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ON APPROXIMATION OF TOPOLOGICAL GROUPS BY FINITE QUASIGROUPS AND FINITE SEMIGROUPS

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ABSTRACT. It is known that any locally compact group that is approximable by finite groups must be unimodular. However, this condition is not sufficient. For example, simple Lie groups are not approximable by finite ones. In this paper we consider the approximation of locally compact groups by more general finite algebraic systems. We prove that a locally compact group is approximable by finite semigroups iff it is approximable by finite groups. Thus, there exist some locally compact groups and even some compact groups that are not approximable by finite semigroups. We prove also that whenever a locally compact group is approximable by finite quasigroups (latin squares) it is unimodular. The converse theorem is also true: any unimodular group is approximable by finite quasigroups and even by finite loops. In this paper we prove this theorem only for discrete groups. For the case of non-discrete groups the proof is rather long and complicated and is given in a separate paper.

1. Introduction

In this paper we discuss the notion of approximation of a topological group by finite groups that was introduced by the second author [8]. The case of locally compact abelian (LCA) groups was investigated in detail in [8]. The case of discrete groups and the case of nilpotent locally compact groups were treated in [21].

The approximations constructed in these papers have interesting applications in various areas of mathematics. In [3] new finite-dimensional approximations of pseudodifferential operators in Hilbert spaces of functions on LCA groups were constructed using approximations of these groups by finite ones. Approximations of discrete groups have some interesting applications in the ergodic theory of group actions [21], [1], and in symbolic dynamics [10].

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The approximability of any LCA group by finite abelian groups is proved in [8] and the approximability of a large class of nilpotent Lie groups by finite nilpotent groups is proved in [21]. The class of discrete approximable groups is a proper extension of the class of locally residually finite groups. There do exist some non-approximable groups: the Baumslag-Solitar groups, finitely presented infinite simple groups, and some others [21]. It was proved in [9] that all approximable locally compact groups are unimodular (i.e., the left and right Haar measures coincide). However, unimodularity is not a sufficient condition for approximability—we mentioned already that there exist non-approximable discrete groups. It was proved in [1] that the simple Lie groups are not approximable by finite groups as topological groups. However, since these groups are locally residually finite, they are approximable as discrete groups.

As was mentioned above, some important groups, e.g., the group $SO(3)$, are not approximable by finite groups. Therefore, it is interesting to investigate more general classes of finite algebras with a binary operation such that some locally compact groups that are non-approximable by finite groups can be approximated by finite algebras from these classes.

The most important and most thoroughly investigated extensions of the class of finite groups are the classes of finite *semigroups* (see, for example, [20]) and the class of finite *quasigroups*, which are the same as *latin squares* (see, for example, [15] and [19]). The goal of this paper is to investigate the approximation of locally compact groups by finite quasigroups and finite semigroups.

We prove (Theorem 4) that if a locally compact group is approximable by finite semigroups then it is approximable by finite groups. This theorem has an interesting corollary concerning computer arithmetic (finite algebras that approximate the field $\mathbf{R}$). In [9] it was shown that the impossibility to approximate non-unimodular groups by finite groups implies the impossibility to construct a computer arithmetic that is a finite field[1]. In a similar way, the results in this paper imply that it is impossible to construct a computer arithmetic that is a finite associative ring. Computer arithmetic will be discussed in detail in another paper.

In contrast with the case of semigroups, the class of groups approximable by finite quasigroups is essentially larger than the class of groups approximable by finite groups. Indeed the following theorem (Theorem 2) holds: *a locally compact group G is approximable by finite quasigroups iff it is unimodular.* In this paper we prove the unimodularity of locally compact groups that are approximable by finite quasigroups, and we prove the converse statement for discrete groups. The proof of approximability of non-discrete unimodular

[1]Recall that numerical systems implemented in working computers are based on a floating point representation of reals. Due to rounding these systems are neither associative nor distributive.

groups by finite quasigroups is rather long and complicated. It involves some non-trivial combinatorics of latin squares and some special properties of the Haar measure. This proof is presented in a companion paper [7].

Theorem 2 seems to be the first known characterization of unimodularity in terms of algebra and topology. The most general sufficient condition for unimodularity that was known up to now is due to Braconnier [4] (see also [12]). It states that if the left uniformity on a locally compact group G is equivalent to the right uniformity, then G is unimodular. A large class of unimodular groups, including all discrete groups and all compact groups, satisfy this condition. Nevertheless, there exist some unimodular groups, e.g., the group $SL(2, \mathbf{R})$, that do not satisfy it[2].

To prove the unimodularity of a group G that is approximable by finite quasigroups we construct an invariant mean on G as a limit of invariant means on finite quasigroups approximating G. This construction is similar to the construction of a left invariant mean on a locally compact group introduced by von Neumann in Chapter VI of [14], in which the finite equidistributed subsets of G were used. The existence of finite equidistributed subsets is proved in [14] with the help of a combinatorial result, Hall's Marriage Lemma (cf., for example, [19]). As we mentioned above, the proof of the existence of finite quasigroups approximating unimodular groups uses more difficult combinatorics. In our context, Hall's Lemma is only enough to prove the approximability of an arbitrary locally compact group G by finite left (right) quasigroups (see Definition 2 and Theorem 3). This is very close to the above mentioned application of Hall's Lemma to von Neumann's proof of the existence of a Haar measure. Though left quasigroups seem to be not as interesting and important finite algebras as quasigroups, we quote Theorem 3 here in order to complete the picture of connections between the properties of finite algebras approximating G and invariant means on G.

The classical proofs of the existence of a Haar measures on locally compact groups (see, for example, the above mentioned proof in [14] or the proof from [11]) can be essentially simplified by using the language of nonstandard analysis (see, for example, [17] and [18]). This language is also very useful for the problems considered in this paper. All main results here are formulated in classical language, but we use nonstandard analysis in the proofs. The main notions of nonstandard analysis can be found in [2] and [13]. We recommend also the preprint [6], where a short review of nonstandard analysis is given aimed at understanding the proofs of this paper.

The results of this paper and the paper [7] were presented in the seminar "Nonstandard analysis" at the University of Illinois. The authors are grateful to C.W. Henson, P. Loeb, and the referee for many helpful remarks.

[2]It is easy to see that if a group G satisfies this condition, then any subgroup of G also satisfies it. So the groups that satisfy this conditions are hereditarily unimodular. This is not the case for $SL(2, \mathbf{R})$.

2. Formulation of the main results

Let G be a locally compact group. We denote by $\cdot$ the multiplication in G and use the usual notations

$$\begin{aligned} XY &= \{x \cdot y \mid x \in X,\, y \in Y\}, \\ X^{-1} &= \{x^{-1} \mid x \in X\}, \\ gX &= \{g \cdot x \mid x \in X\} \end{aligned}$$

for $X, Y \subset G,\ g \in G$.

Definition 1. Let $C \subset G$ be a compact set, U a relatively compact neighborhood of the unity in G, and $(H, \odot)$ a finite algebra with a binary operation.

(1) We say that a set $M \subset G$ is a U-grid of C if $C \subset MU$.
(2) A map $j : H \to G$ is called a (C, U)-homomorphism if

$$\forall x, y \in H\ \left((j(x), j(y), j(x) \cdot j(y) \in C) \Rightarrow (j(x \odot y) \in j(x)j(y)U)\right).$$

(3) We say that the pair $\langle H, j\rangle$ is a (C, U)-approximation of G if $j(H)$ is a U-grid of C and $j : H \to G$ is a (C, U)-homomorphism.
(4) Let $\mathcal{K}$ be a class of finite algebras. We say that G is approximable by the systems from the class $\mathcal{K}$ if for any compact set $C \subset G$ and for any neighborhood U of the unity there exists a (C, U)-approximation $\langle H, j\rangle$ of G such that $H \in \mathcal{K}$ and j is an injection.

Remark 1. Since in item (2) the elements $j(x \odot y)$ and $j(x) \cdot j(y)$ are U-close in the *left* uniformity on G, it may seem that the definition of approximability of G by systems of $\mathcal{K}$ depends on which of the two uniformities we consider. However this is not so. Indeed it is clear from the definition that we deal only with the restrictions of the uniformities to compact sets. But it is well known that the restrictions of the left uniformity and of the right uniformity to any compact set are equivalent.

Remark 2. It is easy to see that a similar definition can be formulated for any topological algebra and it is not necessary to assume that the approximating algebras are finite. For example, approximations of discrete groups by amenable ones were introduced in [1]. Approximations of more general algebras will be considered in another paper.

It is easy to see that the following proposition holds.

Proposition 1. *A discrete group G is approximable by algebras from a class $\mathcal{K}$ iff for any finite subset $S \subset G$ there exist an algebra $H \in \mathcal{K}$ and an injection $j : S \to H$ such that*

$$\forall s_1, s_2 \in S \left(s_1 \cdot s_2 \in S \Rightarrow j(s_1 \cdot s_2) = j(s_1) \odot j_(s_2)\right).$$

Definition 2.

(1) We say that an algebra $(A, \circ)$ is a quasigroup if for all $a, b \in A$ each of the equations $a \circ x = b$ and $x \circ a = b$ has a unique solution x.

If this statement holds only for all equations of the form $a \circ x = b$ ($x \circ a = b$), then we say that an algebra $(A, \circ)$ is a left quasigroup (right quasigroup). Some times we will use the abbreviation "l-quasigroup (r-quasigroup)" for "left (right) quasigroup".

(2) We say that an algebra $(A, \circ)$ is a semigroup if the operation $\circ$ satisfies the law of associativity.

There is a huge literature concerning quasigroups, cf., for example, [15]. The operation table of a finite quasigroup is a latin square, i.e., an $n \times n$-table of n elements $\{a_1, \dots, a_n\}$ such that all elements in each row and in each column are distinct. An $n \times n$-table with this property that contains more than n elements is called a latin subsquare. It is known [19] that any $n \times n$ latin subsquare with k distinct elements can be completed to an $r \times r$ latin square, where $r = \max\{2n, k\}$. This fact together with Proposition 1 implies immediately the following proposition.

Proposition 2. *Any discrete group is approximable by finite quasigroups.*

Now we are going to define a (left) invariant mean on a locally compact group G using (left) quasigroups that approximate G.

Let $\mathcal{H}$ be the family of all pairs $\langle C, U, \rangle$ such that $C \subseteq G$ is a compact set and U is a relatively compact neighborhood of the identity in G. Let $\leq$ be the partial order on $\mathcal{H}$ such that

$$\langle C_1, U_1 \rangle \leq \langle C_2, U_2 \rangle \Longleftrightarrow C_1 \supseteq C_2 \wedge U_1 \subseteq U_2.$$

Fix a pair $\langle C, U \rangle \in \mathcal{H}$ and put $\mathcal{H}(C, U) = \{\langle C', U' \rangle \mid \langle (C', U' \rangle \leq \langle C, U \rangle\}$. Obviously, the family $\mathcal{M} = \{\mathcal{H}(C, U) \mid \langle C, U \rangle \in \mathcal{H}\}$ of subsets of $\mathcal{H}$ has the finite intersection property. Thus, there exists an ultrafilter $\mathcal{F}$ on $\mathcal{H}$ such that $\mathcal{F} \supseteq \mathcal{M}$. Fix an arbitrary such ultrafilter $\mathcal{F}$.

Recall that if $\alpha : \mathcal{H} \to X$ is an arbitrary map, X is a Hausdorff space and $a \in X$, then $\lim_{\mathcal{F}} \alpha(C, U) = a$ if for any neighborhood Y of a one has $\{\langle C, U \rangle \mid \alpha(C, U) \in Y\} \in \mathcal{F}$. It is known that if the set $\alpha(H)$ is relatively compact, then the $\lim_{\mathcal{F}} \alpha(C, U)$ exists.

For each pair $\langle C, U \rangle$ fix a finite algebra $H_{C,U}$ that is a (C, U)-approximation of G. Without loss of generality, we may assume that $H_{C,U} \subset G$ as a set. Fix also a compact set $V \subseteq G$ such that its interior is non-empty.

As usual, let $C_0(G)$ be the space of all continuous real-valued functions with compact support on G. For an arbitrary $f \in C_0(G)$ put

$$\Lambda(f) = \lim_{\mathcal{F}} |H_{C,U} \cap V)|^{-1} \sum_{h \in H_{C,U}} f(h) \tag{1}$$

if this limit exists.

Theorem 1. *If for any $\langle C, U\rangle \in \mathcal{H}$ the algebra $H_{C,U}$ is a left (right) quasigroup, then the limit on the right hand side of formula (1) exists for all $f \in C_0(G)$. In this case the functional $\Lambda : C_0(G) \to \mathbf{R}$ is a positive non-zero left (right) invariant functional on $C_0(G)$.*

The following corollary is obvious.

Corollary 1. *If G is approximable by finite quasigroups then it is unimodular.*

Moreover the following theorem is true.

Theorem 2. *A locally compact group is approximable by finite quasigroups iff it is unimodular.*

Theorem 1 is proved in Section 3. The approximability of discrete unimodular groups by finite quasigroups is contained in Proposition 2 above. For non-discrete groups it is proved in the companion paper [7].

Theorem 3. *Any locally compact group G is approximable by finite l-quasigroups (r-quasigroups).*

Theorem 3 is proved in Section 4.

The following theorem deals with the approximation of locally compact groups by finite semigroups.

Theorem 4. *A locally compact group is approximable by finite semigroups iff it is approximable by finite groups.*

Theorem 4 is proved in Section 5.

3. Proof of Theorem 1

Throughout this section G is a fixed locally compact group.

To prove Theorem 1 we use the language of nonstandard analysis. See [2], [13] for the fundamental notions of nonstandard analysis and [5, Chapter 3, Sections 3 and 4] for the background on a nonstandard treatment of topological groups.

First of all we reformulate the notion of approximability in this language.

We deal with a λ^+-saturated nonstandard universe, where λ is greater or equal to the weight of the topology on G (the minimal cardinality of a base of the topology on G).

As usual, we denote by ${}^*S$ the nonstandard extension of a standard set S.

Denote by $\mu(e)$ the monad of the identity e in ${}^*G$. We say that $g_1, g_2 \in {}^*G$ are left (right) infinitesimally close and write $g_1 \approx_L g_2$ ($g_1 \approx_R g_2$) if $g_1 \in$

$g_2 \cdot \mu(e)$ ($g_1 \in \mu(e) \cdot g_2$). Since $\mu(e)$ is a normal subgroup in $\mathrm{ns}({}^*G)$ (the external subgroup of all nearstandard elements of ${}^*G$), we have

$$\forall g_1, g_2 \in \mathrm{ns}({}^*G)\ (g_1 \approx_L g_2 \Longleftrightarrow g_1 \approx_R g_2).$$

In this case we write simply $g_1 \approx g_2$.

THEOREM 5. *The group G is approximable by finite algebras from a class $\mathcal{K}$ iff there exists a hyperfinite system $H \in {}^*\mathcal{K}$ and an internal injection $j : H \to G$ that satisfy the following conditions:*

(1) $\forall g \in G\, \exists h \in H\ (j(h) \approx g)$.
(2) $\forall h_1, h_2 \in j^{-1}(\mathrm{ns}({}^*G))\, (j(h_1 \odot h_2) \approx j(h_1) \circ j(h_2))$.

Proof. (a) Let G be approximable by finite systems from the class $\mathcal{K}$. Since the group G is locally compact, there exists a *-compact set $C \supset \mathrm{ns}({}^*G)$. Let $U \subset \mu(e)$ be an infinitesimal neighborhood of the identity. By the transfer principle of nonstandard analysis, there exists a hyperfinite algebra $H \in {}^*K$ and an internal injection $j : H \to {}^*G$ such that $\langle H, j\rangle$ is a (C, U)-approximation of ${}^*G$. Since $\mathrm{ns}({}^*G) \subset C$, the conditions (1) and (2) of the theorem obviously hold.

(b) Assume that a hyperfinite algebra $H \in {}^*K$ and an internal injection $j : H \to {}^*G$ that satisfy conditions (1) and (2) of the theorem exist. Then it is obvious that for any standard pair $(C, U) \in \mathcal{H}$ the pair $\langle H, j\rangle$ is a $({}^*C, {}^*U)$-approximation of ${}^*G$. By the transfer principle (applied in this case in the opposite direction), we see that G is approximable by algebras from the class $\mathcal{K}$. □

Let $\langle H, j\rangle$ be a hyperfinite l-quasigroup and an injection, respectively, that satisfy Theorem 5. In this case we say that $\langle H, j\rangle$ is a *hyperfinite approximation of G*. Let $V \subset G$ be a compact set with nonempty interior. By the regularity of the topological space G, there exists an open set W such that $\overline{W} \subset V$.

We write $W \sqsubset D$ if $\overline{W}$ is a subset of the interior of D. In what follows we use the following obvious fact:

LEMMA 1. *If $W \sqsubset D$, $x \in {}^*W$ and $y \approx x$, then $y \in {}^*D$.*

Let $\Delta^{-1} = |j^{-1}({}^*V)|$. Define the functional $I(f)$ for $f \in C_0(G)$ as follows:

$$I(f) = {}^\circ\left(\Delta \sum_{h \in H} {}^*f(j(h))\right). \tag{2}$$

PROPOSITION 3. *Formula (2) defines a non-zero positive left-invariant functional on $C_0(G)$.*

To prove Proposition 3 we need three technical lemma, which we prove next.

LEMMA 2. *Let $D \subseteq G$ be compact and $U \sqsubset D$ be an open set. Then for all $a \in G$ the following inequality holds:*

$$|j^{-1}(a \cdot {}^*U)| \leq |j^{-1}({}^*D)|.$$

Proof. Let $x \in j^{-1}(a \cdot {}^*U)$, i.e., $j(x) \in a \cdot {}^*U \subset \mathrm{ns}({}^*G)$. Then $a^{-1} \cdot j(x) \in {}^*U$. By Theorem 5, there exists $\beta \in H$ such that $a^{-1} \approx j(\beta)$. So, $a^{-1} \cdot j(x) \approx j(\beta \odot x) \in {}^*D$, since $U \sqsubset D$. Consequently, $\beta \odot (j^{-1}(a \cdot {}^*U)) \subset j^{-1}({}^*D)$, but the function $l_\beta(x) = \beta \odot x$ is an injection, since H is an l-quasigroup. □

LEMMA 3. *Let $X, Y \subset G$ be compact sets and suppose that Y has nonempty interior. Then there exists $0 < C_{X,Y} \in \mathbf{R}$, such that*

$$\frac{|j^{-1}({}^*X)|}{|j^{-1}({}^*Y)|} \leq C_{X,Y}.$$

Proof. Take an open set $U \sqsubset Y$. Let $l_a : G \to G$ be a left shift on G, which is a homeomorphism for any $a \in G$, since $l_{a^{-1}}$ is the inverse mapping to l_a. Thus, $l_a(U) = a \cdot U$ is an open set for any $a \in G$. Since X is a compact set, there exists a finite set $F \subset G$ such that $X \subset F \cdot U$. This means that ${}^*X \subset F \cdot {}^*U$ (${}^*F = F$). Consequently,

$$|j^{-1}({}^*X)| \leq \sum_{\alpha \in F} |j^{-1}(\alpha \cdot {}^*U)|.$$

By Lemma 2, we have $|j^{-1}({}^*X)| \leq |F| \cdot |j^{-1}({}^*Y)|$. So, one can take $C_{X,Y} = |F|$. □

LEMMA 4. *Let an internal function $\phi : H \to {}^*\mathbf{R}$ satisfy the following conditions:*

1. $\forall h \in H\ \varphi(h) \geq 0$;
2. $j(\mathrm{supp}(\varphi)) \subset {}^*S$, *where $S \subset G$ is a compact set;*
3. *there exist a compact set $D \subseteq G$ with nonempty interior and a positive real $\alpha \in \mathbf{R}$ such that $\forall h \in j^{-1}({}^*D)\ \varphi(h) > \alpha$.*

Then

$$\frac{1}{C_{V,D}} \alpha \leq \Delta \sum_{h \in H} \phi(h) \leq C_{S,V} \sup(\phi). \tag{3}$$

Proof. Recall that $\Delta^{-1} = |j^{-1}({}^*V)|$. By Lemma 3 we have

$$\Delta \sum_{h \in H} \phi(h) \geq \Delta \sum_{j(h) \in {}^*D} \phi(h) \geq \frac{\alpha}{C_{V,D}}.$$

This proves the first of the inequalities (3). The second inequality is obtained as follows:

$$\Delta \sum_{h \in H} \phi(h) = \Delta \sum_{j(h) \in {}^*S} \phi(h) \leq C_{S,V} \cdot \sup_x \phi(x).$$ □

Proof of Proposition 3. Let I be the functional defined on $C_0(G)$ in equation (2) above. We need to prove that it is non-zero, positive, and left-invariant.

Lemma 4 implies immediately that $I(f)$ is a non-zero positive bounded functional defined on $C_0(G)$. Indeed, for any $0 < f \in \mathbf{C}_0(G)$ put $\varphi(h) = {}^*f(j(h))$. Then φ satisfies the conditions of Lemma 4. Obviously, we have $\varphi(h) \geq 0$ and $S = \mathrm{supp}(f)$. Since $f > 0$, there exists a point $a \in A$ such that $f(a) > 0$. Thus, there exist an open set $U \ni a$ and a positive α such that $\forall b \in U\ f(b) > \alpha$. Take any relatively compact open set W such that $\overline{W} \subset U$. Then $D = \overline{W}$ satisfies condition (3) of Lemma 4. By (2) and the first inequality in (3), we have $I(f) \neq 0$. By the second inequality in (3), the linear functional I is bounded.

It remains to prove that the functional $I(f)$ is left-invariant.

Obviously, it is enough to prove that I satisfies the inequality $I(f) \geq I(l_a(f))$ for any non-negative $f \in C_0(G)$.

Let $S \subset G$ be a compact set such that there exists an open set $U \subset G$ with the property $a^{-1} \cdot \mathrm{supp}(f) \subset U \sqsubset S$. Let $h \in H$ be such that $j(h) \approx a \in G$. Then the following equality holds:

$$(4) \qquad {}^\circ\left(\Delta \sum_{x \in H} {}^*f(a \cdot j(x)) - \Delta \sum_{x \in H} {}^*f(j(h) \cdot j(x))\right) = 0.$$

To prove this, put $\varphi(x) = |\,{}^*f(a \cdot j(x)) - {}^*f(j(h) \cdot j(x))|$ and apply Lemma 4 as follows. By the continuity of the multiplication operation and the operation of taking the inverse element in G, we have

$$a \cdot j(x) \in \mathrm{ns}({}^*G) \Leftrightarrow j(x) \in \mathrm{ns}({}^*G) \Leftrightarrow j(h) \cdot j(x) \in \mathrm{ns}({}^*G).$$

We next show that $j(\mathrm{supp}(\varphi)) \subset {}^*S$. It is enough to show that

$$(5) \qquad j(x) \notin {}^*S \Longrightarrow {}^*f(a \cdot j(x)) = {}^*f(j(h) \cdot j(x)) = 0.$$

Assume that $a \cdot j(x) \in {}^*\mathrm{supp}(f)$. Thus, $j(x) \in a^{-1} \cdot \mathrm{supp}(f) \subset {}^*S$. This proves the first of the equalities (5).

Assume that $j(h) \cdot j(x) \in {}^*\mathrm{supp}(f)$. Then $j(x) \in j(h)^{-1} \cdot \mathrm{supp}(f) \approx a^{-1} \cdot \mathrm{supp}(f) \subset {}^*U$. But $U \sqsubset S$ and $j(x) \in {}^*S$. We get a contradiction, which proves the second of the equalities (5).

Since $a \cdot j(x) \approx j(h) \cdot j(x)$ if $j(x) \in \mathrm{ns}({}^*G)$ and $\mathrm{supp}(\varphi) \in j^{-1}({}^*S) \subset j^{-1}(\mathrm{ns}({}^*G))$, we have $\mathrm{supp}(\varphi) \approx 0$ and, by the second of the inequalities (3), $\Delta \sum_{h \in H} \varphi(h) \approx 0$. This proves the equality (4).

Let us now show that the following inequality holds:

$$(6) \qquad {}^\circ\left(\Delta \sum_{x \in H} {}^*f(j(h \odot x)) - \Delta \sum_{x \in H} {}^*f(j(h) \cdot j(x))\right) \geq 0.$$

By (5), we have

$$\Delta\sum_{x\in H} {}^*f(j(h\odot x)) - \Delta\sum_{x\in H} {}^*f(j(h)\cdot j(x))$$
$$= \Delta\sum_{j(x)\notin {}^*S} {}^*f(j(h\odot x)) + \Delta\sum_{j(x)\in {}^*S} ({}^*f(j(h\odot x)) - {}^*f(j(h)\cdot j(x)))$$

Obviously,

$$\Delta\sum_{j(x)\notin {}^*S} {}^*f(j(h\odot x)) = c \geq 0.$$

But

$$\Delta\sum_{j(x)\in {}^*S} ({}^*f(j(h\odot x)) - {}^*f(j(h)\cdot j(x))) \approx 0.$$

Indeed, since $j(h), j(x) \in \mathrm{ns}({}^*A)$ (because $j(x) \in {}^*S$), we have $j(h\odot x) \approx j(h)\cdot j(h)$ by Theorem 5. Thus, by the continuity of f, we have ${}^*f(j(h\odot x)) \approx {}^*f(j(h)\cdot j(h))$. Hence $\beta = \sup_{j(x)\in {}^*S} |{}^*f(j(h\odot x)) - {}^*f(j(h)\cdot j(x))| \approx 0$. By Lemma 3, we have

$$\left|\Delta\sum_{j(x)\in {}^*S} ({}^*f(j(h\odot x)) - {}^*f(j(h)\odot j(x)))\right| \leq C_{S,V}\beta \approx 0.$$

Since $\{h\odot x \mid x\in H\}$ is a permutation of H, we have

$$\Delta\sum_{x\in H} {}^*f(j(x)) = \Delta\sum_{x\in H} {}^*f(j(h\odot x)).$$

Now it is easy to see that the following equalities hold:

$$I(f) - I(l_a(f)) = {}^\circ\left(\Delta\sum_{x\in H} {}^*f(j(x)) - \Delta\sum_{x\in H} {}^*f(a\cdot j(x))\right)$$
$$= {}^\circ\left(\left(\Delta\sum_{x\in H} {}^*f(j(h\odot x)) - \Delta\sum_{x\in H} {}^*f(j(h)\cdot j(x))\right)\right.$$
$$\left. + \left(\Delta\sum_{x\in H} {}^*f(j(h)\cdot j(x)) - \Delta\sum_{x\in H} {}^*f(a\cdot j(x))\right)\right).$$

The first term on the right hand side of this equality is positive by (6), and the second one is infinitesimal by (4). Thus, we have $I(f) - I(l_a(f)) \geq 0$.

This completes the proof of Proposition 3. □

For the proof of Theorem 1 we reformulate the definition of a limit over an ultrafilter using nonstandard language. We say that a pair $(C_0, U_0) \in {}^*\mathcal{H}$ is infinite if $\forall (C,U) \in \mathcal{H}\ (C_0, U_0) \leq ({}^*C, {}^*U)$. We say that an infinite pair (C_0, U_0) dominates a standard ultrafilter $\mathcal{F}$ if ${}^*\mathcal{H}(C_0, U_0) \in \mathcal{F}$. It follows from the λ^+-saturation of the nonstandard universe that for any standard ultrafilter $\mathcal{F}$ on $\mathcal{H}$ there exists an infinite pair (C_0, U_0) that dominates $\mathcal{F}$.

The following result is immediate:

LEMMA 5. *For a standard function $\alpha : \mathcal{H} \to \mathbf{R}$ and a standard ultrafilter $\mathcal{F}$ over H, $\lim_{\mathcal{F}} \alpha(C, U) = a$ iff for any infinite $(C_0, U_0) \in \mathcal{H}$ that dominates $\mathcal{F}$ one has ${}^*\alpha(C_0, U_0) \approx a$.*

Now we complete the proof of Theorem 1.

For each $(C, U) \in \mathcal{H}$ let $H_{C,U}$ be a left (right) quasigroup that is a (C, U)-approximation of G. As before, we may assume that $H_{C,U} \subseteq G$ as a set.

If an infinite pair (C_0, U_0) dominates $\mathcal{F}$ and $H = H_{C_0,U_0}$, then H is a hyperfinite approximation of G, when we take $j : H \to {}^*G$ to be an inclusion. Let $I(f)$ be the functional defined by formula (2) for this H. Then, by Lemma 5, we have $\Lambda(f) = I(f)$. Proposition 3 thus yields Theorem 1. □

4. Proof of Theorem 3

LEMMA 6. *For any neighborhood U of the identity in G and any compact set $C \subseteq G$ there exist a finite set $F \subset G$ and a collection $\{A_{g,h} \subset F\ ;\ g, h \in F\}$ satisfying the following conditions:*

(1) *F is a U-grid of C;*
(2) *if $g, h \in C \cap F$, then $A_{g,h} \subset ghU$;*
(3) *$\forall g \in F\, \forall S \subset F\ \ |\bigcup_{h \in S} A_{g,h}| \geq |S|$.*

To prove Lemma 7 we need two technical lemmas, whose proofs are routine. To state them, we need some notation. We take $\mathcal{O}$ to be a neighborhood of the identity in G. For $A \subseteq G$, let $(A : \mathcal{O})$ denote the minimum cardinality of a set $F \subseteq G$ satisfying $A \subseteq F\mathcal{O}$. Note that $(A : \mathcal{O})$ is finite if A is compact. We take K to be a compact subset of G and F a finite subset of G such that $|F| = (K : \mathcal{O})$ and $K \subseteq F\mathcal{O}$ (F is an optimal $\mathcal{O}$-grid of K).

LEMMA 7. *Let $S \subset F$. Then $(S\mathcal{O} : \mathcal{O}) = |S|$.*

LEMMA 8. *Let $M \subset K$. Then $|M\mathcal{O}^{-1} \cap F| \geq (M : \mathcal{O})$.*

Proof of Lemma 6. Given a neighborhood of the identity $U \subset G$ and a compact $C \subset G$, one can chose a neighborhood of the identity $\mathcal{O}$ and a compact set K such that

- $\mathcal{O}\mathcal{O}^{-1} \subset U$;
- $C^2 \subset K$;
- $CU \subset K$.

Let F be an optimal $\mathcal{O}$-grid of K. Define the sets $A_{g,h}$ as follows:

$$A_{g,h} = \begin{cases} gh\mathcal{O}\mathcal{O}^{-1} \cap F, & \text{if } g, h \in C, \\ F, & \text{otherwise..} \end{cases}$$

It is easy to see that F is U-grid of C and item (2) of Lemma 6 is also satisfied.

We now prove item (3). The nontrivial case occurs when $g \in C$ and $S \subset C$. By Lemma 7, $(S\mathcal{O} : \mathcal{O}) = |S|$. Consequently, $(gS\mathcal{O} : \mathcal{O}) = |S|$. Then, by Lemma 8,

$$|S| \leq |gS\mathcal{O}\mathcal{O}^{-1} \cap F| = |\bigcup_{h \in S} A_{g,h}|. \qquad \square$$

Proof of Theorem 3. Lemma 6 (3) implies that the set F can be equipped with an operation $\odot$ satisfying the definition of l-quasigroup. Indeed, by condition (3) the system $\{A_{g,h} \mid h \in F\}$ satisfies Hall's Theorem (Marriage Lemma) for any fixed $g \in F$. Thus, for any $g, h \in F$ there exists $g \odot h \in A_{g,h}$ such that for any $g \in F$ $\{g \odot h \mid h \in F\}$ is a permutation of F. Thus, $\langle F, \odot \rangle$ is an l-quasigroup. The conditions (1) and (2) of Lemma 6 imply that the l-quasigroup $\langle F, \odot \rangle$ with the inclusion is a (C, U)-approximation of G; see Definition 1(3). $\square$

5. Proof of Theorem 4

First of all we formulate some necessary results about the structure of finite semigroups. (See [16] for these results and their proofs.)

Let S be a finite semigroup.

Definition 3.

(1) An $x \in S$ is said to be a zero ($x = 0$) if $\forall y \in S$ $xy = yx = x$. (Obviously if a zero exists it is unique.)
(2) A set $I \subseteq S$ is said to be a left (right) ideal if $SI \subseteq I$ ($IS \subseteq I$). A set $I \subseteq S$ is said to be an ideal if I is a left and a right ideal. (Obviously an ideal (a left or a right ideal) is a subsemigroup.)
(3) The semigroup S is said to be 0-simple if it has no proper ideals except $\{0\}$ and $\emptyset$.
(4) The semigroup S is a zero semigroup iff $\forall s, t \in S$ $st = 0$.
(5) Let $I \subset S$ be an ideal in $(S, \cdot)$. The quotient semigroup S/I is the set $(S \backslash I) \cup \{0\}$ with the multiplication "$*$" defined by

$$s_1 * s_2 = \begin{cases} s_1 \cdot s_2, \text{ if } s_1 \cdot s_2 \notin I, \\ 0, \text{ if } s_1 \cdot s_2 \in I. \end{cases}$$

(6) A maximal sequence of ideals in S is an ordered sequence of ideals of S,

$$S = I_0 \supset I_1 \supset I_2 \cdots I_n \supset I_{n+1} = \emptyset, \tag{7}$$

such that for any $k = 0, \ldots, n$ and for any ideal I' of S, if $I_k \supset I' \supset I_{k+1}$, then either $I' = I_k$ or $I' = I_{k+1}$.

It is clear that any finite semigroup has a maximal sequence of ideals.

THEOREM 6. *If the sequence (7) is a maximal sequence of ideals in S, then for any i, $1 \leq i \leq n+1$, the semigroup I_{r-1}/I_r is 0-simple or zero.*

Let $n, m \in \mathbf{N}$, H be a group, $\rho : \{1,..,n\} \times \{1,..,m\} \to H \cup \{0\}$. The Rees semigroup $S(n,m,H,\rho)$ is defined as follows:

$$S(n,m,H,\rho) = \{(i,j,h) \mid i = 1,\dots,n;\ j = 1,\dots,m;\ h \in H\} \cup \{0\},$$

$$(i_1,j_1,h_1)(i_2,j_2,h_2) = \begin{cases} (i_1,j_2,h_1\rho(i_2,j_1)h_2), \text{ if } \rho(i_2,j_1) \in H, \\ 0, \text{ if } \rho(i_2,j_1) = 0. \end{cases}$$

The Rees semigroup $S(n,m,H,\rho)$ is called regular if $\forall i\ \exists\ j\ \rho(i,j) \neq 0$ and $\forall j\, \exists i\ \rho(i,j) \neq 0$.

THEOREM 7. *Any finite 0-simple semigroup S (with a zero) is isomorphic to a regular Rees semigroup.*

(If S is a semigroup without a zero, we may add a zero to S or remove the zero from the Rees semigroup.)

This theorem implies the following corollary.

COROLLARY 2. *Let S be a 0-simple finite semigroup, $0 \neq s \in S$ and $F = sSs$. Then F is a zero subsemigroup or $F\backslash\{0\}$ is a group.*

Proof. Let $s = (i_s,j_s,h_s)$. If $F \neq \{0\}$, then $F = \{sas,\ a \in S\} = \{(i_s,j_s,h),\ h \in H\} \cup \{0\}$. If $\rho(i_s,j_s) = 0$, then F is a zero semigroup. If $\rho(i_s,j_s) = g$, then the map $\phi : F\backslash\{0\} \to H$ such that $\phi(i_s,j_s,h) = hg$ is an isomorphism. □

We are able now to prove Theorem 4.

Let G be a locally compact group that is approximable by finite semigroups and let $\langle S, \phi\rangle$ be a hyperfinite approximation of G by a hyperfinite semigroup S. (Such an approximation exists by Theorem 5.) The operations in G and in S will both be denoted by $\cdot$, since this does not lead to any misunderstanding.

First of all, we show that if $0 \in S$ then $\phi(0) \notin \mathrm{ns}({}^*G)$. Suppose that $\phi(0) \in \mathrm{ns}({}^*G)$. If $\phi(0) \approx e$, then $\forall x \in G\ xe \approx e$, which is impossible. If $\phi(0) \approx x$ and $x \neq e$, then there exists y such that $\phi(y) \approx x^{-1}$. Now

$$e = xx^{-1} \approx \phi(0)\phi(y) \approx \phi(0 * y) = \phi(0) \approx x.$$

This is impossible since x, e are standard and $x \neq e$.

Consider an internal hyperfinite maximal sequence of ideals in S (see Definition 3(6)),

$$S = I_0 \supset I_1 \supset I_2 \supset \cdots \supset I_n \supset I_{n+1} = \emptyset,$$

which exists by the transfer principle.

By assumption, $\phi(S) \cap \mathrm{ns}({}^*G) \neq \emptyset$. Hence there exists $r \in {}^*\mathbf{N}$ such that $\phi(I_{r-1}) \cap \mathrm{ns}({}^*G) \neq \emptyset$ and $\phi(I_r) \cap \mathrm{ns}({}^*G) = \emptyset$. There are two cases.

(1) $I_r = \emptyset$. Then put $F = I_{r-1}$ and $\psi = \phi|_F$.
(2) $I_r \neq \emptyset$. Then put $F = I_{r-1}/I_r$ and define $\psi : F \to G$ by $\psi(x) = \phi(x)$ for $x \neq 0$ and $\psi(0) = g \notin \mathrm{ns}({}^*G) \cup \mathrm{Im}(\phi)$.

Such an element g exists since G is not compact. Indeed, otherwise ${}^*G = \mathrm{ns}({}^*G)$. However, $\phi(I_r) \subseteq {}^*G \setminus \mathrm{ns}({}^*G)$ and $\phi(I_r) \neq \emptyset$, since $I_r \neq \emptyset$.

It is easy to see that there exists an internal compact set $D \supset \mathrm{ns}({}^*G)$. The set ${}^*G \setminus {}^*D$ is non-compact and thus, is non-hyperfinite. So, ${}^*G \setminus (\mathrm{ns}({}^*G) \cup \mathrm{Im}(\phi)) \neq \emptyset$.

Next we prove that $\langle F, \psi\rangle$ approximates G in the sense of Theorem 5. Denote the operation on F by $\circ$.

First we show that ψ is an almost homomorphism, i.e.,

$$\forall x, y \in F\ (\psi(x), \psi(y) \in \mathrm{ns}({}^*G) \Longrightarrow \psi(x \circ y) \approx \psi(x)\psi(y)).$$

Let $x, y \in F$ and $\psi(x), \psi(y) \in \mathrm{ns}({}^*G)$. We have to prove that $\psi(x \circ y) \approx \psi(x)\psi(y)$. For case (1) this is trivial, since ψ is a restriction of ϕ to a subsemigroup. Consider now case (2). Since $\psi(x), \psi(y) \in \mathrm{ns}({}^*G)$, one has $x, y \neq 0$. Thus, $\psi(x) = \phi(x)$ and $\psi(y) = \phi(y)$. Then $\mathrm{ns}({}^*G) \ni \phi(x)\phi(y) \approx \phi(xy)$. Hence $\phi(xy) \in \mathrm{ns}({}^*G)$ and thus, $xy \notin I_r$. By the definition of the operation in a quotient semigroup, we have $x \circ y = xy \neq 0$, and by the construction of ψ, we have $\psi(x \circ y) = \phi(xy)$.

It remains to prove that $\forall g \in G\ \exists x \in F\ g \approx \psi(x)$, or equivalently, that $\forall g \in G\ \exists x \in I_{r-1}\ g \approx \phi(x)$. Since $\phi(I_{r-1}) \cap \mathrm{ns}({}^*G) \neq \emptyset$, there exists an element $x \in I_{r-1}$ such that $\phi(x) \in \mathrm{ns}({}^*G)$. Since ${}^{-1}$ is a continuous operation, $(\phi(x))^{-1} \in \mathrm{ns}({}^*G)$ and there exists $y \in S\ \phi(y) \approx (\phi(x))^{-1}$. Hence $e \approx \phi(y)\phi(x) \approx \phi(yx)$. Notice that $yx \in I_{r-1}$ since I_{r-1} is an ideal and $x \in I_{r-1}$. Let $g \in G$ and $s \in S$ be such that $\phi(s) \approx g$. Then $\phi(yxs) \approx g$ and $yxs \in I_{r-1}$.

Obviously, a zero semigroup can never approximate an infinite group. Thus, the hyperfinite semigroup F is 0-simple by Theorem 6.

Let $s \in F$ be such that $\psi(s) \approx e$. Consider the semigroup $T = sFs$. It is easy to see that if $j = \psi|_T$ then the pair $\langle T, j\rangle$ approximates G.

By Corollary 2 of Theorem 7, $H = T \setminus \{0\}$ is a hyperfinite group. This completes the proof since $j(0) \notin \mathrm{ns}({}^*G)$. □

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