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ON SUMMING SEQUENCES IN $\mathbb{R}^d$

M. ANOUSSIS AND D. GATZOURAS

Abstract. We give a necessary and sufficient condition on a sequence of convex sets in $\mathbb{R}^d$ for the corresponding sequence of measures to be a summing sequence.

1. Introduction

In this note we give a geometric characterization of summing sequences consisting of convex sets in $\mathbb{R}^d$.

Definition 1. A sequence of regular Borel probability measures $\{\mu_n\}$ on $\mathbb{R}^d$ is a summing sequence if $\widehat{\mu}_n(\chi) \longrightarrow 0$ as $n \longrightarrow \infty$, for every character χ of $\mathbb{R}^d$ not identically equal to one.

Throughout, we shall restrict attention to sequences of the form

$$\mu_n(B) := \frac{|B \cap G_n|}{|G_n|}, \tag{1}$$

where $\{G_n\}$ is a sequence of Borel sets in $\mathbb{R}^d$ of positive and finite Lebesgue measure. Here, and throughout the paper, $|\ |$ denotes Lebesgue measure on $\mathbb{R}^d$.

In this sense, summing sequences were introduced by Blum and Eisenberg in [2] under the name "generalized summing sequences" and used to produce mean ergodic theorems in locally compact abelian groups. In case $\mu_n = n^{-1} \sum_{k=1}^{n} \delta_{x_k}$, $\{\mu_n\}$ is a summing sequence means exactly that $\{x_n\}$ is (Hartman) uniformly distributed. Such sequences are studied extensively in [4] ([4, Ch. 4, Sect. 5])). Summing sequences also appear in [5], [6], [7], and [8]. The most well-known examples of sequences of sets producing summing sequences are Følner sequences [2, Corollary 2].

The inradius of a convex set in $\mathbb{R}^d$ is the radius of the largest ball contained in it. For convex sets G_n in $\mathbb{R}^d$, Day [3] has shown that if the inradii $\varrho(G_n)$ of

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the G_n tend to infinity, then the sequence $\{G_n\}$ is a Følner sequence. In fact, it is not hard to see that a sequence of convex sets G_n is a Følner sequence iff $\varrho(G_n) \longrightarrow \infty$.

In this note we give a necessary and sufficient condition on a sequence of convex sets for the corresponding sequence of measures to be a summing sequence. We also present an example of a sequence of convex sets G_n in $\mathbb{R}^d$ which produces a summing sequence, yet $\varrho(G_n) \longrightarrow 0$.

2. The main result

Definition 2. Let G be a Borel set in $\mathbb{R}^d$ with $0 < |G| < \infty$. For $\boldsymbol{u} \in \mathbb{S}^{d-1}$, the width of G in the direction $\boldsymbol{u}$ is the number

$$w_G(\boldsymbol{u}) := \sup_{\boldsymbol{x}\in G} \boldsymbol{x}\cdot\boldsymbol{u} - \inf_{\boldsymbol{x}\in G} \boldsymbol{x}\cdot\boldsymbol{u}.$$

Theorem. *Let G_n, $n \in \mathbb{N}$, be Borel sets in $\mathbb{R}^d$ with $0 < |G_n| < \infty$ for all n and $\{\mu_n\}$ be the sequence of measures defined by $\mu_n(B) := |B \cap G_n| \,/\, |G_n|$.*

(1) *If $\{\mu_n\}$ is a summing sequence, then*

$$w_{G_n}(\boldsymbol{u}) \longrightarrow \infty \quad \forall\, \boldsymbol{u} \in \mathbb{S}^{d-1}.$$

(2) *Assume that G_n is convex for every $n \in \mathbb{N}$. Then if*

$$w_{G_n}(\boldsymbol{u}) \longrightarrow \infty \quad \forall\, \boldsymbol{u} \in \mathbb{S}^{d-1}$$

the sequence $\{\mu_n\}$ is a summing sequence.

Proof. (1) Suppose that for some $\boldsymbol{u} \in \mathbb{S}^{d-1}$, $w_{G_n}(\boldsymbol{u})$ does not tend to ∞. By passing to a subsequence if necessary, we may then assume that

$$B := \sup_{n\in\mathbb{N}} w_{G_n}(\boldsymbol{u}) < \infty.$$

We shall show that, for some $\boldsymbol{\xi} \neq \boldsymbol{0}$, $\widehat{\mu}_n(\boldsymbol{\xi}) \nrightarrow 0$.

Let $\boldsymbol{c}_n \in \bar{G}_n$ be such that $\boldsymbol{c}_n \cdot \boldsymbol{u} = \inf_{\boldsymbol{x}\in G_n} \boldsymbol{x}\cdot\boldsymbol{u}$ (notice that the condition $\sup_{n\in\mathbb{N}} w_{G_n}(\boldsymbol{u}) < \infty$ guarantees that $\inf\limits_{\boldsymbol{x}\in G_n} \boldsymbol{x}\cdot\boldsymbol{u} > -\infty$). Then

$$0 \leqslant \boldsymbol{u}\cdot(\boldsymbol{x} - \boldsymbol{c}_n) \leqslant \sup_{\boldsymbol{y}\in G_n} \boldsymbol{y}\cdot\boldsymbol{u} - \boldsymbol{c}_n\cdot\boldsymbol{u} = w_{G_n}(\boldsymbol{u})$$

for all $\boldsymbol{x} \in G_n$. Choose $\delta > 0$ so that $\left|e^{is} - 1\right| \leqslant \frac{1}{2}$, say, for $|s| \leqslant \delta$, and set $\xi := \delta/B$ and $\boldsymbol{\xi} := \xi\boldsymbol{u}$. Then it follows that

$$\left| |G_n|^{-1} \int_{G_n} e^{i\boldsymbol{\xi}\cdot(\boldsymbol{x}-\boldsymbol{c}_n)}\, d\boldsymbol{x} - 1 \right| \leqslant |G_n|^{-1} \int_{G_n} \left| e^{i\boldsymbol{\xi}\cdot(\boldsymbol{x}-\boldsymbol{c}_n)} - 1 \right| d\boldsymbol{x} \leqslant \tfrac{1}{2},$$

and hence

$$|\widehat{\mu}_n(\boldsymbol{\xi})| = |G_n|^{-1} \left| \int_{G_n} e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}}\, d\boldsymbol{x} \right| = |G_n|^{-1} \left| e^{-i\boldsymbol{\xi}\cdot\boldsymbol{c}_n} \int_{G_n} e^{i\boldsymbol{\xi}\cdot(\boldsymbol{x}-\boldsymbol{c}_n)}\, d\boldsymbol{x} \right| \geqslant \tfrac{1}{2}$$

for all $n \in \mathbb{N}$.

(2) We shall need to consider both d-dimensional and $(d-1)$-dimensional Lebesgue measure in the following proof, so we switch to the notation $|\ |_m$ for m-dimensional Lebesgue measure.

Assume that the G_n are convex and

$$w_{G_n}(\boldsymbol{u}) \longrightarrow \infty \quad \forall\, \boldsymbol{u} \in \mathbb{S}^{d-1}.$$

We shall show that $\widehat{\mu}_n(\boldsymbol{\xi}) \longrightarrow 0$ for all $\boldsymbol{\xi} \in \mathbb{R}^d \smallsetminus \{\mathbf{0}\}$.

Fix $\boldsymbol{\xi} \in \mathbb{R}^d \smallsetminus \{\mathbf{0}\}$ and write $\boldsymbol{\xi} = \xi \boldsymbol{u}$ with $\xi > 0$ and $\boldsymbol{u} \in \mathbb{S}^{d-1}$. Using coordinates with respect to an orthonormal basis of which $\boldsymbol{u}$ is a member, one sees that

$$\widehat{\mu}_n(\boldsymbol{\xi}) = |G_n|_d^{-1} \int_{G_n} e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}}\, d\boldsymbol{x} = \int_{\mathbb{R}} e^{i\xi x} f_n(x)\, dx = \widehat{f}_n(\xi),$$

where f_n is the probability density function on $\mathbb{R}$ given by

$$f_n(x) := \frac{\big|G_n \cap (\boldsymbol{u}^{\perp} + x\boldsymbol{u})\big|_{d-1}}{|G_n|_d},$$

and where $\boldsymbol{u}^{\perp}$ denotes the hyperplane perpendicular to $\boldsymbol{u}$. Thus it suffices to show that

$$\widehat{f}_n(\xi) \longrightarrow 0 \qquad \forall\, \boldsymbol{\xi} \in \mathbb{R} \smallsetminus \{0\}\,.$$

Set $a_n = \inf_{\boldsymbol{x}\in G_n} \boldsymbol{x}\cdot\boldsymbol{u}$ and $b_n = \sup_{\boldsymbol{x}\in G_n} \boldsymbol{x}\cdot\boldsymbol{u}$, and note that a_n and b_n are finite since we are assuming that the G_n are convex and of positive and finite measure (and hence necessarily pre-compact). Furthermore, since G_n is convex, the function $f_n^{1/(d-1)}$ is concave in $[a_n, b_n]$, by the Brunn–Minkowski inequality; hence f_n is continuous on $[a_n, b_n]$, and unimodal, i.e., there exists $c_n \in [a_n, b_n]$ such that f_n is non-decreasing on $[a_n, c_n]$ and non-increasing on $[c_n, b_n]$. Now

$$\widehat{f}_n(\xi) = \int_{a_n}^{b_n} \cos(\xi x) f_n(x)\, dx + i \int_{a_n}^{b_n} \sin(\xi x) f_n(x)\, dx$$

and, writing $G(x) := \xi^{-1}\sin(\xi x)$, integration by parts yields

$$\begin{aligned}\int_{a_n}^{b_n} \cos(\xi x) f_n(x)\, dx &= \int_{a_n}^{b_n} G'(x) f_n(x)\, dx \\ &= G(b_n) f_n(b_n) - G(a_n) f_n(a_n) - \int_{a_n}^{b_n} G(x)\, df_n(x),\end{aligned}$$

where the last integral is a Riemann–Stieltjes integral, and similarly for the other integral (see, e.g., [1, Theorem 18.4]). It follows that

$$\left|\widehat{f}_n(\xi)\right| \leqslant \frac{8}{\xi} \max_{x\in[a_n,b_n]} f_n(x) = \frac{8}{\xi} f_n(c_n).$$

Finally, the concavity of the function $x \longmapsto \left|G_n \cap (\boldsymbol{u}^\perp + x\boldsymbol{u})\right|_{d-1}^{1/(d-1)}$, $x \in [a_n, b_n]$, also implies that

$$\frac{w_{G_n}(\boldsymbol{u})}{d} \max_x \left|G_n \cap (\boldsymbol{u}^\perp + x\boldsymbol{u})\right|_{d-1} \leqslant |G_n|_d \leqslant w_{G_n}(\boldsymbol{u}) \max_x \left|G_n \cap (\boldsymbol{u}^\perp + x\boldsymbol{u})\right|_{d-1},$$

whence

$$\frac{1}{w_{G_n}(\boldsymbol{u})} \leqslant \max_x f_n(x) \leqslant \frac{d}{w_{G_n}(\boldsymbol{u})} \longrightarrow 0 \qquad (n \longrightarrow \infty).$$

We conclude that

$$\widehat{f}_n(\xi) \longrightarrow 0 \qquad \forall \boldsymbol{\xi} \in \mathbb{R} \setminus \{0\}. \qquad \square$$

The second assertion of the theorem is not valid if we do not assume that the sets G_n are convex. This may be easily seen by considering, for example, the sets $G_n := [-n, n]^d \setminus [-n+1, n-1]^d$ in $\mathbb{R}^d$.

3. An example

The following is an example of a sequence $\{G_n\}$ of convex sets in $\mathbb{R}^d$, for which the corresponding measures (1) form a summing sequence in $\mathbb{R}^d$, yet $\varrho(G_n) \longrightarrow 0$.

Example. Consider the ellipsoids $G_n := \left\{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{x}' Q_n \boldsymbol{x} \leqslant 1\right\}$ in $\mathbb{R}^d$ determined by

$$Q_n := \begin{pmatrix} \boldsymbol{u}_1(n) & \ldots & \boldsymbol{u}_d(n) \end{pmatrix} \begin{pmatrix} a_1(n)^{-2} & & \\ & \ddots & \\ & & a_d(n)^{-2} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_1(n)' \\ \vdots \\ \boldsymbol{u}_d(n)' \end{pmatrix},$$

where $a_1(n) > \ldots > a_d(n) > 0$ are positive numbers and $\boldsymbol{u}_1(n), \ldots, \boldsymbol{u}_d(n)$ are unit column-vectors in $\mathbb{R}^d$ forming an orthonormal basis and where $\boldsymbol{u}_1(n)', \ldots$ $\ldots, \boldsymbol{u}_d(n)'$ denote the corresponding row-vectors. Choose $a_1(n), \ldots, a_d(n)$ and

$$\boldsymbol{u}_1(n) := (u_{11}(n), \ldots, u_{d1}(n))'$$

so that

$$u_{i1}(n) > 0 \quad \text{for all} \quad i, \tag{2}$$

$$\frac{u_{i1}(n)}{u_{k1}(n)} \longrightarrow 0 \quad \text{for all} \quad 1 \leqslant k < i \leqslant d, \tag{3}$$

$$a_1(n) \cdot \min_{1 \leqslant i \leqslant d} u_{i1}(n) \longrightarrow \infty, \tag{4}$$

and

$$a_1(n) \cdots a_d(n) \longrightarrow 0, \tag{5}$$

as $n \longrightarrow \infty$. Let also μ_n be the measures (1) corresponding to these G_n. If $\boldsymbol{u} \in \mathbb{S}^{d-1}$ is a fixed unit vector,

(6)

$$w_{G_n}(\boldsymbol{u}) \geqslant |a_1(n)(\boldsymbol{u} \cdot \boldsymbol{u}_1(n)) - a_1(n)(\boldsymbol{u} \cdot (-\boldsymbol{u}_1(n)))| = 2\, a_1(n)\, |\boldsymbol{u} \cdot \boldsymbol{u}_1(n)|\,,$$

since the vectors $\pm a_1(n)\boldsymbol{u}_1(n)$ belong to G_n. Writing $\boldsymbol{u} = (u_1, \ldots, u_d)'$, and denoting by k the least i for which $u_i \neq 0$, one has that

$$\boldsymbol{u} \cdot \boldsymbol{u}_1(n) = \sum_{i=k}^{d} u_i u_{i1}(n) = u_k\, u_{k1}(n) \left(1 + \sum_{i=k+1}^{d} \frac{u_i}{u_k} \frac{u_{i1}(n)}{u_{k1}(n)}\right),$$

whence $w_{G_n}(\boldsymbol{u}) \longrightarrow \infty$ by (3), (4) and (6). Thus the measures μ_n corresponding to these G_n form a summing sequence in $\mathbb{R}^d$. On the other hand

$$\varrho(G_n) = a_d(n) \longrightarrow 0,$$

by (5).

It remains to show that there exist numbers $a_1, \ldots, a_d(n)$ and $u_{11}(n), \ldots, u_{d1}(n)$ satisfying (2)–(5). For an example let c_d be any positive number satisfying $c_d < (d-1)^{-1}$, and set

$$u_{i1}(n) := \sqrt{\frac{c_d}{n^{i-1}}} \quad \text{for} \quad 1 < i \leqslant d$$

and

$$u_{11}(n) := \sqrt{1 - \frac{c_d}{n} - \cdots - \frac{c_d}{n^{d-1}}}\ ;$$

then choose $a_1(n), \ldots, a_d(n)$ accordingly.

Notice that in the above example

$$|G_n| = \gamma_d\, a_1(n) \cdots a_d(n) \longrightarrow 0,$$

where γ_d denotes the d-dimensional measure of the unit ball in $\mathbb{R}^d$. However, it is not hard to see that if we assume that $G_n \subseteq G_{n+1}$, then the condition $|G_n| \longrightarrow |G|$ is necessary for the corresponding sequence of measures (1) to be a summing sequence, in any locally compact abelian group.

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M. Anoussis, Department of Mathematics, University of the Aegean, 832 00 Karlovasi, Samos, Greece

E-mail address: mano@aegean.gr

D. Gatzouras, Laboratory of Mathematics and Statistics, Agricultural University of Athens, Iera Odos 75, 118 55 Athens, Greece

E-mail address: gatzoura@aua.gr