Illinois Journal of Mathematics
Volume 49, Number 1, Spring 2005, Pages 17–31
S 0019-2082

ON APPROXIMATION OF UNIMODULAR GROUPS BY FINITE QUASIGROUPS

L. YU. GLEBSKY, E. I. GORDON, AND C. J. RUBIO

Abstract. Recall that a locally compact group G is called unimodular if the left Haar measure on G is equal to the right one. It is shown that G is unimodular iff it is approximable by finite quasigroups (Latin squares).

1. Introduction

This paper is a continuation of [3]. We prove here Theorem 2 of that paper, which is the following result. (See Definitions 2 and 3 in [3].)

Theorem. *A locally compact group G is unimodular if and only if it is approximable by finite quasigroups.*

The right to left direction of this result was proved in [3, Corollary 1 of Theorem 1]. Also proved in [3, Proposition 2] was the fact that any discrete group is approximable by finite quasigroups. Here we prove only the remaining part of this theorem, which is the following:

Theorem 1. *Any non-discrete locally compact unimodular group G is approximable by finite quasigroups.*

The proof of Theorem 1 presented in this paper is based on two results, formulated as Theorem 2 and Theorem 3. Theorem 2 is a result about the topological and measure theoretic structure of locally compact groups. It is new, to our knowledge, and seems interesting in its own right, independent of its role in proving Theorem 1. The proof of Theorem 2 depends on a Theorem of Rado [8] and on Lemma 3, whose lengthy proof is given in Section 3. Theorem 3 is a combinatorial existence result based on ideas of A.J.W. Hilton and D. de Werra, [2], [7], [10]; it is proved in Section 4.

Received April 21, 2003; received in final form March 20, 2005.
2000 *Mathematics Subject Classification.* Primary 26E35, 03H05. Secondary 28E05, 42A38.

2. Reduction of Theorem 1 to Theorems 2 and 3

Consider a locally compact group G and fix a left Haar measure ν on G. For the moment we do not assume that G is unimodular (and, of course, a similar treatment can be given based on a right Haar measure). If G is compact, we normalize ν so that $\nu(G) = 1$.

Let C be a compact subset of G, let U be a relatively compact neighborhood of the identity in G and let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a measurable partition of C.

The partition $\mathcal{P}$ is said to be *equisize* if $\nu(P_i)$ are all the same for $i = 1, \ldots, n$ (which obviously means that they all have the value $\nu(C)/n$).

The partition $\mathcal{P}$ is said to be *U-fine* if for every $i = 1, \ldots, n$ there exists $g \in G$ such that $P_i \subseteq gU$.

Theorem 2. *For any neighborhood of the identity U in G and any compact subset B of G there exist U-fine equisize partition of some compact set C, which satisfies $B \subseteq C \subseteq G$.*

Proof. See Section 3.

Next we formulate Theorem 3, for which we need a brief discussion of quasigroups.

Let $\circ : \mathrm{dom}(\circ) \to Q$ be a partial binary operation on a set Q, i.e., $\mathrm{dom}(\circ) \subseteq Q \times Q$. We say that Q is a partial quasigroup if for any $a, b \in Q$ each of the equations $a \circ x = b$ and $x \circ a = b$ has at most one solution. Further, Q is a quasigroup if the operation $\circ$ is totally defined on Q and if each of the equations $a \circ x = b$ and $x \circ a = b$ has a unique solution. (See [3, Definition 2].)

Lemma 1. *Any finite partial quasigroup Q can be completed to a finite quasigroup, i.e., there exists a finite quasigroup $(Q', \circ')$ such that $Q \subseteq Q'$ and $\circ \subseteq \circ'$.*

The proof of this lemma follows immediately from the fact that any Latin subsquare can be completed to a Latin square [9]. We used this fact in [3] to prove the approximability of discrete groups by finite quasigroups (Proposition 2 of [3]).

Let σ be an equivalence relation on a partial quasigroup Q, identified with the partition $\{Q_1, \ldots, Q_n\}$ of Q by σ-equivalence classes.

Denote by Q/σ the subset of $\{1, \ldots, n\}^3$ such that $\langle ijk \rangle \in Q/\sigma$ iff there exist $q \in Q_i$ and $q' \in Q_j$ with $q \circ q' \in Q_k$. Notice that if σ is a congruence relation on Q (i.e., it preserves the operation $\circ$) and Q is a quasigroup, then the above set is exactly the graph of an operation on $\{1, \ldots, n\}$ making it a quasigroup.

In combinatorics this construction is called *an amalgamation*; see [7], [10].

THEOREM 3. *Let a non-negative three-index matrix $w = \langle w_{ijk} \mid 1 \leq i,j,k \leq n\rangle$, sets $S, S', S'' \subset \{1,\dots,n\}^2$ and a positive real l satisfy the following conditions:*

1. $\sum_{i=1}^n w_{ijk}, \sum_{j=1}^n w_{ijk}, \sum_{k=1}^n w_{ijk} \leq l$*;*
2. $\forall\ \langle i,j\rangle \in S\ \ \sum_{k=1}^n w_{ijk} = l$*,* $\forall\ \langle i,k\rangle \in S'\ \ \sum_{j=1}^n w_{ijk} = l$ *and* $\forall\ \langle j,k\rangle \in S''\ \ \sum_{i=1}^n w_{ijk} = l$*.*

Then there exists a finite partial quasigroup $(Q,\circ)$ and a partition $\sigma = \{Q_1,\dots,Q_n\}$ of Q that satisfy the following conditions:

1. $\bigcup_{\langle i,j\rangle\in S} Q_i \times Q_j \subseteq \mathrm{dom}(\circ)$*;*
2. *the equation $a \circ x = b$ ($x \circ a = b$) has a solution for any $\langle a,b\rangle \in \bigcup_{\langle i,j\rangle\in S'} Q_i \times Q_j$ (for any $\langle a,b\rangle \in \bigcup_{\langle i,j\rangle\in S''} Q_i \times Q_j$);*
3. $Q/\sigma \subseteq \mathrm{supp}\ w$*, where* $\mathrm{supp}\ w = \{\langle i,j,k\rangle \mid w_{ijk} > 0\}$.

Proof. See Section 4.

Now we turn to the proof of Theorem 1, using Theorems 2 and 3.

Let G be a locally compact unimodular group, and let C be a compact subset of G with a U-fine equisize partition $\mathcal{P} = \{P_1,\dots,P_n\}$. Consider the three-index matrix $w = \langle w_{ijk} \mid 1 \leq i,j,k \leq n\rangle$, where

$$w_{ijk} = \iint\limits_{C\times C} \chi_i(xy^{-1})\chi_j(y)\chi_k(x)d\nu(x)d\nu(y). \tag{1}$$

We use the notation χ_i for the characteristic function χ_{P_i}. Let $S = \{\langle i,j\rangle \mid P_i \cdot P_j \subset C\}$.

LEMMA 2. *The three-index matrix w_{ijk} has the following properties:*

1. $\sum_{i=1}^n w_{ijk}, \sum_{j=1}^n w_{ijk}, \sum_{k=1}^n w_{ijk} \leq \nu(C)^2/n^2 i$*;*
2. $\forall\ \langle i,j\rangle \in S\ \ \sum_{k=1}^n w_{ijk} = \nu(C)^2/n^2$*;*
3. $\forall \langle i,j\rangle \in S\, \exists k\ w_{ijk} > 0$ *and* $\forall \langle i,j,k\rangle w_{i,j,k} > 0 \Longrightarrow \nu((P_i \cdot P_j) \cap P_k) > 0$.

Proof. Note that, since $\mathcal{P}$ is a partition of C, we have $\sum_{i=1}^n \chi_i(t) = \chi_C(t)$. Since the partition $\mathcal{P}$ is equisize, we have $\forall i \leq n\ \nu(P_i) = \nu(C)/n$.

Now

$$\begin{aligned}\sum_{i=1}^n w_{ijk} &= \iint\limits_{C\times C} \chi_C(xy^{-1})\chi_j(y)\chi_k(x)d\nu(x)d\nu(y)\\ &\leq \iint\limits_{G\times G} \chi_j(x)\chi_k(y)d\nu(x)d\nu(y) = \frac{\nu(C)^2}{n^2}.\end{aligned}$$

The second and the third inequalities in (1) can be proved similarly.

To prove equality (2) note that since $P_i \cdot P_j \subseteq C$ the equality $\chi_i(xy^{-1})\chi_j(y) = 1$ implies $\chi_C(x) = 1$. Thus,

$$\sum_{k=1}^{n} w_{ijk} = \iint_{C\times C} \chi_i(xy^{-1})\chi_j(y)\chi_C(x)d\nu(x)d\nu(y)$$
$$= \iint_{G\times G} \chi_i(xy^{-1})\chi_j(y)d\nu(x)d\nu(y) = \frac{\nu(C)^2}{n^2}.$$

Here the last equality follows from the right invariance of ν. The first part of statement (3) follows immediately from statement (2) and the second from Fubini's Theorem. □

REMARK 1. Lemma 2 is the only place in the proof of Theorem 1 where the unimodularity of G is used.

REMARK 2. If G is a compact group and $\mathcal{P}$ is a U-fine equisize partition of G, then the set S contains all pairs $\langle i, j\rangle$ and the three-index matrix w satisfies the conditions of Theorem 3 with $S = S' = S'' = \{1, \dots, n\}^2$ and $l = 1/n^2$. After multiplication of w by n^2, we obtain the three-stochastic matrix w'. It is known that a two-index bistochastic matrix p satisfies Birkhoff's theorem (cf., for example, [9]), which states that p is a convex combination of permutation matrices. Hence, for any bistochastic matrix p there exists a permutation π such that $\operatorname{supp} \pi \subseteq \operatorname{supp} p$.

Obviously, an analog of a permutation for a three-index matrix is a three-index matrix $q = (q_{ijk})$ such that each q_{ijk} is either 0 or 1 and $\operatorname{supp} q$ is the graph of an operation of a quasigroup. Thus, if Birkhoff's theorem could be extended to three-index matrices, then the support of any three-index stochastic matrix would contain the graph of the operation of some quasigroup. However, it is well-known that Birkhoff's theorem fails for three-index matrices. Therefore, the above mentioned particular case of Theorem 3 can be considered as a weaker version of Birkhoff's theorem that holds for three-index matrices. This particular case of Theorem 3 follows easily from the results of A.J.W. Hilton [7].

To complete the proof of Theorem 1 we use the nonstandard characterization of approximability (Theorem 5 of [3]).

Recall that $\mathrm{ns}({}^*G)$ is the set of all nearstandard elements of ${}^*G$. It is enough to prove the existence of a hyperfinite quasigroup $(Q', \circ)$ and an internal map $\alpha : Q' \to {}^*G$ that satisfy the following conditions:

(i) $\forall g \in G\, \exists q \in Q'\, (\alpha(q) \approx g)$;

(ii) $\forall q_1, q_2 \in Q'\, (\alpha(q_1), \alpha(q_2) \in \mathrm{ns}({}^*G) \to \alpha(q_1 \circ q_2) \approx \alpha(q_1) \cdot \alpha(q_2))$.

Since G is a locally compact group, there exists an internal compact set $C \supseteq \mathrm{ns}({}^*G)$. By Theorem 2 and the transfer principle, we may assume that C has a hyperfinite U-fine equisize partition $\mathcal{P} = \{P_1, \dots, P_N\}$ for some infinitesimal neighborhood U of the unity in ${}^*G$. Let $w = \langle w_{ijk} \mid 1 \le i, j, k, \le N \rangle$ be the internal three-index matrix defined by formula (1) with this partition $\mathcal{P}$. Notice that if P_i contains at least one nearstandard point then $P_i \subseteq \mathrm{ns}({}^*G)$, and if $P_i, P_j \subseteq \mathrm{ns}({}^*G)$ then $P_i \cdot P_j \subseteq \mathrm{ns}({}^*G)$. So if $S \subseteq \{1, \dots, N\}^2$ is the set defined before Lemma 2, then

$$S \supseteq \{\langle i, j \rangle \mid P_i, P_j \subseteq \mathrm{ns}({}^*G)\}.$$

Denote by $\mathcal{P}_{\mathrm{ns}}$ the (external) set of all $X \in \mathcal{P}$ such that $X \subseteq \mathrm{ns}({}^*G)$.

Let $(Q, \circ)$ be a hyperfinite partial quasigroup and $\sigma = \{Q_1, \dots, Q_N\}$ a partition of Q that satisfy the conditions of Theorem 3 for $S' = S'' = \emptyset$. Let Q' be a hyperfinite quasigroup that completes Q (see Lemma 1). Now consider an arbitrary internal injection $\alpha : Q' \to {}^*G$ such that $\alpha(Q' \setminus Q) \subseteq {}^*G \setminus C$, $\alpha(Q_i) \subseteq P_i$, $i = 1, \dots, N$.

If $q_i \in Q_i$ and $q_j \in Q_j$ and $\langle i, j \rangle \in S$, then, by the definition of S, there exists a k such that $q_i \circ q_j \in Q_k$. Thus $w_{ijk} > 0$ and

$$(P_i \cdot P_j) \cap P_k \neq \emptyset \tag{2}$$

by Lemma 2.

It is easy to see that all elements of any $X \in \mathcal{P}_{\mathrm{ns}}$ are infinitesimally close to each other since U is infinitesimal and $\mathcal{P}$ is U-fine. Thus, if $X, Y \in \mathcal{P}_{\mathrm{ns}}$, then all elements of $X \cdot Y$ are infinitesimally close to each other.

Let $g \in G$. Since $\mathcal{P}$ is a partition of ${}^*G \supset G$ there exists $P \in \mathcal{P}_{\mathrm{ns}}$ such that $g \in P$. By the construction of α, there exist $q \in Q$ such that $\alpha(q) \in P$. So $\alpha(q) \approx g$ and (i) is proved.

Let $\alpha(q_i) \in P_i \in \mathcal{P}_{\mathrm{ns}}$, $\alpha(q_j) \in P_j \in \mathcal{P}_{\mathrm{ns}}$, $\alpha(q_i \circ q_j) \in P_k$. Then $\langle i, j \rangle \in S$ and, by (2), there exists $h \in (P_i \cdot P_j) \cap P_k$. Then $\alpha(q_1) \cdot \alpha(q_2) \approx h$ and $\alpha(q_1 \circ q_2) \approx h$. This completes the proof of Theorem 1 using Theorems 2 and 3. □

3. Proof of Theorem 2

To prove Theorem 2 we need the following results.

Theorem (Rado [8]). *Let S be a measurable space with a finite non-atomic measure μ, $\{S_1, \dots, S_n\}$ - a collection of subsets of S such that $\bigcup_{i=1}^n S_i = S$, $\langle \varepsilon_1, \dots, \varepsilon_n \rangle \in \mathbf{R}^n$, $\varepsilon_i > 0$, $\sum_{i=1}^n \varepsilon_i = \mu(S)$. Then the following two statements are equivalent:*

1. *there exists a partition $\{P_1, \dots, P_n\}$ of S such that $\mu(P_i) = \varepsilon_i$, $P_i \subseteq S_i$, $i = 1, \dots, n$;*
2. *for any $I \subseteq \{1, \dots, n\}$ we have $\mu\left(\bigcup_{i \in I} S_i\right) \ge \sum_{i \in I} \varepsilon_i$.*

LEMMA 3. *For any compact set $B \subseteq G$ and for any neighborhood of the unity U there exist a compact set $C \supseteq B$ and a finite set $F \subset C$ such that $C \subset FU$ and $\forall I \subseteq F$ $\nu((IU) \cap C) \geq \frac{|I|}{|F|}\nu(C)$*

Theorem 2 follows immediately from Rado's Theorem and Lemma 3. Indeed, let F satisfy Lemma 3, $F = \{h_1, \ldots, h_n\}$. Consider the collection $\{(h_1U) \cap C, \ldots, (h_nU) \cap C\}$ of subsets of G and put $\varepsilon_i = n^{-1}\nu(C),\ i = 1, \ldots, n$. Then condition (2) of Rado's Theorem is equivalent to condition (2) of Lemma 3. Thus, there exists the partition $\mathcal{P}$ that satisfies the conditions of Rado's Theorem. Obviously this partition satisfies Theorem 2. □

The rest of this section is devoted to the proof of Lemma 3. First we prove some lemmas.

Throughout this section U is a relatively compact and symmetric ($U^{-1} = U$) neighborhood of the identity $e \in G$.

We say that a set $S \subseteq G$ is U-disconnected if there exists a set $A \subseteq S,\ A \neq \emptyset,\ A \neq S$, such that $AU \cap S = A$. Otherwise S is called U-connected.

LEMMA 4. *If a set $K \subseteq G$ is such that $U^n \subseteq K \subseteq U^{n+1}$ for some $n \geq 0$, then K is U-connected.*

Proof. Assume that a U-disconnected set $K \subseteq G$ satisfies the condition of Lemma 4. Then there exists a set $X \subset K$ such that $\emptyset \neq X,\ XU \cap K = X,\ Y = K \setminus X \neq \emptyset$. Thus, $\{X, Y\}$ is a partition of K and $XU \cap Y = \emptyset$. The last equality implies that $X \cap YU^{-1} = \emptyset$ and since U is symmetric, we have $X \cap YU = \emptyset$. Thus, $YU \cap K = Y$. Consider the map $\Phi : 2^K \to 2^K$ defined by $\Phi(A) = AU \cap K$. This map obviously has the following properties:

- if $A \subseteq B$, then $\Phi(A) \subseteq \Phi(B)$;
- $\Phi^{n+1}(\{e\}) = K$;
- $\Phi(X) = X,\ \Phi(Y) = Y$.

On the other hand, $e \in X$ or $e \in Y$. This contradiction completes the proof. □

We say that a compact set C is *regular* if it is equal to the closure of its interior.

LEMMA 5. *Let C be a regular compact set and C_0 its interior. If C_0 is U-connected, $A \cap C_0 \neq \emptyset$, and $C \setminus \overline{A} \neq \emptyset$, then $\nu(\overline{A} \cap C) < \nu(AU \cap C)$.*

Proof. Since $\overline{A} \subseteq AU$, we have $\nu(\overline{A} \cap C) \leq \nu(AU \cap C)$. It only remains to prove that this inequality is strict. It is enough to prove that $\nu(\overline{A} \cap C_0) < \nu(AU \cap C_0)$. This will be proved if we show that $C_0 \cap AU \neq C_0 \cap \overline{A}$. Indeed, in this case the set $(C_0 \cap AU) \setminus (C_0 \cap \overline{A})$ is equal to $(C_0 \cap AU) \setminus \overline{A} \neq \emptyset$. Since this set is open, we have $\nu((C_0 \cap AU) \setminus (C_0 \cap \overline{A})) > 0$.

Since U is an open set, we have $AU = \overline{A}U$. Thus, the following inclusions hold:

$$C_0 \cap \overline{A}U \supseteq C_0 \cap (\overline{A} \cap C_0)U \supseteq C_0 \cap \overline{A}.$$

Suppose that the last inclusion is an equality. Then $\overline{A} \cap C_0 = C_0$, because C_0 is U-connected. So, $C_0 \subseteq \overline{A}$, thus $\overline{C}_0 \subseteq \overline{A}$, and, since C is regular, we have $C \subseteq \overline{A}$. This contradicts the condition $C \setminus \overline{A} \neq \emptyset$. $\square$

LEMMA 6. *For every $n \in \mathbf{N}$ there exists a regular compact set K such that $U^n \subseteq K \subseteq U^{n+1}$ and $\nu(\partial K) = 0$.*

Proof. Let $(K)_0$ denote the interior of a set K. Consider the family $\mathcal{K}$ of all compact sets K such that $U^n \subseteq K \subseteq U^{n+1}$. Consider the partial order $\prec$ on $\mathcal{K}$ such that $K_1 \prec K_2$ iff $K_1 \subseteq (K_2)_0$. Let Ξ be a maximal chain in $\mathcal{K}$ with respect to this partial order. If Ξ is uncountable, then Z contains at least one compact set K with $\nu(\partial K) = 0$ since $\nu(U^{n+1})$ is finite. So, we assume that Ξ is countable. There are three possibilities under this assumption:

(1) There exist $X, Y \in \Xi$ such that $X \prec Y$, but there does not exist $Z \in \Xi$ such that $X \prec Z \prec Y$. In this case, due to the regularity of G, there exist an open set W and a compact set K such that $X \subseteq W \subseteq K \subseteq (Y)_0$. Due to the maximality of Ξ, either $K = X$ and thus X is a clopen set, or $K = Y$ and Y is a clopen set. Since the boundary of a clopen set is the empty set, we are done in this case.

(2) The maximal chain Ξ contains the maximal element $X \subseteq U_{n+1}$. Then a similar argument shows that X is clopen.

(3) The order type of Ξ is either η or $1+\eta$, where η is the order type of $\mathbf{Q}$. Let us show that this case is impossible. We consider the case of η. In the case of $1+\eta$ the argument is similar.

Let $\Xi = \{X_\alpha \mid \alpha \in \mathbf{Q}\}$, and $X_\alpha \prec X_\beta$ iff $\alpha < \beta$. Fix an arbitrary irrational number a and put $Y = \overline{\bigcup_{\alpha<a} X_\alpha}$. Then it is easy to see that for all $\alpha < a$ one has $X_\alpha \prec Y$ and for $\alpha > a$ one has $Y \prec X_\alpha$. This contradicts the maximality of Ξ.

We have proved that there exists a compact set K such that $U^n \subseteq K \subseteq U^{n+1}$ and $\nu(\partial K) = 0$. If K is not regular, we can consider $K' = \overline{(K)_0}$. It is well known that K' is always regular. Obviously $\partial K' \subseteq \partial K$. Thus, $\nu(\partial K') = 0$ and $K' \subseteq U^{n+1}$. On the other hand, we have $U^n \subseteq K$ and thus $U^n \subseteq (K)_0 \subseteq K'$. $\square$

Let

$$\Gamma(U) = \bigcup_{n=1}^{\infty} U^n.$$

It is well known (cf., for example, [6]) that $\Gamma(U)$ is a complete, and thus a clopen subgroup of G.

Lemma 7. *For any symmetric relatively compact neighborhood V of the unity and for any compact set $B \subseteq G$ there exist a regular compact set C' and a finite set $\{g_1, \dots, g_n\} \subset G$ such that*

- *the interior $(C')_0$ of C' is V-connected;*
- *$\nu(\partial C') = 0$;*
- *$B \subseteq \bigcup_{i=1}^n g_i C'$;*
- *if $i \neq j$, then $g_i C' V \cap g_j C' V = \emptyset$.*

Proof. Consider the decomposition of G into the family of left cosets of $\Gamma(V)$. Since the cosets are clopen sets, there exist only finitely many cosets $\Gamma_1 = g_1\Gamma(V), \dots, \Gamma_n = g_n\Gamma(V)$ such that $\Gamma_i \cap B \neq \emptyset$, $i = 1, \dots, n$ and there exists an $m \in \mathbf{N}$ such that for all $i \leq n$ we have $\Gamma_i \cap B \subseteq g_i V^m$. By Lemma 6, there exists a regular compact set C' with $\nu(\partial C') = 0$ such that $V^m \subseteq C' \subseteq V^{m+1}$. By Lemma 4, the set C' is V-connected. It is easy to see that C' satisfies all other conditions of this lemma. For example, $g_i\Gamma(V) \cdot V = g_i\Gamma(V)$ and, thus, the last condition holds. □

For $A \subseteq {}^*G$ denote by $\mathrm{st}(A)$ the set $\{ {}^\circ g \in G \mid g \in A\}$.

Lemma 8. *Let $C \subseteq G$ be a compact set, $W \subseteq G$ an open relatively compact set, $U \subseteq G$ a relatively compact neighborhood of the unity, and $I \subseteq {}^*C$ an internal set. Then:*

1. *${}^*(\mathrm{st}(I)W \cap C) \subseteq I\,{}^*W\,{}^*U \cap {}^*C$;*
2. *$I \subseteq {}^*(\mathrm{st}(I)U \cap C)$.*

Proof. (1) We prove the stronger inclusion ${}^*(\overline{\mathrm{st}(I)W}) \subseteq I\,{}^*W\,{}^*U$. Since $\mathrm{st}(I)$ is a closed set for any internal set I (this follows from the saturation principle—see, for example, [1]), we have

$$\overline{\mathrm{st}(I)W} = \overline{\mathrm{st}(I)}\,\overline{W} = \mathrm{st}(I)\overline{W}. \tag{3}$$

Let $x \in {}^*(\overline{\mathrm{st}(I)W})$. Since $\overline{\mathrm{st}(I)W}$ is a compact set ($\mathrm{st}(I) \subseteq C$), there exists an element $b \in \overline{\mathrm{st}(I)W}$ such that $x \approx b$. By (3), the element b can be represented in the form $b = {}^\circ i \cdot a$ for some $i \in I$ and $a \in \overline{W}$. By the nonstandard characteristic of the closure of a standard set, there exists $w \in {}^*W$ such that $w \approx a$. Thus $x \approx iw$. This implies that $x \in iw\,{}^*U \subseteq I\,{}^*W\,{}^*U$.

We show that $I \subseteq \mathrm{st}(I)\,{}^*U \subseteq {}^*(\mathrm{st}(I)U)$. Indeed, since C is a compact set and $I \subseteq {}^*C$, for any $i \in I$ there exists ${}^\circ i \in \mathrm{st}(I)$. Since $i \approx {}^\circ i$, we have $i \in {}^\circ i \cdot {}^*U \subseteq \mathrm{st}(I)\,{}^*U$. □

Recall that $I \subseteq G$ is called a (left) O-grid of K if $K \subseteq IO$. A left O-grid I of K called optimal iff it has the minimal cardinality among all left O-grids of K

Lemma 9. *Let $V \subseteq G$ be a compact set with non-empty interior, O an infinitesimal neighborhood of the unity in ${}^*G$, K an internal compact set such* *that $G \subseteq K \subseteq {}^*G$, and $H \subseteq {}^*G$ a hyperfinite set that is an optimal left O-grid of K, $\Delta = |H \cap V|^{-1}$. Consider the functional $\Lambda_V(f)$ defined by the formula*

$$\Lambda_V(f) = {}^\circ\left(\Delta \sum_{h \in H} {}^*f(h)\right). \tag{4}$$

Then the following statements hold:

1. *$\Lambda_V(f)$ is a left invariant finite positive functional on $C_0(L)$ and, thus, it defines the left Haar measure ν_V on L.*
2. *If $C \subseteq G$ is a compact set, then $\nu_V(C) \geq \Lambda_V(\chi_C)$.*
3. *If $C \subseteq G$ is a compact set and $\nu_V(\partial C) = 0$, then $\nu_V(C) = \Lambda_V(\chi_C)$; in particular, if $\nu_V(\partial V) = 0$, then $\nu_V(V) = 1$.*

Proof. Statement (1) is a slight modification of the nonstandard version of Theorem 1 in [3]. See Lemma 1 and the proof of Theorem 3 in [3].

Statement (2) follows from the general theory of integration on locally compact spaces. See, e.g., [5].

Statement (3) follows immediately from Proposition 1.2.18 of [4]. □

We now complete the proof of Lemma 3.

Fix a compact set $B \subseteq G$ and a neighborhood U of the identity. Without loss of generality we assume that the neighborhood U is relatively compact and symmetric.

We show that there exist a compact set $C \supseteq B$ and a hyperfinite set $F \subset {}^*C$ such that ${}^*C \subset F\,{}^*U$ and for any internal set $I \subseteq F$ the inequality ${}^*\nu(I\,{}^*U \cap {}^*C) \geq \frac{|I|}{|F|}\nu(C)$ holds. Lemma 3 follows from this statement by the transfer principle.

Let $U = U_1U_2U_3$. Without loss of generality we assume that U_2 is symmetric. Let C' and $g_1, \dots, g_n$ satisfy the conditions of Lemma 7 for B and $V = U_2$. Put $C = g_1C' \cup \dots \cup g_nC'$. Consider an internal compact set K satisfying $\Gamma(U_2) \subseteq K \subseteq {}^*\Gamma(U_2)$.

There exists a hyperfinite set M such that

- $\{g_1, \dots, g_n\} \subseteq M$;
- $G \subseteq MK$;
- for any $m_1 \neq m_2 \in M$ one has $m_1K\,{}^*U_2 \cap m_2K\,{}^*U_2 = \emptyset$.

Indeed, let $G \subseteq X \subseteq {}^*G$ be an internal compact set and let $D = \{E \in {}^*G/{}^*\Gamma(U_2) \mid E \cap X \neq \emptyset\}$ (${}^*G/{}^*\Gamma(U_2)$ is the set of left classes of ${}^*\Gamma(U_2)$). Then the set D is hyperfinite and for any $g \in G$ one has ${}^*(g\Gamma(U_2)) \in D$. Consider an internal set M' of representatives of the internal family $D \setminus \{g_1\,{}^*\Gamma(U_2), \dots, g_n\,{}^*\Gamma(U_2)\}$ and put $M = M' \cup \{g_1, \dots, g_n\}$. If $m_1 \neq m_2 \in M$, then $m_1\,{}^*\Gamma(U_2)\,{}^*U_2 \cap m_2\,{}^*\Gamma(U_2)\,{}^*U_2 = \emptyset$, since $\Gamma(U_2)U_2 = \Gamma(U_2)$ and thus $m_1K\,{}^*U_2 \cap m_2K\,{}^*U_2 = \emptyset$.

Let $O \subseteq {}^*G$ be an infinitesimal neighborhood of the identity and let the hyperfinite set H be an optimal O-grid of K. Then obviously MH is an optimal O-grid of MK. Put $F = MH \cap C$. By Lemma 9(1),(3), the functional

$$\Lambda(f) = {}^{\circ}\left(\frac{1}{|F|}\sum_{x\in MH} {}^{*}f(x)\right)$$

restricted to $C_0(G)$ is an invariant functional, which induces the Haar measure ν_C on G with $\nu_C(C) = 1$. (It is easy to see that $\nu(\partial C) = 0$.) In what follows we identify ν_C and ν.

Fix an arbitrary internal $I \subseteq F$. We have to prove that ${}^{*}\nu((I \cdot {}^{*}U) \cap {}^{*}C) \geq |I|/|F|$.

Let $A = \mathrm{st}(I) \cdot U_1$. There are two possibilities.

(1) *The set* $\overline{A} \cap C$ *is the union of* $k \leq n$ *sets* $g_i C'$, *say,* $\overline{A} \cap C = \bigcup_{i=1}^{k} g_i C'$. Then $A \cdot U_2 \cap C = \overline{A} \cap C$. Indeed, we have

$$\overline{A} \cap C = \overline{A \cap C} \subseteq (A \cap C) \cdot U_2 \cap C \subseteq \left(\bigcup_{i=1}^{k} g_i C' U_2\right) \cap C = \overline{A} \cap C.$$

The last equality holds since $g_i C' U_2 \cap g_j C' = \emptyset$ for $i \neq j$.

So we have $\nu((AU_2) \cap C) = k/n$, since $n\nu(C') = \nu(C)$.

Let $F_i = F \cap g_i\, {}^{*}C'$. Then $F_i = MH \cap g_i\, {}^{*}C' = g_i H \cap g_i\, {}^{*}C' = g_i({}^{*}C' \cap H)$. Therefore, all F_i have the same cardinality and so $|F_i| = |F|/n$. By Lemma 8, we have $I \subseteq ({}^{*}\mathrm{st}(I)\, {}^{*}U_1) \cap {}^{*}C = {}^{*}(\overline{A} \cap C)$, thus $|I| \leq k|F|/n$.

Again, by Lemma 8, ${}^{*}((\mathrm{st}(I) \cdot U_1 \cdot U_2) \cap C) \subseteq (I\, {}^{*}U) \cap {}^{*}C$. Thus, we have

$$\begin{aligned}{}^{*}\nu((I\, {}^{*}U) \cap {}^{*}C) &\geq {}^{*}\nu({}^{*}((\mathrm{st}(I) \cdot U_1 \cdot U_2) \cap C)) \\ &= \nu((\mathrm{st}(I) \cdot U_1 \cdot U_2) \cap C) = \frac{k}{n} \geq \frac{|I|}{|F|}.\end{aligned}$$

(2) *For some* $i \leq n$ *the sets* $\mathrm{st}(I)U_1$, $g_i C'$ *and* U_2 *satisfy the conditions of Lemma 5 for* A, C *and* U, *respectively.* Using Lemma 5, Lemma 8 and Lemma 9 (3), we obtain

$$\begin{aligned}{}^{*}\nu((I \cdot {}^{*}U) \cap C) &\geq \nu((\mathrm{st}(I) \cdot U_1 \cdot U_2) \cap C) > \nu(\overline{(\mathrm{st}(I) \cdot U_1) \cap C}) \\ &\geq \Lambda(\chi_{(\mathrm{st}(I)\cdot U_1)\cap C}) = {}^{\circ}\left(\frac{|{}^{*}((\mathrm{st}(I) \cdot U_1) \cap C) \cap F|}{|F|}\right) \\ &\geq {}^{\circ}\left(\frac{|I|}{|F|}\right) \approx \frac{|I|}{|F|}.\end{aligned}$$

4. Proof of Theorem 3

Without loss of generality we may assume that in Theorem 3 w is a non-negative integer matrix and $l = k^2$, $k \in \mathbf{N}$. Indeed, let w and l satisfy the conditions of the theorem. Consider non-zero elements of w and l as variables. Since the theories of ordered abelian groups $\mathbf{R}$ and $\mathbf{Q}$ are elementary equivalent, this system of equalities and inequalities has a positive rational solution. Multiplying this solution by a proper integer, we construct an integer matrix w', $\operatorname{supp}(w') = \operatorname{supp}(w)$, satisfying the conditions of Theorem 3 with $l = k^2$.

Denote by $T(m,n)$ the set of all nonnegative integer $m \times n$ matrices. The following Proposition is a reformulation of De Werra's theorem on balanced edge-colorings of a finite bipartite multigraph.

Let Γ be a bipartite multigraph, $E(\Gamma)$ the set of its edges, and $V(\Gamma)$ the set of its vertices. A k-edge-coloring of Γ is a map $f : E(\Gamma) \to \{1,\dots,k\}$. For $s \le k,\ u, v \in V(\Gamma)$, denote by $E_s(u,v)$ the set of all edges of color s connecting u and v, and by $E_s(v)$ the set of all edges of color s incident with v. We say that the edge-coloring f is balanced if the following conditions hold for any $u, v \in V(\Gamma)$ and $r, s \le k$:

(1) $||E_r(u,v)| - |E_s(u,v)|| \le 1$;
(2) $||E_r(u)| - |E_s(u)|| \le 1$.

Theorem (De Werra [10]). *For any bipartite multigraph Γ and for any natural k there exists a balanced k-edge-coloring of Γ.*

Proposition 1. *Let $M \in T(n,m)$ and $k \in \mathbf{N}$. Then $M = M_1 + M_2 + \dots + M_k$, where the matrices $M_i \in T(n,m),\ i = 1,\dots,k$, satisfy the following conditions:*

- $M_i \in T(n,m)$,
- $\forall i,j,p,r\ |M_i(p,r) - M_j(p,r)| \le 1$,
- $\forall i,j,p\ |\sum_{r=1}^{m} M_i(p,r) - \sum_{r=1}^{m} M_j(p,r)| \le 1$,
- $\forall i,j,p\ |\sum_{r=1}^{n} M_i(r,p) - \sum_{r=1}^{n} M_j(r,p)| \le 1$.

Proof. Let Γ be the bipartite multigraph with the set of vertices $\{u_1,\dots,u_n\} \cup \{v_1,\dots,v_m\}$ and $M(i,j)$ edges connecting the vertices u_i and v_j, $\ i = 1,\dots,n,\ j = 1,\dots,m$. By De Werra's theorem, there exists a balanced k-edge-coloring of Γ. For all $s \le k$ let $M_s(i,j) = E_s(u_i,v_j)$ be the number of edges connecting the vertices u_i and v_j of color s. □

Let $M \in T(n,m)$, $\alpha \in \mathbf{N}$, $I \subseteq \{1,2,\dots,m\}$ ($I \subseteq \{1,2,\dots,n\}$). We say that M satisfies the $(1,\alpha,I)$-property (resp. $(2,\alpha,I)$-property) if $\sum_{i=1}^{n} M(i,j) \le \alpha$ for all $j \le m$, and $\sum_{i=1}^{n} M(i,j) = \alpha$ for $j \in I$ (resp. $\sum_{j=1}^{m} M(i,j) \le \alpha$ for all $i \le n$, and $\sum_{j=1}^{m} M(i,j) = \alpha$ for $i \in I$).

Proposition 1 implies:

Corollary 1. *Let $M \in T(n,m)$ satisfy both the $(1,\alpha k,I)$-property and the $(2,\beta k,I')$-property, where $I \subseteq \{1,2,\dots,m\}$ and $I' \subseteq \{1,2,\dots,n\}$, $\alpha,\beta,k \in \mathbf{N}$. Then $M = M_1 + M_2 + \dots + M_k$, where each $M_i \in T(n,m)$ satisfies both the $(1,\alpha,I)$-property and the $(2,\beta,I')$-property.*

We introduce the following notations.

Denote by (n) the set $\{1, 2, \dots, n\}$. Let $M : (n_1) \times \cdots \times (n_k) \to \mathbf{R}$ be a k-index matrix and $\sigma = \{P_1, \dots, P_r\}$ be an (ordered) partition of (n_i). We define the quotient matrix

$$M \circ_i \sigma : (n_1) \times \cdots \times (n_{i-1}) \times (r) \times (n_{i+1}) \times \cdots \times (n_k) \to \mathbf{R},$$

by the formula

$$M \circ_i \sigma(x_1, \dots, x_i, \dots, x_k) = \sum_{y \in P_{x_i}} M(x_1, \cdots, y, \cdots x_k).$$

Let L be a three-index $n_1 \times n_2 \times n_3$-matrix, $I \subseteq (n_2) \times (n_3)$, $\alpha \in \mathbf{N}$. We say that L satisfies the $(1, \alpha, I)$-property if $\sum_{i=1}^{n} L(i, j, k) \leq \alpha$ for all $\langle j, k \rangle$ and $\sum_{i=1}^{n} L(i, j, k) = \alpha$ for $\langle j, k \rangle \in I$. Similarly we define the $(2, \alpha, I)$- and $(3, \alpha, I)$-properties of a three-index matrix.

Let $T(n_1, n_2, n_3)$ denote the set of all non-negative integer $n_1 \times n_2 \times n_3$-matrices.

Proof of Theorem 3. Let $w \in T(n, n, n)$, $S, S', S'' \subseteq \{1, 2, \dots, n\}^2$ and $l = k^2$, $k \in \mathbf{N}$, satisfy the conditions of Theorem 3. Then conditions (1) and (2) of Theorem 3 are equivalent to the condition that w satisfies the $(3, k^2, S)$-, $(2, k^2, S')$-, and $(1, k^2, S'')$-properties. Take $m = kn$ and a partition $\sigma = \{Q_1, Q_2, \cdots, Q_n\}$ of (m) such that $|Q_i| = k$. Without loss of generality we assume that $Q_i = \{(i-1)k + 1, (i-1)k + 2, \dots, ik\}$. We will below construct $w_1 \in T(n, n, m), w_2 \in T(n, m, m)$ and $w_3 \in T(m, m, m)$ such that:

- $w = w_1 \circ_3 \sigma,\ w_1 = w_2 \circ_2 \sigma,\ w_2 = w_3 \circ_1 \sigma.$
- w_1 satisfies the $(1, k, \bigcup_{\langle i,j \rangle \in S''} \{i\} \times Q_j)$-, $(2, k, \bigcup_{\langle i,j \rangle \in S'} \{i\} \times Q_j)$- and $(3, k^2, S)$-properties.
- w_2 satisfies the $(1, 1, \bigcup_{\langle i,j \rangle \in S''} Q_i \times Q_j)$-, $(2, k, \bigcup_{\langle i,j \rangle \in S'} \{i\} \times Q_j)$- and $(3, k, \bigcup_{\langle i,j \rangle \in S} \{i\} \times Q_j)$-properties.
- w_3 satisfies the $(1, 1, \bigcup_{\langle i,j \rangle \in S''} Q_i \times Q_j)$-, $(2, 1, \bigcup_{\langle i,j \rangle \in S'} Q_i \times Q_j)$- and $(3, 1, \bigcup_{\langle i,j \rangle \in S} Q_i \times Q_j)$-properties.

Now it is easy to check that the three-index matrix w_3 is a graph of a partial quasigroup operation $(Q, \circ)$, which satisfies Theorem 3.

Construction of w_1. For every fixed $c \in (n)$ the matrix $w(\cdot, \cdot, c) \in T(n, n)$ satisfies the $(1, k^2, I_1)$- and $(2, k^2, I_2)$-properties, where $I_1 = \{i \ : \ \langle i, c \rangle \in S''\}$, $I_2 = \{i \ : \ \langle i, c \rangle \in S'\}$. By Corollary 1, we have

$$w(\cdot, \cdot, c) = M_{j_1} + \cdots + M_{j_k},$$

where $j_i = (c-1)k + i$ and each $M_{j_i} \in T(n, n)$ satisfies the $(1, k, I_1)$- and $(2, k, I_2)$-properties. Doing the same for all c, we obtain matrices $M_1, \dots, M_m$ $\in T(n, n)$. For every $r \in (m)$ put $w_1(i, j, r) = M_r(i, j)$.

Construction of w_2. For every fixed c the matrix $w_1(\cdot, c, \cdot) \in T(n, m)$ satisfies $(1, k, I_1)$- and $(2, k^2, I_2)$-properties, where $I_1 = \bigcup_{\langle c,j \rangle \in S''} Q_j$, $I_2 =$

$\{i \,:\, \langle i,c\rangle \in S\}$. By Corollary 1 we have

$$w_1(\cdot,c,\cdot) = M_{j_1} + \cdots + M_{j_k},$$

where $j_i = (c-1)k+i$ and each $M_{j_i} \in T(n,m)$ satisfies the $(1,1,I_1)$- and $(2,k,I_2)$-properties. Doing the same for all c, we obtain matrices $M_1,\ldots,M_m \in T(n,m)$. For every $r \in (m)$ put $w_2(i,r,j) = M_r(i,j)$.

The construction of w_3 is similar. □

5. Approximation of unimodular groups by loops

In this section we sketch a proof of a slightly stronger result than Theorem 1, namely:

THEOREM 4. *Any locally compact unimodular group G is approximable by finite loops.*

Recall that an element e of a quasigroup $(Q,\circ)$ is called *an identity* if $\forall a \in Q\; a\circ e = e\circ a = a$. A quasigroup with an identity is called *a loop.*

Recently, Miloš Ziman [11] proved that any discrete group is approximable by loops (with some additional properties), so we only have to prove the following result.

PROPOSITION 2. *Any non-discrete locally compact unimodular group is approximable by finite loops.*

To prove this proposition, we need the following variant of Lemma 2.

LEMMA 10. *In conditions of Lemma 2 the three-index matrix w_{ijk} has the following properties:*

(1) $\sum_{i=1}^n w_{ijk}, \sum_{j=1}^n w_{ijk}, \sum_{k=1}^n w_{ijk} \le \nu(C)^2/n^2$;

(2) $\forall\ \langle i,j\rangle \in S\ \sum_{k=1}^n w_{ijk} = \nu(C)^2/n^2$, $\forall\ \langle i,k\rangle \in S'\ \sum_{j=1}^n w_{ijk} = \nu(C)^2/n^2$ *and* $\forall\ \langle j,k\rangle \in S''\ \sum_{i=1}^n w_{ijk} = \nu(C)^2/n^2$;

(3) $\forall\langle i,j,k\rangle\ w_{ijk} > 0 \Longrightarrow \nu(\ (P_i\cdot P_j)\cap P_k) > 0$,

where $S = \{\langle i,j\rangle \mid P_iP_j \subseteq C\}$, $S' = \{\langle i,k\rangle \mid P_i^{-1}\cdot P_k \subseteq C\}$, *and* $S'' = \{\langle j,k\rangle \mid P_k\cdot P_j^{-1} \subseteq C\}$.

The proof is the same as that of Lemma 2 (see Section 2).

Using Lemma 10 and Theorem 3 one immediately obtains that the quasigroup Q' and the map $\alpha : Q' \to {}^*G$ constructed in the proof of Proposition 1 for the general case (see the very end of Section 2) satisfy the following condition.

(I) If $\alpha(x),\alpha(z) \in \mathrm{ns}({}^*G)$ and $(x\cdot y = z$ or $y\cdot x = z)$, then $\alpha(y) \in \mathrm{ns}({}^*G)$.

Now we introduce a new loop operation $*$ on Q' such that $(Q',*)$ approximates G with the same α. The construction is as follows:

- Take $q_0 \in Q'$, such that $\alpha(q_0) \approx e$ ($e \in G$ is the unity).
- Construct a permutation $a : Q' \to Q'$ such that $q_0 \circ a(x) = x$. By property (I), $\alpha(a(x)) \in \mathrm{ns}({}^*G)$ if and only if $\alpha(x) \in \mathrm{ns}({}^*G)$. So, if $x \in \mathrm{ns}({}^*G)$, then $\alpha(a(x)) \approx \alpha(x)$.
- Construct a permutation $b : Q' \to Q'$ such that $b(x) \circ a(q_0) = x$. By the same argument, $\alpha(b(x)) \approx \alpha(x)$ for $x \in \mathrm{ns}({}^*G)$. It easy to check that $b(q_0) = q_0$.
- Define the operation $x * y = b(x) \circ a(y)$. It is easy to see that $(Q', *)$ is a loop with unity q_0 and $((Q', *), \alpha)$ is a hyperfinite approximation of G.

This proves Proposition 2. □

Acknowledgement. The authors are grateful to C.W. Henson and the referee for their very significant help in the preparation of the manuscript for publication.

References

[1] S. Albeverio, R. Høegh-Krohn, J. E. Fenstad, and T. Lindstrøm, *Nonstandard methods in stochastic analysis and mathematical physics*, Pure and Applied Mathematics, vol. 122, Academic Press Inc., Orlando, FL, 1986. MR 859372 (88f:03061)

[2] J. K. Dugdale, A. J. W. Hilton, and J. Wojciechowski, *Fractional Latin squares, simplex algebras, and generalized quotients*, J. Statist. Plann. Inference **86** (2000), 457–504. MR 1768286 (2001k:05043)

[3] L. Yu. Glebsky and E. I. Gordon, *On approximation of topological groups by finite quasigroups and semigroups*, Illinois J. Math. **49** (2005), 1–16.

[4] E. I. Gordon, *Nonstandard methods in commutative harmonic analysis*, Translations of Mathematical Monographs, vol. 164, American Mathematical Society, Providence, RI, 1997. MR 1449873 (98f:03056)

[5] P. R. Halmos, *Measure theory*, Springer-Verlag, New York, 1974. MR 0033869 (11,504d)

[6] E. Hewitt and K. Ross, *Abstract harmonic analysis. Volume I*, Springer-Verlag, Berlin, 1963. MR 0156915 (28 #158)

[7] A. J. W. Hilton, *Outlines of Latin squares*, Ann. Discrete Math. **34** (1987), 225–241. MR 920647 (89a:05037)

[8] R. Rado, *A theorem on general measure function*, Proc. London Math. Soc **44** (1938), 61–91.

[9] H. J. Ryser, *Combinatorial mathematics*, The Carus Mathematical Monographs, No. 14, Mathematical Association of America, 1963. MR 0150048 (27 #51)

[10] D. de Werra, *A few remarks on chromatic scheduling*, Combinatorial programming: methods and applications (Proc. NATO Advanced Study Inst., Versailles, 1974), D. Reidl, Dordrecht, 1975, pp. 337–342. MR 0401530 (53 #5357)

[11] M. Ziman, *Extensions of Latin subsquares and local embeddability of groups and group algebras*, Quasigroups Related Systems **11** (2004), 115–125. MR 2064165

L. Yu. Glebsky, IICO-UASLP, Av. Karakorum 1470, Lomas 4ta Session, San Luis Potosi SLP, 78210, Mexico

E-mail address: glebsky@cactus.iico.uaslp.mx

E. I. Gordon, Department of Mathematics and Computer Science, Eastern Illinois University, 600 Lincoln Avenue, Charleston, IL 61920-3099, USA

E-mail address: cfyig@eiu.edu

C. J. Rubio, IICO-UASLP, Av. Karakorum 1470, Lomas 4ta Session, San Luis Potosi SLP, 78210, Mexico

E-mail address: jacob@cactus.iico.uaslp.mx