets of G defined over A and containing the identity element of G and there is a unique topology τ on G such that: (i) $\{V_s : s \in S\}$ is a basis for the τ-open neighbourhoods of the identity element of G; (ii) (G, τ) is a topological group; (iii) every generic element of G has an open definable neighbourhood $U \subseteq N^k$ such that $U \cap G$ is τ-open and the topology which $U \cap G$ inherits from τ agrees with the topology it inherits from N^k.*

In Theorem 2.4, by a uniformly definable family $\{V_s : s \in S\}$ of definable subsets of G defined over A we mean that S is definable over A and there is a definable subset of $N^k \times S$ over A such that the fiber over s is V_s for each $s \in S$.

As in [9, Lemma 2.6] we see that the following result holds.

THEOREM 2.5. *Let G be a locally definable group over A and H a locally definable subgroup of G over A. Then the following holds: (i) the τ-topology on H is the subspace topology induced by the τ-topology on G; (ii) H is closed in G in the τ-topology; (iii) H is open in G in the τ-topology if and only if $\dim H = \dim G$.*

The proof of [9, Lemma 2.8] gives the following theorem.

THEOREM 2.6. *Any locally definable homomorphism between locally definable groups is continuous with respect to the τ-topology.*

Theorems 2.4, 2.5 and 2.6 are called property (TOP) for locally definable groups since they generalise the corresponding property for definable groups. *From now on, whenever we use topological notions on a locally definable group we are referring to the τ-topology.*

The following easy result will be useful later.

LEMMA 2.7. *Let G be a locally definable group over A. Then there exists a collection $\{X_i : i \in I\}$ of open definable subsets of G over A such that: (i) $G = \cup\{X_i : i \in I\}$; (ii) there is $I_0 \subseteq I$ with $|I_0| < \aleph_1$ and $G = \cup\{X_i : i \in I_0\}$; (iii) for every $i, j \in I$ there is $k \in I$ such that $X_i \cup X_j \subseteq X_k$.*

Proof. Let $\{Z_i : i \in I\}$ be the collection of definable subsets of G over A such that: (i) $G = \cup\{Z_i : i \in I\}$; (ii) there is $I_0 \subseteq I$ with $|I_0| < \aleph_1$ and $G = \cup\{Z_i : i \in I_0\}$; (iii) for every $i, j \in I$ there is $k \in I$ such that $Z_i \cup Z_j \subseteq Z_k$. Let U be an open definable neighbourhood in G of e_G over A. For each $i \in I$, let X_i be the open definable subset UZ_i of G over A. Since, for each $i \in I$, we have $Z_i \subseteq X_i$, the collection $\{X_i : i \in I\}$ satisfies the lemma. $\square$

We will now introduce the notion of compatible locally definable subsets of a locally definable group. This notion will be very useful later.

DEFINITION 2.8.   Let G be a locally definable group over A and let H be a locally definable subgroup (resp., subset) of G over A. We say that H is a *compatible locally definable subgroup (resp., subset)* if for every open definable subset U of G over A, the set $H \cap U$ is a definable subset of G over A.

For example, if H is a definable subgroup (resp., subset) of G over A, then H is a compatible locally definable subgroup (resp., subset) of G. We now prove some lemmas on the notion of compatible locally definable subsets, which will be used quite often later.

LEMMA 2.9.   *Let G be a locally definable group over A and let X be a locally definable subset of G over A. Then X is compatible if and only if for every definable subset Z of G, not necessarily over A, the intersection $Z \cap X$ is a definable subset of G.*

Proof. Suppose that X is a compatible locally definable subset of G over A and let Z be a definable subset of G. By Lemma 2.7 and saturation, there are open definable subsets $U_1, \ldots, U_l$ of G over A such that $Z \subseteq U_1 \cup \cdots \cup U_l$. But then $Z \cap X = Z \cap (U_1 \cap X \cup \cdots \cup U_l \cap X)$ is definable since each $U_i \cap X$ is definable. The converse is clear. $\square$

LEMMA 2.10.   *Let G, H and K be locally definable groups over A. The following hold:*
   (i) *G is a compatible locally definable subgroup of G over A;*
  (ii) *if K is a compatible locally definable subgroup of G over A and H is a locally definable subgroup of G over A containing K, then K is a compatible locally definable subgroup of H over A;*
 (iii) *if K is a compatible locally definable subgroup of H over A and H is a compatible locally definable subgroup of G over A, then K is a compatible locally definable subgroup of G over A.*

Proof. (i) is obvious. For (ii), let U be an open definable subset of H over A. Then, by Lemma 2.7 and saturation, there is an open definable subset V of G over A such that $U \subseteq V$. But then $U \cap K = U \cap (V \cap K)$ is definable over A. For (iii), let U be an open definable subset of G over A. Then $U \cap H$ is an open definable subset of H over A. Hence $U \cap K = (U \cap H) \cap K$ is definable over A. $\square$

LEMMA 2.11.   *Let $\alpha : G \longrightarrow H$ be a locally definable map over A between locally definable groups over A. If Y is a compatible locally definable subset of H over A, then $\alpha^{-1}(Y)$ is a compatible locally definable subset of G over A.*

Proof. Let Z be an open definable subset of G over A. Then $\alpha(Z)$ is a definable subset of H over A and, since there is a uniformly definable basis $\{V_s : s \in S\}$ for the τ-open neighbourhoods of the identity element of H,

there is an open definable subset X of H over A such that $\alpha(Z) \subseteq X$. But clearly $Z \cap \alpha^{-1}(Y) = \alpha_{|Z}^{-1}(X \cap Y)$. Thus, since Y is compatible, $X \cap Y$ is definable. Hence $\alpha_{|Z}^{-1}(X \cap Y)$ is definable since $\alpha_{|Z}$ is definable. $\square$

LEMMA 2.12.   *Suppose that $f, g : G \longrightarrow H$ are locally definable maps over A between locally definable groups over A. Then $\{x \in G : f(x) = g(x)\}$ is a compatible locally definable subset of G over A.*

Proof. Consider the locally definable map $\alpha : G \longrightarrow H$ given by $\alpha(x) = f(x)g(x)^{-1}$. By Lemma 2.11, $\{x \in G : f(x) = g(x)\} = \alpha^{-1}(e_H)$ is a compatible locally definable subset of G over A. $\square$

Lemmas 2.10 and 2.11 will be used quite often in the paper without mentioning it.

LEMMA 2.13.   *Let G be a locally definable group over A. If H is a compatible locally definable subgroup of G over A and X an open definable subset of G over A, then the equivalence relation on X given by $x \simeq y$ if and only if $xH = yH$ is definable over A.*

Proof. Let $\theta : X \times X \longrightarrow \theta(X \times X)$ be the map given by $\theta(x, y) = x^{-1}y$. Then, by the definition of locally definable groups and saturation, θ is a definable map over A and $\theta(X \times X)$ is an open definable subset over A. Since H is a compatible locally definable subgroup of G over A, the set $Z = \theta(X \times X) \cap H$ is a definable subset of H over A. But, for all $x \in X$, we have $xH \cap X = xZ \cap X$. Thus the equivalence relation on X given by $x \simeq y$ if and only if $xH = yH$ is definable since $x \simeq y$ if and only if there is $z \in Z$ such that $y = xz$. $\square$

The next result is the generalization of [9, Lemma 2.15(i)].

PROPOSITION 2.14.   *Let G be a locally definable group over A and let H be a compatible locally definable subgroup of G over A. Then the following are equivalent: (i) H is open in G; (ii) $\dim H = \dim G$; (iii) $(G : H) < \aleph_1$.*

Proof. By Theorem 2.5, H is open in G if and only if $\dim H = \dim G$. On the other hand, if $(G : H) < \aleph_1$, then by compactness we clearly have $\dim H = \dim G$.

Suppose that $\dim H = \dim G$. We must show that $(G : H) < \aleph_1$, i.e., we must show that there is a locally definable subset $\{z_s : s \in S\}$ of G over A such that $G = \cup\{z_s H : s \in S\}$.

Let Z be an open definable subset of G over A. We must show that Z is covered by finitely many cosets of H, all defined over A. By Lemma 2.13, the equivalence relation on Z given by $x \simeq y$ if and only if $xH = yH$ is definable over A. But since $xH = yH$ if and only if $xH \cap Z = yH \cap Z$, we

see that the equivalence classes of $\simeq$ in Z have dimension $\dim H = \dim G$. Therefore, there are finitely many equivalence classes of $\simeq$ in Z for otherwise, by [3, Chapter IV (1.5)], the definable set Z would have dimension greater than $\dim G$, which is a contradiction. So there are finitely many elements $u_1, \ldots, u_{r_Z}$ of Z defined over A such that $Z \subseteq \cup\{u_l H : l = 1, \ldots, r_Z\}$.

Let $\{V_j : j \in J\}$ be the collection of open definable subsets of G over A given by Lemma 2.7. Let $S = \{(j,l) : j \in J, \ l = 1, \ldots, r_{V_j}\}$ and for $s = (j,l) \in S$, let z_s be the element u_l obtained as above with $Z = V_j$. Then by Lemma 2.7, $G = \cup\{z_s H : s \in S\}$. Also $\{z_s : s \in S\}$ is a locally definable subset of G over A since each z_s is defined over A and $\{z_s : s \in S\}$ is the union of the collection of all finite subsets of $\{z_s : s \in S\}$. $\square$

The following corollary of the proof of Proposition 2.14 will be used quite often.

COROLLARY 2.15.   *Let G be a locally definable group over A and let H be a compatible locally definable subgroup of G over A. If $(G : H) < \aleph_1$, then there is a locally definable subset $\{z_s : s \in S\}$ of G over A such that $G = \cup\{z_s H : s \in S\}$ (disjoint union).*

The following definition is the analogue of [9, Definition 2.12].

DEFINITION 2.16.   Let G be a locally definable group over A. We say that a set $Z \subseteq G$ is *definably connected* if there is no definable subset $U \subseteq G$ over A such that $U \cap Z$ is a nonempty proper subset of Z which is closed and open in the topology induced on Z by G.

The next remark can be proved in exactly the same way as [9, Lemmas 2.13 and 2.14].

REMARK 2.17.   Let G be a locally definable group over A. Then the following hold:
  (1) Every definable open subset $Z \subseteq G$ over A can be partitioned into finitely many definably connected definable subsets of G over A.
  (2) There is a locally definable subgroup G' of G over A which is definably connected and such that $\dim G' = \dim G$.

As pointed out in [9], the definably connected locally definable subgroups given by Remark 2.17 (2) are not unique. In fact, let $\mathcal{N}$ be a nonstandard model of the theory of the ordered additive group of real numbers, $G = (N^2, +)$, $G' = \{(x,y) \in N^2 :$ there exists $n \in \mathbb{N}$ such that $-n < x < n\}$ and $G'' = \{(x,y) \in N^2 :$ there exists $n \in \mathbb{N}$ such that $-n < y < n\}$. Then G' and G'' are two distinct definably connected locally definable subgroups of G over $\mathbb{Z}$.

Nevertheless, we have the following generalization of [9, Lemma 2.15(iii)].

PROPOSITION 2.18. *Let G be a locally definable group over A. Then there is a unique definably connected compatible locally definable normal subgroup G^0 of G over A with dimension $\dim G$. Moreover, the following hold:*

(i) *G^0 contains all definably connected locally definable subgroups of G over A;*

(ii) *G^0 is the smallest compatible locally definable subgroup of G over A such that $(G : G^0) < \aleph_1$;*

(iii) *there is a locally definable subset $\{x_s : s \in S\}$ of G over A such that $G = \cup\{x_s G^0 : s \in S\}$ (disjoint union).*

Proof. By Lemma 2.7, let $\{Z_k : k \in K\}$ be a collection of open definable subsets of G over A such that: (i) $G = \cup\{Z_k : k \in K\}$; (ii) there is $K_0 \subseteq K$ with $|K_0| < \aleph_1$ and $G = \cup\{Z_k : k \in K_0\}$; (iii) for every $i, j \in K$ there is $k \in K$ such that $Z_i \cup Z_j \subseteq Z_k$. By the definition of locally definable groups, we may assume that each Z_k contains the identity 1 of G. By Remark 2.17 (1), each Z_k can be partitioned into finitely many definably connected components. For each such Z_k, let Z_k^0 be the definably connected component of Z_k which contains 1.

We claim that $G^0 = \cup\{Z_k^0 : k \in K\}$ is a compatible locally definable subgroup of G over A. Indeed, given $i, j \in K$, we have $Z_i \cup Z_j \subseteq Z_k$ for some $k \in K$, hence $Z_i^0 \cup Z_j^0 \subseteq Z_k$. But $Z_i^0 \cup Z_j^0$ is a definably connected set which contains 1. Hence it must be contained in Z_k^0. Similarly, $Z_i^0 \cdot Z_j^0$ and $(Z_i^0)^{-1}$ are contained in some Z_k^0. Thus G^0 is a locally definable subgroup of G over A which, by construction, is obviously compatible, definably connected and satisfies $\dim G^0 = \dim G$.

By Proposition 2.14, we have $(G : G^0) < \aleph_1$ and so, by Corollary 2.15, $G = \cup\{z_s G^0 : s \in S\}$ (disjoint union) for some locally definable subset $\{z_s : s \in S\}$ of G over A. Thus to show that G^0 is normal, it is enough to show that for each z_s with $s \in S$, $z_s G^0 (z_s)^{-1} = G^0$. But this is obvious since, for every Z_i^0, the open definably connected definable set $z_s Z_i^0 (z_s)^{-1}$ over A containing the identity of G is necessarily contained in a set of the form Z_j^0.

As $G = \cup\{z_s G^0 : s \in S\}$ (disjoint union) for some locally definable subset $\{z_s : s \in S\}$ of G over A, we see that G^0 contains all definably connected locally definable subgroups of G over A.

By Proposition 2.14, if H is a compatible locally definable subgroup of G such that $(G : H) < \aleph_1$, then $\dim H = \dim G$. Let H^0 be the compatible, definably connected locally definable subgroup of H over A obtained from H in the same way as we obtained G^0 from G. Then, by Lemma 2.10 (iii), H^0 is a compatible definably connected locally definable subgroup of G over A such that $\dim H^0 = \dim G$ and so, by (i), $H^0 \subset G^0$. Hence, by Proposition 2.14, H^0 is open in G^0. Therefore, again by Proposition 2.14, we have $(G^0 : H^0) < \aleph_1$ and so, by Corollary 2.15, $G^0 = \cup\{z_s H^0 : s \in S\}$ (disjoint union) for some

locally definable subset $\{z_s : s \in S\}$ of G^0 over A. But both H^0 and G^0 are definably connected, so $|S| = 1$, $H^0 = G^0$ and $G^0 \subseteq H$. $\square$

3. Locally definable covering groups

In this section we present the theory of locally definable covering groups. We will often follow the topological case in Fulton's book [7] and the definable case treated in [6].

3.1. Locally definable covering homomorphisms.

DEFINITION 3.1.   A locally definable homomorphism $p : H \longrightarrow G$ over A between locally definable groups over A is called a *locally definable covering homomorphism* if p is surjective and there is a family $\{U_l : l \in L\}$ of open definable subsets of G over A such that $G = \cup \{U_l : l \in L\}$ and, for each $l \in L$, the compatible locally definable subset $p^{-1}(U_l)$ of H over A is a disjoint union of open definable subsets of H over A, each of which is mapped homeomorphically by p onto U_l.

We call $\{U_l : l \in L\}$ a *p-admissible family of definable neighbourhoods* over A.

If $p : H \longrightarrow G$ is a locally definable covering homomorphism over A, then for each $x \in G$, the fibre $p^{-1}(x)$ is a compatible locally definable subset of H over A of dimension zero. Furthermore, $p : H \longrightarrow G$ is an open surjection. In fact, let V be an open definable subset of H over A and, for each $l \in L$, let $\{U_s^l : s \in S_l\}$ be the collection of open disjoint definable subsets of H over A such that $p^{-1}(U_l) = \cup \{U_s^l : s \in S_l\}$ and $p_{|U_s^l} : U_s^l \longrightarrow U_l$ is a definable homeomorphism over A for every $s \in S_l$. Since $|A| < \aleph_1$, by saturation, there is $\{W_1, \ldots, W_m\} \subseteq \{U_s^l : l \in L, \ s \in S_l\}$ such that $V \subseteq \cup \{W_i : i = 1, \ldots, m\}$. But then $V = \cup \{V \cap W_i : i = 1, \ldots, m\}$ and $p(V) = \cup \{p(V \cap W_i) : i = 1, \ldots, m\}$ is open.

If $q : K \longrightarrow H$ is another locally definable covering homomorphism over A, then $p \circ q : K \longrightarrow G$ is also a locally definable covering homomorphism over A. Indeed, if $\{V_r : r \in R\}$ is a q-admissible family of open neighbourhoods of H over A, then $\{p(V_r \cap U_s^l) : r \in R, \ l \in L, \ s \in S_l\}$ is a $p \circ q$-admissible family of open neighbourhoods of G over A.

The next remark follows from Lemma 2.12 and the argument in the proof of [7, Lemma 11.5].

LEMMA 3.2.   *Let $p : H \longrightarrow G$ be a locally definable covering homomorphism over A and let $f, g : K \longrightarrow H$ be two continuous locally definable maps over A between locally definable groups over A such that $p \circ f = p \circ g$. If K is definably connected and $f(z) = g(z)$ for some $z \in K$, then $f = g$.*

Proof. Let $\{U_l : l \in L\}$ be a p-admissible family of open neighbourhoods of G over A. Let $\{U_s^l : s \in S_l\}$ be the collection of open disjoint definable subsets of H over A such that $p^{-1}(U_l) = \cup\{U_s^l : s \in S_l\}$ and $p_{|U_s^l} : U_s^l \longrightarrow U_l$ is a definable homeomorphism over A for every $s \in S_l$.

By continuity, there is a cover of K by a family $\{V_k^l : l \in L, \ k \in M_l\}$ of open definably connected definable subsets of K over A such that we have $(p \circ f)(V_k^l) = (p \circ g)(V_k^l) \subseteq U_l$ for every $l \in L$ and $k \in M_l$. Note that if $v \in V_k^l$ and $f(v) \in U_s^l$, then $f_{|V_k^l} = (p_{|U_s^l})^{-1} \circ (p \circ f)_{|V_k^l}$. Thus, by Lemma 2.12, the set $\{z \in K : f(z) = g(z)\}$ is an open and closed compatible locally definable subset of K over A. Since K is definably connected, we have $f = g$. $\qquad\square$

DEFINITION 3.3.   Consider a locally definable covering homomorphism $p : H \longrightarrow G$ over A. The group $\mathrm{Aut}(H/G)$ of *locally definable covering transformations over* A is the group of all locally definable homeomorphisms $\phi : H \longrightarrow H$ over A such that $p \circ \phi = p$.

Note that $\mathrm{Aut}(H/G)$ acts on $p^{-1}(e_G)$ and, if H is definably connected, then by Lemma 3.2, $\phi \in \mathrm{Aut}(H/G)$ is uniquely determined by $\phi(e_H)$.

PROPOSITION 3.4.   *Let $h : H \longrightarrow G$ be a locally definable covering homomorphism over A. Suppose that H is definably connected. Then*

$$\mathrm{Ker}\, h \simeq \mathrm{Aut}(H/G)$$

and $\mathrm{Aut}(H/G)$ is abelian.

Proof. For $y \in \mathrm{Ker}\, h$, let $l_y : H \longrightarrow H$ be the locally definable homeomorphism over A given by $l_y(z) = yz$. Clearly $l_y \in \mathrm{Aut}(H/G)$ and this correspondence determines an injective homomorphism $\mathrm{Ker}\, h \longrightarrow \mathrm{Aut}(H/G)$. We now show that this homomorphism is also surjective. Take $\phi \in \mathrm{Aut}(H/G)$ and fix $z \in H$. As $h = h \circ \phi$, there is $y \in \mathrm{Ker}\, h$ such that $l_y(z) = yz = \phi(z)$. But then, by Lemma 3.2, we have $\phi = l_y$.

We now show that $\mathrm{Ker}\, h \subseteq Z(H)$, from which it follows that $\mathrm{Aut}(H/G)$ is abelian. Let $y \in \mathrm{Ker}\, h$. Then we have a locally definable map $\sigma_y : H \longrightarrow \mathrm{Ker}\, h$ over A given by $\sigma_y(x) = xyx^{-1}$ for every $x \in H$. Since $\mathrm{Ker}\, h$ has dimension zero, by Lemma 2.11, $(\sigma_y)^{-1}(y)$ is an open and closed compatible locally definable subset of H over A containing e_H. Since H is definably connected, we have $H = (\sigma_y)^{-1}(y)$ and the result follows. $\qquad\square$

The analogues of the previous result and the next result for definable covering homomorphisms are proved in [6]. The arguments are quite similar. We first need the following lemma.

LEMMA 3.5.   *Let H be a locally definable group over A and V an open compatible locally definable subset of H over A such that $\dim H \setminus V < \dim H$.*

Then there is a set $\{y_s \in H : s \in S\}$ whose elements are defined over A such that $H = \cup\{y_s V : s \in S\}$

Proof. Let $\mathcal{K}$ be the prime model of $\mathrm{Th}_A(\mathcal{N})$ and suppose that $\{X_i : i \in I\}$ (resp., $\{V_j : j \in J\}$) is the collection of all open definable subsets of H (resp., V) over A. Let $\mathcal{M}$ be a sufficiently saturated model of $\mathrm{Th}_A(\mathcal{N})$, $i \in I$, $a \in X_i(M)$, and let $c \in X_i(M)$ be a generic point over K such that $\mathrm{tp}(c/Ka)$ is finitely satisfiable in K. Then c is a generic point of $X_i(M)$ over Ka (see the proof of [11, Lemma 2.4]). Since $\dim H \setminus V < \dim H$, we also have that $\dim X_i \setminus (X_i \cap V) < \dim X_i$. Thus $(X_i \cap V)(M)$ is a large definable subset of $X_i(M)$ over A, i.e., every generic point of $X_i(M)$ over A is contained in $(X_i \cap V)(M)$. But we also have that $\dim H \setminus V a^{-1} < \dim H$. So $X_i(M) \cap V(M)a^{-1}$ is a large definable subset of $X_i(M)$ over $A \cup \{a\}$ (to see that $X_i \cap V a^{-1}$ is a definable set, we use the fact that, by Lemma 2.9, $X_i a \cap V$ is a definable set). Therefore, by definition, $c \in V_j(M)a^{-1}$ and $a \in c^{-1}V_j(M)$ for some $j \in J$. Since $\mathrm{tp}(c/Ka)$ is finitely satisfiable over K, there is $b \in X_i(K)$ such that $a \in b^{-1}V_j(M)$ for some $j \in J$. Therefore, by the compactness theorem, for each $i \in I$, there are $b_1, \ldots, b_{r_i} \in X_i(K)$ and $j_1, \ldots, j_{r_i} \in J$ such that, for every $a \in X_i(M)$, we have $a \in \cup\{(b_l)^{-1}V_{j_l}(M) : l = 1, \ldots, r_i\}$. $\square$

THEOREM 3.6. *Let $h : H \longrightarrow G$ be a surjective locally definable homomophism over A between locally definable groups over A. If $\mathrm{Ker}\, h$ has dimension zero, then $h : H \longrightarrow G$ is a locally definable covering homomorphism over A.*

Proof. We start by proving that there exists a locally definable map $\alpha : G \longrightarrow H$ over A such that $h \circ \alpha = 1_G$.

Let $\{X_i : i \in I\}$ (resp., $\{Y_j : j \in J\}$) be the collection of all open definable subsets of H (resp., G) over A. Fix $j \in J$. Since $\{h(X_i) : X_i \subseteq h^{-1}(Y_j)\}$ is a cover of Y_j by definable subsets over A, by saturation, there is a finite subset $I(j)$ of I such that $\{h(X_i) : i \in I(j)\}$ is a cover of Y_j by definable subsets over A. Let $U_j = \cup\{X_i : i \in I(j)\}$. Then $h^{-1}(y) \cap U_j$ is finite for every $y \in Y_j$. Moreover, since by o-minimality for each $i \in I(j)$ there is a uniform bound for $|h^{-1}(y) \cap X_i|$, there is a uniform bound for $|h^{-1}(y) \cap U_j|$. So there is a definable map $s_j : Y_j \longrightarrow U_j$ over A such that $h_{|U_j} \circ s_j = 1_{Y_j}$.

Let $\kappa \le \omega$ be an enumeration of J and without loss of generality assume that $\{Y_j : j \in \kappa\}$ is an increasing sequence. We define the locally definable map $\alpha : G \longrightarrow H$ of the claim in the following way. Put $\alpha_{|Y_0} = s_0$; assuming that $\alpha_{|Y_j}$ has been defined, we define $\alpha_{|Y_{j+1}}$ by $\alpha_{|Y_{j+1}}(y) = \alpha_{|Y_j}(y)$ for $y \in Y_j$ and $\alpha_{|Y_{j+1}}(y) = s_{j+1}(y)$ for $y \in Y_{j+1} \setminus Y_j$. Clearly, by construction, we have $h \circ \alpha = 1_G$.

Let $\alpha : G \longrightarrow H$ be the locally definable map over A given above. For each $a \in \mathrm{Ker}\, h$, let $\alpha_a : G \longrightarrow H$ be the locally definable map given by $\alpha_a(x) = a\alpha(x)$. Then for each $x \in G$ we have $h^{-1}(x) = \{\alpha_a(x) : a \in \mathrm{Ker}\, h\}$.

For $a \in \operatorname{Ker} h$, let $S_a = \alpha_a(G)$. Then $S_a \cap S_b = \emptyset$ if and only if $a \neq b$. In fact, $y \in S_a \cap S_b$ if and only if there are $u, v \in G$ such that $\alpha_a(u) = y = \alpha_b(v)$ if and only if $u = h(y) = v$ if and only if $a = b$.

By o-minimality there is an open compatible locally definable subset U of G such that $\dim G \setminus U < \dim G$ and $\alpha : U \longrightarrow H$ is continuous. But then $\alpha_a : U \longrightarrow H$ is continuous for each $a \in \operatorname{Ker} h$. By the above and Lemma 2.11, $V = h^{-1}(U)$ is an open compatible locally definable subset of H which is a disjoint union of open locally definable subsets $V_a = \alpha_a(U)$ with $a \in \operatorname{Ker} h$ such that $h_{|V_a} : V_a \longrightarrow U$ is a locally definable homeomorphism over A for every $a \in \operatorname{Ker} h$. Since $h(H \setminus h^{-1}(U)) \subseteq G \setminus U$ and $\dim \operatorname{Ker} h = 0$, we also have $\dim H \setminus V < \dim H$.

By Lemma 3.5 there is a set $\{y_s \in H : s \in S\}$ over A such that $H = \cup\{y_s V : s \in S\}$. Hence it follows that $G = \cup\{h(y_s)U : l = s \in S\}$. So, for each $s \in S$, we have that $h^{-1}(h(y_s)U)$ is the disjoint union of the open locally definable sets $y_s V_a$ with $a \in \operatorname{Ker} h$. Furthermore, $h_{|y_s V_a} : y_s V_a \longrightarrow h(y_s)U$ is a locally definable homeomorphism for every $a \in \operatorname{Ker} h$. Hence $h : H \longrightarrow G$ is a locally definable covering homomorphism over A. $\square$

COROLLARY 3.7. *Suppose that $h : H \longrightarrow G$ is a locally definable covering homomophism over A. If G is definably connected, then $h_{|H^0} : H^0 \longrightarrow G$ is a locally definable covering homomorphism over A.*

Proof. Clearly, $\operatorname{Ker} h_{|H^0}$ has dimension zero. So by Theorem 3.6 we need to show that $h_{|H^0} : H^0 \longrightarrow G$ is surjective. Note that by Proposition 2.18(iii) there is a locally definable subset $\{x_s : s \in S\}$ of H over A such that $H = \cup\{x_s H^0 : s \in S\}$ (disjoint union). Let $s, t \in S$ and suppose that $h(x_t H^0) \cap h(x_s H^0) \neq \emptyset$. Then there are $u, v \in H^0$ such that $h(x_t)h(u) = h(x_s)h(v)$. Thus $h(x_s) = h(x_t)h(uv^{-1}) \in h(x_t H^0)$ and $h(x_t) = h(x_s)h(vu^{-1}) \in h(x_s H^0)$. Hence, $h(x_s H^0) = h(x_t H^0)$. So there is a subset $S_0 \subset S$ such that $G = \cup\{h(x_s)h(H^0) : s \in S_0\}$ (disjoint union). Since h is an open mapping, $h(H^0)$ and so each $h(x_s)h(H^0)$ is open in G. Thus each $h(x_s)h(H^0)$ is open and closed in G. If $h_{|H^0} : H^0 \longrightarrow G$ is not surjective, then $|S_0| > 1$ and this implies that G is not definably connected. In fact, there is a definable subset U of G, which can only be covered by more than one of the sets $h(x_s)h(H^0)$ with $s \in S_0$. By saturation, there are $s_1, \ldots, s_l \in S_0$ and definable subsets $U_i \subseteq h(x_{s_i})h(H^0)$ with $i = 1, \ldots, l$, such that U is the disjoint union $(U \cap U_1) \cup \cdots \cup (U \cap U_l) = (U \cap h(x_{s_1})h(H^0)) \cup \cdots \cup (U \cap h(x_{s_l})h(H^0))$ of open and closed subsets of U. $\square$

3.2. The universal covering homomorphism. The goal now is to show that for a definably connected locally definable group G there exists a universal covering homomorphism $\widetilde{p} : \widetilde{G} \longrightarrow G$ of G. Universal here means that if $h : H \longrightarrow G$ is a locally definable covering homomophism (over some A with

$|A| < \aleph_1$), then there exists a covering homomorphism $\widetilde{p}_{(H,h)} : \widetilde{G} \longrightarrow H$ such that $\widetilde{p} = h \circ \widetilde{p}_{(H,h)}$.

We denote by $\mathrm{Cov}(G)$ the category whose objects are locally definable covering homomorphisms $p : H \longrightarrow G$ (over some A with $|A| < \aleph_1$) and whose morphisms are surjective locally definable homomorphisms $r : H \longrightarrow K$ (over some A with $|A| < \aleph_1$) such that $q \circ r = p$, where $q : K \longrightarrow G$ is a locally definable covering homomorphism (over some A with $|A| < \aleph_1$). Let $p : H \longrightarrow G$ and $q : K \longrightarrow G$ be locally definable covering homomorphisms. If $r : H \longrightarrow K$ is a morphism in $\mathrm{Cov}(G)$, then by Theorem 3.6, $r : H \longrightarrow K$ is a locally definable covering homomorphism. Here is another immediate consequence of Theorem 3.6.

LEMMA 3.8.   *If $p : H \longrightarrow G$ and $q : K \longrightarrow G$ are locally definable covering homomorphisms over A, then $p \times_G q : H \times_G K \longrightarrow G$, $\pi_H : H \times_G K \longrightarrow H$ and $\pi_K : H \times_G K \longrightarrow K$, where π_H and π_K are the obvious projections, are locally definable covering homomorphisms over A such that $p \times_G q = p \circ \pi_H = q \circ \pi_K$.*

Lemma 3.8 shows that $\mathrm{Cov}(G)$ is an inverse system. Given a definably connected locally definable group G, we denote by $\mathrm{Cov}^0(G)$ the full subcategory of $\mathrm{Cov}(G)$ whose objects are locally definable covering homomorphisms $p : H \longrightarrow G$ with H definably connected. Corollary 3.7 and Lemma 3.8 shows that $\mathrm{Cov}^0(G)$ is an inverse system.

DEFINITION 3.9.   Let G be a definably connected locally definable group. The inverse limit $\widetilde{p} : \widetilde{G} \longrightarrow G$ of the inverse system $\mathrm{Cov}^0(G)$ is called the *(o-minimal) universal covering homomorphism of G.*

The kernel of the universal covering homomorphism $\widetilde{p} : \widetilde{G} \longrightarrow G$ of G is called the *(o-minimal) fundamental group of G* and is denoted by $\pi(G)$.

By the definition of an inverse limit, the elements of $\widetilde{G}$ are sequences $x = (x_{(K,k)})$ with $x_{(K,k)} \in K$, $k : K \longrightarrow G \in \mathrm{Cov}^0(G)$ such that $l(x_{(L,k \circ l)}) = x_{(K,k)}$ for all $l : L \longrightarrow K \in \mathrm{MorCov}^0(G)$. In $\widetilde{G}$ the product $z = xy$ of $x = (x_{(K,k)})$ and $y = (y_{(K,k)})$ is given by the sequence $(z_{(K,k)})$ with $z_{(K,k)} \in K$, $k : K \longrightarrow G \in \mathrm{Cov}^0(G)$ such that $z_{(K,k)}$ is the product $x_{(K,k)} y_{(K,k)})$ of the elements $x_{(K,k)}$ and $y_{(K,k)}$ in K with the same $k : K \longrightarrow G \in \mathrm{Cov}^0(G)$ as for $z_{(K,k)}$. If $h : H \longrightarrow G \in \mathrm{Cov}^0(G)$, then there exists a covering homomorphism $\widetilde{p}_{(H,h)} : \widetilde{G} \longrightarrow H$ such that $\widetilde{p} = h \circ \widetilde{p}_{(H,h)}$ and sending the sequence $x = (x_{(K,k)})$ into the element $x_{(H,h)} \in H$.

By construction we have the following theorem.

THEOREM 3.10.   *Let G be a definably connected locally definable group. Then we have the short exact sequence*

$$1 \to \pi(G) \to \widetilde{G} \xrightarrow{\widetilde{p}} G \to 1.$$

We do not know if in general this short exact sequence exists in the category of locally definable groups.

3.3. The o-minimal fundamental group. In this subsection we develop the theory of o-minimal fundamental groups of definably connected locally definable groups.

PROPOSITION 3.11.   *Let G be a definably connected locally definable group. Then $\pi(G)$ is abelian.*

Proof. As we saw after Definition 3.9, a point $x \in \widetilde{G}$ is a sequence $x = (x_{(K,k)})$ with $x_{(K,k)} \in K$ and $k : K \longrightarrow G \in \mathrm{Cov}^0(G)$ such that for every morphism $l : L \longrightarrow K$ in $\mathrm{Cov}^0(G)$ we have $l(x_{(L,k \circ l)}) = x_{(K,k)}$. Moreover, the product xy in $\widetilde{G}$ of $x = (x_{(K,k)})$ and $y = (y_{(K,k)})$ is given by $(x_{(K,k)} y_{(K,k)})$ with $x_{(K,k)}, y_{(K,k)} \in K$ and $k : K \longrightarrow G \in \mathrm{Cov}^0(G)$. Thus if $x \in \pi(G)$, then $\widetilde{p}_{(K,k)}(x) = x_{(K,k)} \in \mathrm{Ker}(k : K \longrightarrow G)$ for every $k : K \longrightarrow G \in \mathrm{Cov}^0(G)$ since $\widetilde{p} = k \circ \widetilde{p}_{(K,k)}$. So, by Proposition 3.4, $\pi(G)$ is abelian. $\qquad\square$

Let $h : H \longrightarrow G$ be a locally definable covering homomorphism. Suppose that H and G are definably connected and let $\widetilde{q} : \widetilde{H} \longrightarrow H$ and $\widetilde{p} : \widetilde{G} \longrightarrow G$ be the o-minimal universal covering homomorphisms. Then we have an isomorphism

$$\widetilde{h} : \widetilde{H} \longrightarrow \widetilde{G}$$

such that for every $K \overset{k}{\to} H \in \mathrm{Cov}^0(H)$ and $K \overset{h \circ k}{\to} G \in \mathrm{Cov}^0(G)$ we have $\widetilde{q}_{(K,k)} = \widetilde{p}_{(K,h \circ k)} \circ \widetilde{h}$, i.e., for $x = (x_{(K,k)}) \in \widetilde{G}$, the coordinate $\widetilde{h}(x)_{(K,h \circ k)}$ of $\widetilde{h}(x)$ is equal to the coordinate $x_{(K,k)}$ of x. Since the collection of all $K \overset{h \circ k}{\to} G \in \mathrm{Cov}^0(G)$ for $K \overset{k}{\to} H \in \mathrm{Cov}^0(H)$ is cofinal in $\mathrm{Cov}^0(G)$, the coordinates $\widetilde{h}(x)_{(K,h \circ k)}$ determine the element $\widetilde{h}(x)$. Let us verify why $\widetilde{h} : \widetilde{H} \longrightarrow \widetilde{G}$ is an isomorphism. If $z = (z_{(L,l)}) \in \widetilde{H}$ and $x = (x_{(K,k)}) \in \widetilde{G}$ are elements such that $\widetilde{h}(z) = x$, then $z_{(L,l)} = \widetilde{q}_{(L,l)}(z) = \widetilde{p}_{(L,h \circ l)}(x) = x_{(L,h \circ l)}$. But $x_{(L,h \circ l)} = \pi_L(x_{(K \times_G L,(h \circ l) \circ \pi_L)})$ and so we have $\pi_L(x_{(K \times_G L,(h \circ l) \circ \pi_L)}) = z_{(L,l)}$ for all $k : K \longrightarrow G \in \mathrm{Cov}^0(G)$ and $l : L \longrightarrow H \in \mathrm{Cov}^0(H)$. Consequently, $\widetilde{h} : \widetilde{H} \longrightarrow \widetilde{G}$ is indeed an isomorphism.

The isomorphism $\widetilde{h} : \widetilde{H} \longrightarrow \widetilde{G}$ restricts to an injective homomorphism $h_* : \pi(H) \longrightarrow \pi(G)$.

PROPOSITION 3.12.   *Let $h : H \longrightarrow G$ be a locally definable covering homomorphism. Suppose that H and G are definably connected. Then the following hold:*

(i) *$h_* : \pi(H) \longrightarrow \pi(G)$ is an injective homomorphism;*
(ii) *$\pi(G)/h_*(\pi(H)) \simeq \mathrm{Aut}(H/G)$.*

Proof. As we saw above, $h_* : \pi(H) \longrightarrow \pi(G)$ is injective. We prove (ii). First we define a group homomorphism $\theta : \pi(G) \longrightarrow \mathrm{Aut}(H/G)$, with $\theta(x) :$ $H \longrightarrow H : w \mapsto \theta(x)(w) = \phi_{\widetilde{p}_{(H,h)}(x)}(w)$, where $\phi_{\widetilde{p}_{(H,h)}(x)}$ is the unique locally definable covering transformation such that $\phi_{\widetilde{p}_{(H,h)}(x)}(e_H) = \widetilde{p}_{(H,h)}(x)$. Since $\widetilde{p}_{(H,h)}(x) \in \mathrm{Ker}\, h$, we have that $\phi_{\widetilde{p}_{(H,h)}(x)}$ is indeed in $\mathrm{Aut}(H/G)$. Since $\widetilde{p}_{(H,h)}(xy) = \widetilde{p}_{(H,h)}(x)\widetilde{p}_{(H,h)}(y)$, it follows that θ is a homomorphism.

Let $\widetilde{q} : \widetilde{H} \longrightarrow H$ be the o-minimal universal covering homomorphism of H. Then we have $\widetilde{q} = \widetilde{p}_{(H,h)} \circ \widetilde{h}$, where $\widetilde{h}$ is an isomorphism. So by the definition of $h_* : \pi(H) \longrightarrow \pi(G)$, the kernel of θ is $h_*(\pi(H))$. It remains to show that θ is surjective. So let $\phi \in \mathrm{Aut}(H/G)$. Since $\widetilde{p}_{(H,h)} : \widetilde{G} \longrightarrow H$ is surjective, there is $x \in G$ such that $\widetilde{p}_{(H,h)}(x) = \phi(e_H)$. But $\widetilde{p}(x) = h \circ \widetilde{p}_{(H,h)}(x) = h \circ \phi(e_H) = h(e_H) = e_G$. So $x \in \pi(G)$ and $\theta(x) = \phi$. $\square$

Let $h : H \longrightarrow G$ be a locally definable covering homomorphism. Suppose that H and G are definably connected. Then by Proposition 3.12 we have a short exact sequence

$$1 \to h_*(\pi(H)) \to \pi(G) \xrightarrow{\theta} \mathrm{Aut}(H/G) \to 1.$$

If $k : K \longrightarrow H$ is another locally definable covering homomorphism with K definably connected, then we have an obvious commutative diagram

$$
\begin{array}{ccccccccc}
1 & \to & h_*(\pi(H)) & \to & \pi(G) & \xrightarrow{\theta} & \mathrm{Aut}(H/G) & \to & 1 \\
 & & \uparrow & & \uparrow_{\mathrm{id}} & & \uparrow_{\theta_H^K} & & \\
1 & \to & (h \circ k)_*(\pi(K)) & \to & \pi(G) & \xrightarrow{\theta} & \mathrm{Aut}(K/G) & \to & 1
\end{array}
$$

of homomorphisms.

COROLLARY 3.13. *Let G be a definably connected locally definable group. Then $\pi(G)$ is the direct limit of the family of group homomorphisms*

$$\{\theta_H^K : \mathrm{Aut}(K/G) \longrightarrow \mathrm{Aut}(H/G) \,|\, K \longrightarrow H \in \mathrm{MorCov}^0(G)\}.$$

For the proof of the main theorem of this subsection we will require the following lemma.

LEMMA 3.14. *Let G be a definably connected, abelian locally definable group over A and let $k > 0$. Suppose that G is k-divisible and the subgroup of k-torsion points of G has dimension zero. If $h : H \longrightarrow G$ is a locally definable covering homomorphism over A with H definably connected, then H is k-divisible and the subgroup of k-torsion points of H has dimension zero.*

Proof. Consider the map $p_k : G \longrightarrow G : x \mapsto kx$, which is a locally definable homomorphism over A. By the assumptions on G and Theorem 3.6,

this is also a locally definable covering homomorphism over A. We also have
a commutative diagram

$$
\begin{array}{ccc}
H & \overset{p_k}{\to} & H \\
\downarrow h & & \downarrow h \\
G & \overset{p_k}{\to} & G.
\end{array}
$$

Thus the kernel of $p_k : H \longrightarrow H$ is contained in the kernel of $p_k \circ h : H \longrightarrow G$,
which is a locally definable covering homomorphism over A. So the kernel of
$p_k : H \longrightarrow H$ has dimension zero. It remains to show that $H = p_k(H)$. First
we note that $H = \cup \{ zp_k(H) : z \in \operatorname{Ker} h \}$. In fact, $h_{|p_k(H)} : p_k(H) \longrightarrow G$ is
surjective, thus if $x \in H$, then there is $y \in p_k(H)$ such that $h(y) = h(x)$. So
$z = xy^{-1} \in \operatorname{Ker} h$ and $x = zy \in zp_k(H)$.

Let $z_1, z_2 \in \operatorname{Ker} h$. If $z_1 p_k(H) \cap z_2 p_k(H) \neq \emptyset$, then there are $x_1, x_2 \in p_k(H)$
such that $z_1 x_1 = z_2 x_2$. So $z_2 = z_1(x_1 x_2^{-1}) \in z_1 p_k(H)$ and $z_1 = z_2(x_2 x_1^{-1}) \in$
$z_2 p_k(H)$. Therefore, $z_1 p_k(H) = z_2 p_k(H)$. So there is a subset $S_0 \subset \operatorname{Ker} h$ such
that $H = \cup \{ zp_k(H) : z \in S_0 \}$ (disjoint union). Since $\dim p_k(H) = \dim H$, by
Theorem 2.5, $p_k(H)$ and so each $zp_k(H)$ is open in H. Thus each $zp_k(H)$ is
open and closed in H. If $p_k : H \longrightarrow H$ is not surjective, then $|S_0| > 1$ and
this implies that H is not definably connected. In fact, there is a definable
subset U of H, which can only be covered by more than one of the sets
$zp_k(H)$ with $z \in S_0$. By saturation, there are $z_1, \ldots, z_l \in S_0$ and definable
subsets $U_i \subseteq z_i p_k(H)$ with $i = 1, \ldots, l$, such that U is the disjoint union
$(U \cap U_1) \cup \cdots \cup (U \cap U_l) = (U \cap z_1 p_k(H)) \cup \cdots \cup (U \cap z_l p_k(H))$ of open and
closed subsets of U. $\qquad \square$

The following result is a generalization of [6, Theorem 2.1].

THEOREM 3.15. *Let G be a definably connected, abelian locally definable
group. Suppose that, for each $k > 0$, G is k-divisible and the subgroup of
k-torsion points of G has dimension zero. Then the following hold:*

(i) *the o-minimal universal covering group $\widetilde{G}$ of G is divisible and torsion
free;*

(ii) *the o-minimal fundamental group $\pi(G)$ of G is a torsion-free abelian
group;*

(iii) *the k-torsion subgroup of G is isomorphic to $\pi(G)/k\pi(G)$, for each
$k > 0$.*

Proof. We consider, for each $k > 0$, the map $p_k : G \longrightarrow G : x \mapsto kx$, which
is a locally definable homomorphism. By the assumptions on G and Theorem
3.6, this is also a locally definable covering homomorphism.

By definition, for every $k > 0$, the isomorphism $\widetilde{p_k} : \widetilde{G} \longrightarrow \widetilde{G}$ is given by multiplication by k. In fact, by Lemma 3.14, if $h : H \longrightarrow G \in \mathrm{Cov}^0(G)$, then

$$
\begin{array}{ccc}
H & \overset{p_k}{\to} & H \\
\downarrow h & & \downarrow h \\
G & \overset{p_k}{\to} & G
\end{array}
$$

is a commutative diagram of locally definable covering homomorphisms. So, $\widetilde{p}_{(H,p_k \circ h)}(kx) = k\widetilde{p}_{(H,p_k \circ h)}(x) = kx_{(H,p_k \circ h)} = kx_{(H,h \circ p_k)} = p_k(x_{(H,h \circ p_k)}) = x_{(H,h)}$ and the coordinate $(kx)_{(H,p_k \circ h)}$ of kx is equal to the coordinate $x_{(H,h)}$ of x just like the coordinate $\widetilde{p_k}(x)_{(H,p_k \circ h)}$ of $\widetilde{p_k}(x)$ is equal to the coordinate $x_{(H,h)}$ of x. Thus $kx = \widetilde{p_k}(x)$ and (i) holds.

By Proposition 3.12 (i), $(p_k)_* : \pi(G) \longrightarrow \pi(G)$ is an injective homomorphism for all $k > 0$. Since $(p_k)_* = \widetilde{p}_{k|\pi(G)}$, the homomorphism $(p_k)_* : \pi(G) \longrightarrow \pi(G)$ is given by $(p_k)_*(x) = kx$. Therefore, $\pi(G)$ is a torsion-free abelian group and, by Propositions 3.4 and 3.12 (ii), the k-torsion subgroup of G is isomorphic to $\pi(G)/k\pi(G)$, for each $k > 0$. $\square$

We conjecture that every definably connected locally definable abelian group G, for each $k > 0$, G is k-divisible and the subgroup of k-torsion points of G has dimension zero. For definable groups this is proved in [12]. However, the methods used there do not generalise to the locally definable case.

4. o-minimal expansions of fields

In this section we assume that $\mathcal{N}$ is an o-minimal expansion of a field. In this case, given a definably connected locally definable group G, we can define an o-minimal fundamental group $\pi_1(G)$ of G using definable paths and definable homotopies, adapting the definable case treated in [1]. We will show that $\pi(G)$ and $\pi_1(G)$ are isomorphic.

So let G be a definably connected locally definable group. A map $f : X \longrightarrow G$, where X is a definable set, is a *definable map* if there is a definable subset Y of G such that $f : X \longrightarrow Y$ is a definable map. A *definable path* in G is a continuous definable map $\alpha : [0, 1] \longrightarrow G$.

The *o-minimal fundamental group $\pi_1(G)$ of G* is defined in the usual way except that we use definable paths $\alpha : [0, 1] \longrightarrow G$ in G such that $\alpha(0) = \alpha(1) = e_G$ and definable homotopies. If $f : G \longrightarrow H$ is a continuous locally definable map between definably connected locally definable groups with $f(e_G) = e_H$, then the induced map $f_* : \pi_1(G) \longrightarrow \pi_1(H) : [\sigma] \mapsto [f \circ \sigma]$ is a group homomorphism and we have the usual functorial properties. See [1] for the theory of o-minimal fundamental groups in the category of definable sets with continuous definable maps.

We now generalize the theory of [6, Section 2] to the category of locally definable groups. Since the arguments are similar we will omit the details. We start with the following analogue of [6, Lemma 2.3].

PROPOSITION 4.1.   *Let G be a definably connected locally definable group. Then $\pi_1(G)$ is abelian.*

Let $p : H \longrightarrow G$ be a locally definable covering homomorphism (over some A with $|A| < \aleph_1$). Let Z be a definable set and let $f : Z \longrightarrow G$ be a definable continuous map. A *lifting of f* is a continuous definable map $\widetilde{f} : Z \longrightarrow H$ such that $p \circ \widetilde{f} = f$.

LEMMA 4.2 (Uniqueness of liftings).   *Let $p : H \longrightarrow G$ be a locally definable covering homomorphism over A. Let Z be a definably connected definable set and let $f : Z \longrightarrow G$ be a definable continuous map. If $\widetilde{f}_1, \widetilde{f}_2 : Z \longrightarrow H$ are two liftings of f, then $\widetilde{f}_1 = \widetilde{f}_2$ provided there is a $z \in Z$ such that $\widetilde{f}_1(z) = \widetilde{f}_2(z)$.*

Proof. As in the proof of Lemma 3.2, both sets $\{w \in Z : \widetilde{f}_1(w) = \widetilde{f}_2(w)\}$ and $\{w \in Z : \widetilde{f}_1(w) \neq \widetilde{f}_2(w)\}$ are definable and open and the first one is nonempty. □

LEMMA 4.3 (Path and homotopy lifting).   *Suppose that $p : K \longrightarrow G$ be a locally definable covering homomorphism over A. Then the following hold:*
   (1) *Let γ be a definable path in G and $y \in K$. If $p(y) = \gamma(0)$, then there is a unique definable path $\widetilde{\gamma}$ in K, lifting γ, such that $\widetilde{\gamma}(0) = y$.*
   (2) *Suppose that $H : [a, b] \times [0, 1] \longrightarrow G$ is a definable homotopy between the definable paths γ and σ in G. Let $\widetilde{\gamma}$ be a definable path in K lifting γ. Then there is a unique definable lifting $\widetilde{H}$ of H, which is a definable homotopy between $\widetilde{\gamma}$ and $\widetilde{\sigma}$, where $\widetilde{\sigma}$ is a definable path in K lifting σ.*

Proof. In our category, the path and homotopy liftings can be proved as in [6] by observing that, by saturation, a definable subset of G is covered by finitely many open definable subsets of G. □

NOTATION.   Referring to Lemma 4.3, if γ is a definable path in G and $y \in K$, we denote by $y * \gamma$ the final point $\widetilde{\gamma}(1)$ of the definable lifting $\widetilde{\gamma}$ of γ with initial point $\widetilde{\gamma}(0) = y$.

The following consequence of Lemma 4.3 is proved in exactly the same way as its definable analogue in [6, Corollary 2.9].

REMARK 4.4.   Suppose that $p : H \longrightarrow G$ is a locally definable covering homomorphism over A and let $y \in H$ be such that $p(y) = e_G$. Suppose that H and G are definably connected. Then the following hold:
   (1) If σ is a definable path in G from e_G to e_G, then $y = y * \sigma$ if and only if $[\sigma] \in p_*(\pi_1(H))$.
   (2) If σ and σ' are two definable paths in G from e_G to x, then $y * \sigma = y * \sigma'$ if and only if $[\sigma \cdot \sigma'^{-1}] \in p_*(\pi_1(H))$.

If G is a locally definable group over A, we say that G is *definably path connected* if for every $u, v \in G$ there is a definable path $\alpha : [0, 1] \longrightarrow G$ such that $\alpha(0) = u$ and $\alpha(1) = v$.

LEMMA 4.5.   *Let G be a locally definable group over A. Then the following hold:*

(1) *For every definably connected open definable subset U of G over A there is a definable subset Σ of $U \times U \times [0, 1] \times U$ over A such that for every $(u, v) \in U \times U$ the fiber $\Sigma_{(u,v)} = \{(t, w) \in [0, 1] \times U : (u, v, t, w) \in \Sigma\}$ is the graph of a definable path in G over $A \cup \{u, v\}$ from u to v.*

(2) *G is definably connected if and only if G is definably path connected.*

Proof. By Lemma 2.7, G is covered by the family $\{U_i : i \in I\}$ of all open definably connected definable subsets of G over A. By [3, Chapter VI, Proposition 3.2], property (1) holds for each U_i. Since $U = U_{i_1} \cup \cdots \cup U_{i_l}$, property (1) holds for U.

Fix $i_0 \in I$, $x_i \in U_i$ over A and let $I_0 = \{i \in I :$ there is a definable path in G from x_{i_0} to $x_i\}$. By (1), $\cup\{U_i : i \in I_0\}$ is a nonempty open and closed compatible locally definable subset of G over A, which is definably path connected. Thus G is definably connected if and only if $G = \cup\{U_i : i \in I_0\}$. □

The first part of the next proposition is also a consequence of Lemma 4.3 and is proved in exactly the same way as its definable analogue in [6, Corollary 2.8]. We give here a different proof of the second part.

PROPOSITION 4.6.   *Let $h : H \longrightarrow G$ be a locally definable covering homomorphism. Suppose that H and G are definably connected. Then the following hold:*

(i) *$h_* : \pi_1(H) \longrightarrow \pi_1(G)$ is an injective homomorphism;*

(ii) *$\pi_1(G)/h_*(\pi_1(H)) \simeq \mathrm{Aut}(H/G)$.*

Proof. As we mentioned, (i) is similar to [6, Corollary 2.8]. We prove (ii). First we define a group homomorphism $\psi : \pi_1(G) \longrightarrow \mathrm{Aut}(H/G)$, with $\psi([\sigma]) : H \longrightarrow H : w \mapsto \psi([\sigma])(w) = \phi_{[\sigma]}(w)$, where $\phi_{[\sigma]}$ is the unique locally definable covering transformation such that $\phi_{[\sigma]}(e_H) = e_H * \sigma$. By definition, $e_H * \sigma$ is in $\mathrm{Ker}\, h$ and so $\phi_{[\sigma]}$ is in $\mathrm{Aut}(H/G)$. By Remark 4.4 (2), the map $\phi_{[\sigma]}$ is well defined and depends only on the class $[\sigma]$ and not on the particular representative of this class. Since $e_H * (\sigma \cdot \sigma') = (e_H * \sigma) * \sigma'$, it follows that ψ is a homomorphism. By Remark 4.4 (1), the kernel of ψ is $h_*(\pi_1(H))$. It remains to show that ψ is surjective. So let $\phi \in \mathrm{Aut}(H/G)$. Since H is definably connected, by Lemma 4.5 (2), there is a definable path α in H from e_H to $\phi(e_H)$. Let $\sigma = h \circ \alpha$. Then $[\sigma] \in \pi_1(G)$ and $\psi([\sigma]) = \phi$. □

As in [6, Theorem 2.1] we have the following result.

THEOREM 4.7.   *Let G be a definably connected, abelian locally definable group. Suppose that, for each $k > 0$, G is k-divisible and the subgroup of k-torsion points of G has dimension zero. Then the following hold:*

(i) *the o-minimal fundamental group $\pi_1(G)$ of G is a torsion-free abelian group;*

(ii) *the k-torsion subgroup of G is isomorphic to $\pi_1(G)/k\pi_1(G)$, for each $k > 0$.*

Proof. We consider, for each $k > 0$, the map $p_k : G \longrightarrow G : x \mapsto kx$, which is a locally definable homomorphism. By the assumptions on G and Theorem 3.6, this is also a locally definable covering homomorphism. By Proposition 4.6 (i), $(p_k)_* : \pi_1(G) \longrightarrow \pi_1(G)$ is an injective homomorphism for all $k > 0$. But as in [6, Lemma 2.4] we see that $(p_k)_* : \pi_1(G) \longrightarrow \pi_1(G)$ is the homomorphism given by $(p_k)_*([\sigma]) = k[\sigma]$. Therefore, $\pi_1(G)$ is a torsion-free abelian group and, by Propositions 3.4 and 4.6 (ii), the k-torsion subgroup of G is isomorphic to $\pi_1(G)/k\pi_1(G)$, for each $k > 0$.   □

Theorem 4.7 is also a consequence of Theorem 3.15 and the following result.

THEOREM 4.8.   *Let G be a definably connected locally definable group. Then $\pi(G)$ and $\pi_1(G)$ are isomorphic.*

Proof. If $h : H \longrightarrow G \in \mathrm{Cov}^0(G)$, then by Proposition 4.6 we have a short exact sequence

$$1 \to h_*(\pi_1(H)) \to \pi_1(G) \xrightarrow{\psi} \mathrm{Aut}(H/G) \to 1.$$

If $k : K \longrightarrow H$ is another locally definable covering homomorphism with K definably connected, then we have an obvious commutative diagram

$$
\begin{array}{ccccccc}
1 & \to & h_*(\pi_1(H)) & \to & \pi_1(G) & \xrightarrow{\psi} & \mathrm{Aut}(H/G) & \to & 1 \\
 & & \uparrow & & \uparrow_{\mathrm{id}} & & \uparrow_{\psi_H^K} & & \\
1 & \to & (h \circ k)_*(\pi_1(K)) & \to & \pi_1(G) & \xrightarrow{\psi} & \mathrm{Aut}(K/G) & \to & 1
\end{array}
$$

of homomorphisms. Therefore, $\pi_1(G)$ is the direct limit of the family of group homomorphisms

$$\{\psi_H^K : \mathrm{Aut}(K/G) \longrightarrow \mathrm{Aut}(H/G) \,|\, K \longrightarrow H \in \mathrm{MorCov}^0(G)\}.$$

But by Proposition 3.4 we have a commutative diagram

$$
\begin{array}{ccccc}
\mathrm{Aut}(H/G) & \leftarrow & \mathrm{Ker}\, h & \to & \mathrm{Aut}(H/G) \\
\uparrow_{\theta_H^K} & & \uparrow_k & & \uparrow_{\psi_H^K} \\
\mathrm{Aut}(K/G) & \leftarrow & \mathrm{Ker}(h \circ k) & \to & \mathrm{Aut}(K/G),
\end{array}
$$

where the horizontal arrows are isomorphisms. So $\pi(G)$ is isomorphic to $\pi_1(G)$.   □

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