VERTICES OF SELF-SIMILAR TILES

DA-WEN DENG AND SZE-MAN NGAI

ABSTRACT. The set V_n of n-vertices of a tile T in $\mathbb{R}^d$ is the common intersection of T with at least n of its neighbors in a tiling determined by T. Motivated by the recent interest in the topological structure as well as the associated canonical number systems of self-similar tiles, we study the structure of V_n for general and strictly self-similar tiles. We show that if T is a general self-similar tile in $\mathbb{R}^2$ whose interior consists of finitely many components, then any tile in any self-similar tiling generated by T has a finite number of vertices. This work is also motivated by the efforts to understand the structure of the well-known Lévy dragon. In the case T is a strictly self-similar tile or multitile in $\mathbb{R}^d$, we describe a method to compute the Hausdorff and box dimensions of V_n. By applying this method, we obtain the dimensions of the set of n-vertices of the Lévy dragon for all $n \geq 1$.

1. Introduction

Let $\{f_i\}_{i=1}^q$ be an *iterated function system (IFS)* of injective contractions on $\mathbb{R}^d$ and let T be the unique attractor. That is, T is the unique nonempty compact set satisfying

$$(1.1) \qquad T = \bigcup_{i=1}^q f_i(T).$$

Throughout this paper we assume that $\{f_i\}_{i=1}^q$ satisfies the *open set condition*, i.e., there exists a nonempty bounded open set O such that

$$\bigcup_{i=1}^q f_i(O) \subseteq O \quad \text{and} \quad f_i(O) \cap f_j(O) = \emptyset \text{ if } i \neq j.$$

If T°, the interior of T, is nonempty, we call T a *general self-similar tile*. It is known that $\mathbb{R}^d$ can be tiled by homeomorphic copies of T (see [NT2,

Lemma 2.3]). If each f_i is a contractive similitude, i.e., $f_i(x) = \rho_i R_i(x) + b_i$, where $0 < \rho_i < 1$, R_i is orthogonal, and $b_i \in \mathbb{R}^d$, we say that T is a *(strictly) self-similar tile* if $T^\circ \neq \emptyset$. If each of the f_i in (1.1) is of the form

$$(1.2) \qquad\qquad f_i(x) = A^{-1}(x + d_i),$$

where A is an integer expanding matrix (all eigenvalues have modulus > 1) with $|\det(A)| = q$ and $d_i \in \mathbb{Z}^d$, then T is a called a *self-affine tile* if $T^\circ \neq \emptyset$. $\mathcal{D} = \{d_1, \ldots, d_q\}$ is called a *digit set*.

For any subset $E \subseteq \mathbb{R}^d$, we denote by $E^\circ, \overline{E}, \partial E, E^c$ the interior, closure, boundary, and complement of E, respectively. We denote the cardinality of a set A by $\#A$.

Let T be a general self-similar tile in $\mathbb{R}^d$ and let $\{T_k\}_{k=0}^\infty$ be a self-similar tiling of $\mathbb{R}^d$ defined by homeomorphic copies of T as described in [NT2, Lemma 2.3] and recalled in Section 2, with $T_0 := T$. T_i is said to be a *neighbor* of T if $T_i \neq T$ and $T_i \cap T \neq \emptyset$. For each positive integer n, define

$$V_n(T_i) := \{x \in T_i : x \text{ lies in at least } n \text{ neighbors of } T_i\}.$$

If $V_n(T_i)$ is independent of T_i in the sense that all the $V_n(T_i)$ can be obtained from each other through a rigid motion, we denote it by V_n. We call each $x \in V_n$ an *n-vertex* of T. Note that $V_1 = \partial T$. Also, $V_{n+1} \subseteq V_n$ for all $n \geq 1$ and $V_n = \emptyset$ for all n sufficiently large.

This work is partly motivated by the recent interest in the topological structure of self-similar tiles as well as the canonical number systems associated with the tiles. The topological structure of tiles has been studied extensively recently; see [KL], [BW], [LRT], [NN], [LAT], [NT1], [NT2], [DPPS], [L]. In particular, it is shown in [BW] and [L] that the set V_2 plays a deterministic role in the disk-likeness of certain self-affine tiles.

In the context of number representations, V_n is a subset of T consisting of numbers with at least $n + 1$ different radix expansions. For self-affine tiles arising from quadratic *canonical number systems* (CNS) with base $-m + i$, this problem has been studied extensively, by Kátai and Szabó [KS], Gilbert [G1], [G2], and Akiyama and Thuswaldner [AT1], [AT2], [AT3], [MTT], [ST], [T], [L]. In [MTT], [ST], [T], the box dimension of the boundaries of such tiles are calculated by explicitly finding covers of the boundary. For the Eisenstein set, Benedek and Panzone [BP] computed the Hausdorff and box dimensions of the set V_1 and determined the sets V_n for $n \geq 2$.

Motivated by the above investigations, we study the structure of V_n for general self-similar tiles and self-affine tiles. Since V_1, the boundary of T, has been studied extensively, we will focus on the case $n \geq 2$. We prove that if T is a general self-similar tile in $\mathbb{R}^2$ whose interior T° consists of finitely many components, then in any self-similar tiling it generates, $V_2(T_i)$ is a finite set for any tile T_i (see Theorem 2.1).

In the case V_n is not finite, it is desirable to have a more general method for computing its Hausdorff and box dimensions. In order to obtain the Hausdorff dimension, we restrict our study to the class of strictly self-affine tiles or multitiles. In the case the matrix A in (1.2) is conjugate to a similarity, we describe an algorithm to compute the Hausdorff and box dimensions of the sets V_n. This algorithm is an extension of the one used by Strichartz and Wang to compute the boundary of self-affine tiles (see [SW]). It holds for self-affine multitiles (see Section 3 for the definition) and thus it allows us to study the set V_n for the interesting Lévy dragon. In fact, a main motivation for formulating the above algorithm is the desire to understand the structure of the Lévy dragon. For this dragon, the dimensions of V_1, the boundary of the dragon, were computed by Duvall and Keesling [DK], and Strichartz and Wang [SW]. However, the structure of the dragon is still not completely understood (see [BKS] and the discussion in Section 4). In this paper, we obtain the dimensions of the sets V_n for all $n \geq 1$. This result reveals the complicated way the neighbors are intertwined in the Lévy dragon tiling of the plane.

This paper is organized as follows. In Section 2, we consider the case T is a connected general self-similar tile in $\mathbb{R}^2$ whose interior consists of finitely many components and prove that for every tile T_i in any self-similar tiling generated by T, $V_2(T_i)$ must be finite. In Section 3 we describe an algorithm to compute the dimensions of the sets V_n for the class of self-affine tiles and self-affine multitiles defined by an expansion matrix which is conjugate to a similarity. We then apply the algorithm to study the Lévy dragon in Section 4.

2. Tiles in the plane with interiors having finitely many components

In this section we consider self-similar tilings of the plane and prove a result which only holds on $\mathbb{R}^2$. Let $\{f_i\}_{i=1}^q$ be an IFS of injective contractions on $\mathbb{R}^2$ satisfying the open set condition. Let T be the attractor of the IFS and assume that $T^\circ \neq \emptyset$. It is known that $\mathbb{R}^2$ can be tiled by homeomorphic copies of T (see [NT2, Lemma 2.3]), the tiling being *self-similar* in the sense that it is invariant with respect to blowing up at an interior point of T. The tiling is not unique and T has finitely many neighbors. To see these, recall the construction of the tiling. Let N be sufficiently large and $\alpha \in \{1, \ldots, q\}^N$ so that $f_\alpha(T) \subseteq T^\circ$. Then

$$f_\alpha^{-k} \circ f_\beta(T), \quad \beta \in \{1, \ldots, q\}^{Nk} \text{ and } k \in \mathbb{N},$$

forms a tiling of $\mathbb{R}^2$ which is invariant with respect to blowing up by f_α^{-1} at the fixed point of f_α in T°. By choosing different α or N, we may get non-congruent tilings. To see that a tile T_1 in the tiling has finitely many

neighbors, choose k so large that $f_\alpha^k(T_1) \subseteq T^\circ$. Then T_1 is in the interior of

$$f_\alpha^{-k}(T) = \bigcup_{\beta \in \{1,\dots,q\}^{Nk}} f_\alpha^{-k} \circ f_\beta(T),$$

which is a finite union of tiles. Hence T_1 has finitely many neighbors.

In the case T is a self-affine tile, the above tilings reduce to the familiar tilings obtained by translations of T.

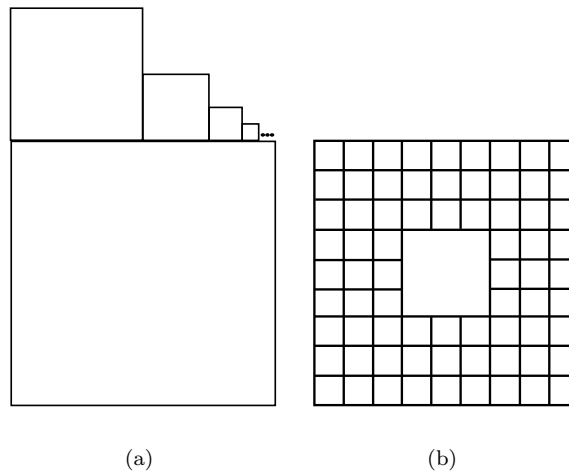(a)   (b)

FIGURE 1. (a) The configuration can be completed to a tiling of the plane by squares. It is a non self-similar tiling by a self-similar tile. Notice that the largest tile has infinitely many vertices. (b) A pattern that generates a self-similar tiling of the plane with different numbers of vertices for different tiles in the tiling.

Notice that there are tilings by self-similar tiles that are not self-similar. For example, let T be the unit square. Border it by a queue of squares decreasing in size, with total length less than one. Complete the configuration to a tiling of $\mathbb{R}^2$ by squares (see Figure 1(a)). As T has infinitely many neighbors, this tiling cannot be generated by the blow-up argument. Notice that $V_2(T)$ is infinite and Theorem 2.1 does not apply.

Notice that tiles in a self-similar tiling generated by self-similar tiles can have different numbers of vertices. Consider a unit square divided into nine squares, with the eight boundary squares each further divided into nine smaller squares. Then this self-similar tile, with $q = 73$, generates self-similar tilings
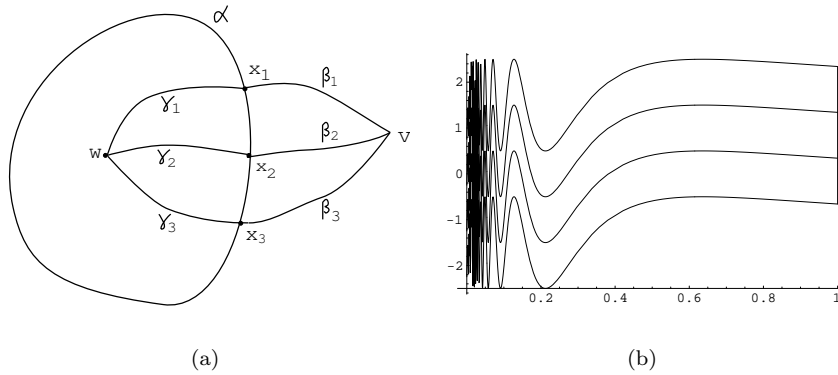
FIGURE 2. (a) Notations used in the proof of Theorem 2.1. As $\alpha \subseteq \partial U$, there are points of U on both sides of $\beta_1 \cup \beta_2 \cup \gamma_1 \cup \gamma_2$, contradicting that U is a component of T°. (b) Three fat topological sine curve tiles reproduced from [NT2]. The tile is not self-similar and its vertex set contains a line segment.

as above, with $V_2(T_i)$ equal to four for some T_i and twelve for others (see Figure 1(b)).

Assume, in addition, that T° has finitely many components. It is shown in [NT2] that if U is a component of T°, then $\overline{U}$, the closure of U, is locally connected. This is established by showing that it has property S, and that it is a finite union of connected sets of arbitrarily small diameter, the closure of the components of the interior of the miniatures of T. It follows that the boundary of every component of $\overline{U}^c$ is a simple closed curve (see [W1]). Moreover, there are finitely many such curves as $\overline{U}$ has at most finitely many holes (see [NT2]). In particular, ∂U is locally connected and thus every point on ∂U is *accessible from all sides of U* (see [W2, Theorem VI.4.2]).

We call a connected open subset of $\mathbb{R}^2$ a *region*.

THEOREM 2.1.   *Let T be the attractor of an IFS of injective contractions on $\mathbb{R}^2$ satisfying the open set condition. Suppose that the set T° is nonempty and consists of finitely many components. Then for each $n \geq 2$ and for any tile T_i in a self-similar tiling by homeomorphic copies of T as described above, the set $V_n(T_i)$ is finite.*

Proof. It suffices to show that $V_2(T)$ is finite. Let U be a component of T°. Let α be the boundary of a component of $\overline{U}^c$. Then α is a simple closed curve. It suffices to show that there are finitely many 2-vertices on α.

Suppose that α contains infinitely many 2-vertices. As T has finitely many neighbors, there is an interior component V of a neighbor T_1 such that $\overline{V}$ contains infinitely many distinct 2-vertices x_n, $n \in \mathbb{N}$. Let $v \in V$. By using the local connectedness of ∂V and [W2, Theorem VI.4.2], we can construct simple arcs β_n joining v to x_n such that $\beta_n \setminus \{x_n\} \subseteq V$, $\beta_n \cap \beta_m = \{v\}$ for $m \neq n$. See Figure 2(a).

As $x_n \in V_2(T)$, the vertices x_n also belong to some other neighbor(s) of T. As the total number of interior components of the finitely many neighbors of T is finite, there is one such component W of a neighbor of T, different from T_1, that contains infinitely many of the x_n's. Assume that $x_i \in \overline{W}$, $i = 1, 2, 3$. Let $w \in W$. Again, as ∂W is accessible from all sides [W2, Theorem VI.4.2], there are arcs $\gamma_i \in \overline{W}$ joining x_i and w, with $\gamma_i \cap \gamma_j = \{w\}$ and $\gamma_i \setminus \{x_i\} \subseteq W$. The last condition implies that $\gamma_i \setminus \{x_i\}$ cannot intersect α nor the β_i's. Hence w and the γ_i's cannot be on the same side of α as v and the β_i's. The region on either side of the simple closed curve $\beta_1 \cup \beta_2 \cup \gamma_1 \cup \gamma_2$ contains points in $\alpha \subseteq \partial U$ and hence also points in U. Thus U is not connected. This contradiction proves the theorem. $\qquad\square$

We remark that local connectedness plays an important role in the proof of Theorem 2.1. In fact, [NT2, Example 5.3] shows a tile and a tiling in which the set $V_2(T)$ contains a line segment (denoted by T_1 in that paper) (see Figure 2(b) above). The interior of the tile T is simply-connected, but T is not self-similar and not locally connected.

For $T \subseteq \mathbb{R}^2$, if the interior of T consists of infinitely many components, V_2 can be complicated, even for strictly self-similar tiles. In fact, for the Eisenstein set, V_2 is countably infinite (see [BP]), and we will show in Section 4 that for the Lévy dragon the Hausdorff dimension of V_2 is $1.7724755691\ldots$.

Theorem 2.1 clearly fails in higher dimensions; the unit cube in $\mathbb{R}^3$ serves as a counterexample.

3. Hausdorff dimension of V_n for self-affine tiles

Let A be a $d \times d$ integer expanding matrix with $|\det A| = q$ and let $\mathcal{D} \subseteq \mathbb{Z}^d$ with $\#\mathcal{D} = q$ be a digit set. Throughout this section we assume that $\mathcal{D}$ is a complete set of coset representatives of $\mathbb{Z}^d / A\mathbb{Z}^d$. Then it is well known that the corresponding IFS in (1.2) satisfies the open set condition and the unique compact set T satisfying

$$(3.1) \qquad\qquad A(T) = \bigcup_{d \in \mathcal{D}} (T + d) = T + \mathcal{D}$$

has nonempty interior and is thus a self-affine tile (see [B], [LW]).

Let V_n be the set of n-vertices of T in a tiling of $\mathbb{R}^d$. We will compute the Hausdorff and box dimensions of V_n in the case A is conjugate to a similarity. This is achieved by generalizing a method used by Strichartz and Wang [SW]

to compute the dimensions of V_1. We describe the method and refer the reader to [SW] for more details. For $E \subseteq \mathbb{R}^d$, let $\dim_\mathrm{H}(E)$, $\dim_\mathrm{B}(E)$, $\mathcal{H}^s(E)$ denote the Hausdorff dimension, box dimension, and s-dimensional Hausdorff measure of E, respectively.

We assume for simplicity that the $\mathbb{Z}^d$ translates of T tile $\mathbb{R}^d$. Let $\alpha_0 := 0$ and for $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}^d)^n$, let

$$(3.2) \quad T_{\boldsymbol{\alpha}} = T_{\alpha_1, \ldots, \alpha_n} := \bigcap_{k=0}^{n} (T + \alpha_k) \quad \text{and} \quad \mathcal{F}_n := \{\boldsymbol{\alpha} \in (\mathbb{Z}^d)^n : T_{\boldsymbol{\alpha}} \neq \emptyset\}.$$

$T_{\boldsymbol{\alpha}}$ is the intersection of T with n of its neighboring tiles. It is clear that $V_n = \bigcup_{\boldsymbol{\alpha} \in \mathcal{F}_n} T_{\boldsymbol{\alpha}}$.

The following proposition shows that V_n can be expressed as the attractor of a graph-directed self-affine set (see [MW]).

PROPOSITION 3.1. *For each integer $n \geq 1$, V_n is a graph-directed self-affine set.*

Proof. Using (3.1) and (3.2) we have, for each $\boldsymbol{\alpha} \in \mathcal{F}_n$,

$$A(T_{\boldsymbol{\alpha}}) = \bigcap_{k=0}^{n} (T + \mathcal{D} + A\alpha_k)$$

$$= \bigcup_{d_0, d_1, \ldots, d_n \in \mathcal{D}} \left[T \cap (T + d_1 - d_0 + A\alpha_1) \right.$$

$$\left. \cap \cdots \cap (T + d_n - d_0 + A\alpha_n) \right] + d_0$$

$$= \bigcup_{d_0, d_1, \ldots, d_n \in \mathcal{D}} T_{d_1 - d_0 + A\alpha_1, \ldots, d_n - d_0 + A\alpha_n} + d_0.$$

For any two elements $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ in $\mathcal{F}_n$, define

$$C(\boldsymbol{\alpha}, \boldsymbol{\beta}) := \Big\{ \boldsymbol{d} = (d_0, d_1, \ldots, d_n) \in \mathcal{D}^{n+1} :$$

$$d_i - d_0 + A\alpha_i = \beta_i \text{ for all } i = 1, \ldots, n \Big\}.$$

Then the above equality can be expressed as

$$(3.3) \qquad T_{\boldsymbol{\alpha}} = \bigcup_{d_0, d_1, \ldots, d_n \in \mathcal{D}} A^{-1} \left(T_{d_1 - d_0 + A\alpha_1, \ldots, d_n - d_0 + A\alpha_n} + d_0 \right)$$

$$= \bigcup_{\boldsymbol{\beta} \in \mathcal{F}_n} \bigcup_{\boldsymbol{d} \in C(\boldsymbol{\alpha}, \boldsymbol{\beta})} A^{-1}(T_{\boldsymbol{\beta}} + d_0).$$

Since $V_n = \bigcup_{\boldsymbol{\alpha} \in \mathcal{F}_n} T_{\boldsymbol{\alpha}}$, this proves the proposition. $\qquad \square$

Define

(3.4) $\mathcal{D}_{1,\boldsymbol{\alpha},\boldsymbol{\beta}} := \big\{ d_0 \in \mathcal{D} : (d_0, d_1, \ldots, d_n) \in C(\boldsymbol{\alpha}, \boldsymbol{\beta}) \big\},$

$\mathcal{D}_{N+1,\boldsymbol{\alpha},\boldsymbol{\beta}} := \bigcup_{\boldsymbol{\gamma} \in \mathcal{F}_n} (A \mathcal{D}_{N,\boldsymbol{\alpha},\boldsymbol{\gamma}} + \mathcal{D}_{1,\boldsymbol{\gamma},\boldsymbol{\beta}}), \quad \text{for all } N \geq 1.$

Then (3.3) can be written as

(3.5) $T_{\boldsymbol{\alpha}} = \bigcup_{\boldsymbol{\beta} \in \mathcal{F}_n} A^{-1}(T_{\boldsymbol{\beta}} + \mathcal{D}_{1,\boldsymbol{\alpha},\boldsymbol{\beta}}).$

It follows by induction that for all $N \geq 1$,

(3.6) $T_{\boldsymbol{\alpha}} = \bigcup_{\boldsymbol{\beta} \in \mathcal{F}_n} A^{-N}(T_{\boldsymbol{\beta}} + \mathcal{D}_{N,\boldsymbol{\alpha},\boldsymbol{\beta}}).$

For $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathcal{F}_n$, let $M_{\boldsymbol{\alpha},\boldsymbol{\beta}} := \#C(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and let M_n be the *weighted incidence matrix* (or *substitution matrix*) $(M_{\boldsymbol{\alpha},\boldsymbol{\beta}})_{\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathcal{F}_n}$.

LEMMA 3.2. *Suppose A is conjugate to a similarity with expansion ratio r. Let ρ_n be the spectral radius of M_n. Then*

$$\dim_{\mathrm{H}}(V_n) \leq \overline{\dim}_{\mathrm{B}}(V_n) \leq \frac{\log \rho_n}{\log r}.$$

Proof. Using the fact that $\mathcal{D}$ is a complete set of coset representatives of $\mathbb{Z}^d / A\mathbb{Z}^d$, it can be verified directly that the sets on the right side of (3.4) are disjoint. Let $\mathcal{N}_\epsilon(T_{\boldsymbol{\alpha}})$ be the smallest number of cubes of side length ϵ needed to cover $T_{\boldsymbol{\alpha}}$ and choose $\delta > 0$ so that any $T_{\boldsymbol{\alpha}}$ can be covered by a single cube of side length δ. Then from (3.6),

$$\mathcal{N}_{r^{-N}\delta}(T_{\boldsymbol{\alpha}}) \leq \sum_{\boldsymbol{\beta}} \#\mathcal{D}_{N,\boldsymbol{\alpha},\boldsymbol{\beta}} = \sum_{\boldsymbol{\beta}} (M_n^N)_{\boldsymbol{\alpha},\boldsymbol{\beta}}.$$

Hence,

$$\varlimsup_{N\to\infty} \frac{\log \mathcal{N}_{r^{-N}\delta}(T_{\boldsymbol{\alpha}})}{-\log(r^{-N}\delta)} \leq \varlimsup_{N\to\infty} \frac{\log \sum_{\boldsymbol{\beta}} (M_n^N)_{\boldsymbol{\alpha},\boldsymbol{\beta}}}{N \log r - \log \delta} \leq \frac{\log \rho_n}{\log r}.$$

Since $V_n = \bigcup_{\boldsymbol{\alpha} \in \mathcal{F}_n} T_{\boldsymbol{\alpha}}$, this proves the lemma.   □

THEOREM 3.3. *Let T be a self-affine tile satisfying $AT = T + \mathcal{D}$, where A is a $d \times d$ integer expanding matrix and $\mathcal{D} \subseteq \mathbb{Z}^d$, with $\#\mathcal{D} = |\det A|$, is a complete set of coset representatives of $\mathbb{Z}^d / A\mathbb{Z}^d$. Suppose A is conjugate to a similarity with expansion ratio r and the $\mathbb{Z}^d$ translates of T tile $\mathbb{R}^d$. Let ρ_n be the spectral radius of M_n. Then*

$$\dim_{\mathrm{B}}(V_n) = \dim_{\mathrm{H}}(V_n) = \frac{\log \rho_n}{\log r}.$$

Moreover, $\mathcal{H}^s(V_n) > 0$, where $s = \dim_{\mathrm{H}}(V_n)$.

Proof. The upper bound is established in Lemma 3.2. The lower bound for the Hausdorff dimension can be obtained by using the method in [SW] as follows. Decompose M_n into block upper triangular matrices with diagonal blocks $M_n^{(1)}, \ldots, M_n^{(k)}$ being irreducible and with the spectral radius of $M_n^{(1)}$ being ρ_n. Let $\widehat{\mathcal{F}}_n$ be the set of indices corresponding to $M_n^{(1)}$. Consider the following sub-IFS of the one in (3.5):

$$(3.7) \qquad A(\widehat{T}_{\boldsymbol{\alpha}}) = \bigcup_{\boldsymbol{\beta} \in \widehat{F}_n} (\widehat{T}_{\boldsymbol{\beta}} + \mathcal{D}_{1,\boldsymbol{\alpha},\boldsymbol{\beta}}), \quad \boldsymbol{\alpha} \in \widehat{F}_n.$$

Clearly, $\widehat{T}_{\boldsymbol{\alpha}} \subset T_{\boldsymbol{\alpha}}$. Since $\bigcup_{\boldsymbol{\alpha}} \mathcal{D}_{1,\boldsymbol{\alpha},\boldsymbol{\beta}} \subseteq \mathcal{D}$, we can augment $\mathcal{D}_{1,\boldsymbol{\alpha},\boldsymbol{\beta}}$ to $\widetilde{\mathcal{D}}_{1,\boldsymbol{\alpha},\boldsymbol{\beta}}$ so that $\bigcup_{\boldsymbol{\alpha}} \widetilde{\mathcal{D}}_{1,\boldsymbol{\alpha},\boldsymbol{\beta}} = \mathcal{D}$. Because $M_n^{(1)}$ is irreducible, the new IFS

$$A(\widetilde{T}_{\boldsymbol{\alpha}}) = \bigcup_{\boldsymbol{\beta} \in \widehat{F}_n} (\widetilde{T}_{\boldsymbol{\beta}} + \widetilde{\mathcal{D}}_{1,\boldsymbol{\alpha},\boldsymbol{\beta}}), \quad \boldsymbol{\alpha} \in \widehat{F}_n,$$

defines self-affine multitiles $\{\widetilde{T}_{\boldsymbol{\alpha}}\}$ such that $\widetilde{T}_{\boldsymbol{\alpha}}^{\circ} \neq \emptyset$ (see [FW]). The family $\{\widetilde{T}_{\boldsymbol{\alpha}}^{\circ}\}$ are graph open set condition sets for the IFS (3.7) and hence by a result of Mauldin and Williams in [MW],

$$\dim_{\mathrm{H}}(\widehat{T}_{\boldsymbol{\alpha}}) = s = \frac{\log \rho_n}{\log r}, \quad \text{and} \quad \mathcal{H}^s(\widehat{T}_{\boldsymbol{\alpha}}) > 0.$$

Since $\widehat{T}_{\boldsymbol{\alpha}} \subseteq V_n$, we have completed the proof of the theorem.   □

The number of types of sets $T_{\boldsymbol{\alpha}}$ and thus the size of M_n can often be reduced significantly by taking advantage of symmetry. We illustrate this and Theorem 3.3 by a simple example. Let $A = 2I$, where I is the 3×3 identity matrix, and let

$$\mathcal{D} = \big\{(i, j, k) : i, j, k \in \{0, 1\}\big\} \subseteq \mathbb{Z}^3.$$

Then the attractor T is the unit cube in $\mathbb{R}^3$ with $\mathcal{D}$ being the set of its corners. To compute the dimensions of V_1, we ignore line and point intersections and only consider neighbors of T that intersect T in one of its six faces. That is, we can replace $\mathcal{F}_1$ by

$$\mathcal{F}_1^* = \{(1, 0, 0), \ (0, 1, 0), \ (0, 0, 1), \ (-1, 0, 0), \ (0, -1, 0), (0, 0, -1)\}.$$

Furthermore, we notice that, by symmetry, all the intersections $T_{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} \in \mathcal{F}_1^*$ are equivalent through a rigid motion. Thus, there is only one type of sets $T_{\boldsymbol{\alpha}}$. Upon an iteration of $T_{\boldsymbol{\alpha}}$ by (3.3) or (3.5), we see that $T_{\boldsymbol{\alpha}}$ is a union of four shrunk copies of itself. Therefore $\#C(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = 4$ and thus $\rho_1 = 4$. Theorem 3.3 now implies that $\dim_{\mathrm{B}}(V_1) = \dim_{\mathrm{H}}(V_1) = \log 4/\log 2 = 2$.

To compute the dimensions of V_2, we ignore point intersections and let $T_{\boldsymbol{\alpha}}$ be the intersection of T with at least two of its neighbors. That is, $T_{\boldsymbol{\alpha}}$ is an edge of T. Again, all intersections are equivalent through a rigid motion. Since $T_{\boldsymbol{\alpha}}$ is the union of two shrunk copies of itself, $\rho_2 = \#C(\boldsymbol{\alpha}, \boldsymbol{\alpha}) = 2$

and $\dim_H(V_2) = 1$. Similarly, one can show that $\dim_H(V_3) = 1$ and that $\dim_H(V_n) = 0$ for $n = 4, \ldots, 7$. (For $n \geq 8$, $V_n = \emptyset$.)

Proposition 3.1 and Theorem 3.3 can be generalized to self-affine multitiles. Suppose $T_1, \ldots, T_\ell$ are compact subsets of $\mathbb{R}^d$, each with a nonempty interior, that satisfy

$$AT_j = \bigcup_{k=1}^{\ell} (T_k + \mathcal{D}_{jk}), \quad 1 \leq j \leq \ell,$$

where $\mathcal{D}_{jk} \subseteq \mathbb{Z}^d$. We assume that the $\mathbb{Z}^d$ translates of T_j, $1 \leq j \leq \ell$, tile $\mathbb{R}^d$. It is proved in [FW] that under this assumption, $\mathcal{D}_k := \bigcup_j \mathcal{D}_{jk}$ is a complete set of coset representatives, and the matrix $(\#\mathcal{D}_{jk})$ is primitive. Let $T = \bigcup_{j=1}^{\ell} T_j$ and let V_n be defined with respect to T as in Section 1.

For $\boldsymbol{j} = (j_0, j_1, \ldots, j_n) \in \{1, \ldots, \ell\}^{n+1}$ and $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in (\mathbb{Z}^d)^{n+1}$ with $\alpha_0 := 0$, define

$$T(\boldsymbol{j}; \boldsymbol{\alpha}) := \bigcap_{m=0}^{n} (T_{j_m} + \alpha_m)$$

and let

$$\mathcal{F}_n := \big\{ (\boldsymbol{j}; \boldsymbol{\alpha}) : T(\boldsymbol{j}; \boldsymbol{\alpha}) \neq \emptyset \text{ and if } j_\ell = j_m \text{ then } \alpha_\ell \neq \alpha_m \big\}.$$

Then

$$AT(\boldsymbol{j}; \boldsymbol{\alpha}) = \bigcap_{m=0}^{n} (AT_{j_m} + A\alpha_m)$$

$$= \bigcap_{m=0}^{n} \bigg(\bigcup_{k_m=1}^{\ell} (T_{k_m} + \mathcal{D}_{j_m k_m} + A\alpha_m) \bigg)$$

$$= \bigcap_{m=0}^{n} \bigg(\bigcup_{k_m=1}^{\ell} \bigcup_{d_{j_m k_m} \in \mathcal{D}_{j_m k_m}} (T_{k_m} + d_{j_m k_m} + A\alpha_m) \bigg)$$

$$= \bigcup_{\boldsymbol{k} \in \{1,\ldots,s\}^{n+1}} \bigcup_{\boldsymbol{d} \in \boldsymbol{\mathcal{D}}_n} T(\boldsymbol{k}; A\boldsymbol{\alpha} + \boldsymbol{d} - d_{j_0 k_0}) + d_{j_0 k_0},$$

where

$$\boldsymbol{k} : = (k_0, \ldots, k_n), \; \boldsymbol{d} := (d_{j_0 k_0}, \ldots, d_{j_n k_n}),$$

$$\boldsymbol{\alpha} : = (\alpha_0, \ldots, \alpha_n), \; \boldsymbol{\mathcal{D}}_n := \mathcal{D}_{j_0 k_0} \times \cdots \times \mathcal{D}_{j_n k_n}.$$

Thus, we get the analog of Proposition 3.1. The analog of Theorem 3.3 can be obtained by using the same argument. We remark that the symmetry argument also applies to multitiles.

4. The Lévy dragon

The Lévy dragon (see Figure 3), introduced by Lévy in 1938 (see [Lé], [E]), has been studied extensively, but is still not completely understood. Duvall and Keesling [DK] and Strichartz and Wang [SW] computed the Hausdorff dimension of the boundary (i.e., V_1) of the dragon. Bailey, Kim and Strichartz [BKS] showed that the interior of the dragon consists of at least 16 shapes, and it remains an open question whether these are the only shapes. The authors [NT2] proved that the closure of each component of the interior of the dragon is a topological disk.
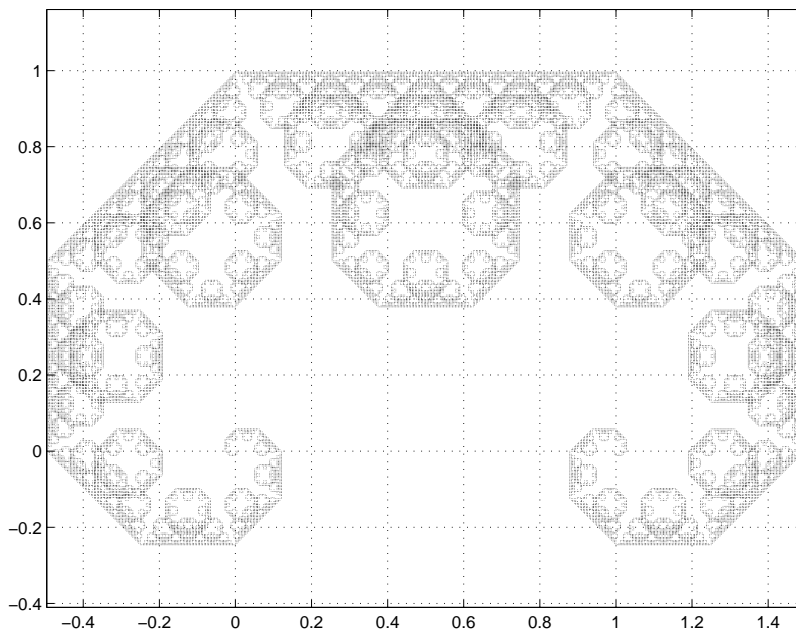

FIGURE 3. The Lévy dragon.

In this section, we will compute the Hausdorff dimension of the set of n-vertices V_n for the Lévy dragon by using the method of Section 3. The Lévy dragon will be denoted by T_0 throughout this section. It is a strictly self-similar tile defined by the similitudes

$$f_1(\mathbf{x}) = \frac{1}{\sqrt{2}} R\left(\frac{\pi}{4}\right) \mathbf{x}, \quad f_2(\mathbf{x}) = \frac{1}{\sqrt{2}} R\left(-\frac{\pi}{4}\right) \mathbf{x} + \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

where $R(\pi/4)$ and $R(-\pi/4)$ are the counterclockwise and clockwise rotations by $\pi/4$, respectively. As is pointed out in [SW], T_0 can be viewed as a part of

a self-affine multitile consisting of 4 prototiles, T_0, T_1, T_2, T_3, where T_j is the rotation of T_0 through the angle $j\pi/2$. Let $A = \left[\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix} \right]$. Then the T_j's satisfy

$$A(T_0) = \big(T_0 + (0,1)\big) \cup \big(T_1 + (-1,0)\big),$$
$$A(T_1) = \big(T_1 + (-1,1)\big) \cup T_2,$$
$$A(T_2) = \big(T_2 + (-1,0)\big) \cup \big(T_3 + (0,1)\big),$$
$$A(T_3) = \big(T_0 + (-1,1)\big) \cup T_3.$$

To compute the dimensions of V_n, we need to first determine the set $\mathcal{F}_n$. We call each element of $\mathcal{F}_n$ a *state*. By symmetry, it suffices to consider the subsystem of the graph-directed system determined by T_0. We still denote the set of all such states by $\mathcal{F}_n$. As explained in [DK] and [SW], tiles in the tiling can intersect only if their convex hulls intersect; moreover, the tiling by $\mathbb{Z}^2$ translates of T_0, T_1, T_2, T_3 can be visualized by considering the tiling by right triangles obtained by drawing the diagonals of each square in the unit square tiling.

For $n = 2$, we can represent each state in $\mathcal{F}_2$ by a triple of isosceles right triangles, with each triangle defining a tile. The first two triangles denote one of the 11 possible nontrivial intersections found in [SW]. The convex hull of the tile defined by the third triangle must intersect the intersection of the convex hulls of the tiles defined by the first two triangles. The states are summarized in Figure 4. In each picture, the black triangles determine two intersecting tiles found in [SW]. Each shaded triangle gives rise to a tile whose convex hull intersects the common intersection of the convex hulls of the tiles generated by the two black triangles. A triangle labeled L (resp., P) generates a tile whose convex hull intersects the tile generated by the black triangles in possibly a line segment (resp., a point). States defining tiles with only point intersections can be excluded from the set $\mathcal{F}_2$ in actual computation. Similarly, we can determine all the possible states in $\mathcal{F}_n$ for $n > 2$.

Next, we determine the matrix M_n. Fix a state $\boldsymbol{\alpha} \in \mathcal{F}_n$. Each triangle defining $\boldsymbol{\alpha}$ generates two smaller triangles, contracted by the factor $1/\sqrt{2}$, rotated by $\pi/4$ or $-\pi/4$, with hypotenuse lying on the right-angled side of the larger triangle, and lying outside of the larger triangle. Pick one small triangle generated by each triangle in $\boldsymbol{\alpha}$ and form a state $\boldsymbol{\gamma}$ and call it a state generated by $\boldsymbol{\alpha}$. If, upon an expansion by the factor $\sqrt{2}$, followed by a possible rigid motion, $\boldsymbol{\gamma}$ equals some $\boldsymbol{\beta} \in \mathcal{F}_n$, then we add 1 to the $(\boldsymbol{\alpha}, \boldsymbol{\beta})$-entry of M_n. Do this for all the states generated by $\boldsymbol{\alpha}$ and each $\boldsymbol{\alpha} \in \mathcal{F}_n$.

Lastly, the dimensions of the set V_n can be computed in terms of the spectral radius ρ_n of M_n. For example, ρ_2 is the largest real zero of an 18th degree polynomial (see Table 1). Since $\rho_2 \approx 1.8483497642\ldots$, we have

$$\dim_{\mathrm{H}}(V_2) = \dim_{\mathrm{B}}(V_2) = \frac{\log \rho_2}{\log \sqrt{2}} \approx 1.7724755691\ldots.$$
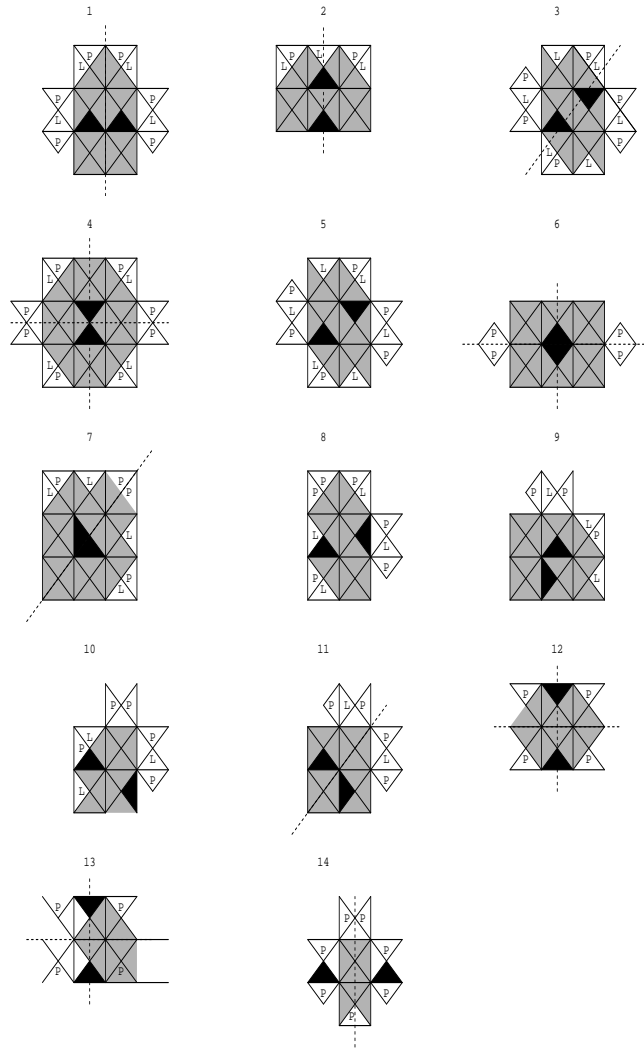
FIGURE 4. Figure showing how all possible states in $\mathcal{F}_2$ are generated.

Table 1 summarizes the results on the dimensions of V_n for the Lévy dragon. The result for $n = 1$ has been obtained in [DK] and [SW] and is included for completeness. We conclude from the results in Table 1 that $\dim_H(V_n) = \dim_B(V_n) = 0$ for all $n \geq 7$.

n	ρ_n	$\dim_{\mathrm{H}}(V_n)$, $\dim_{\mathrm{B}}(V_n)$	Factor of char. poly. determining ρ_n.
1	$1.9547763991\ldots$	$1.9340071829\ldots$	$x^9 - 3x^8 + 3x^7 - 3x^6 + 2x^5 + 4x^4$ $-8x^3 + 8x^2 - 16x + 8$
2	$1.8483497642\ldots$	$1.7724755691\ldots$	$x^{18} - 5x^{17} + 12x^{16} - 21x^{15} + 28x^{14} -$ $25x^{13} + 8x^{12} + 19x^{11} - 53x^{10} + 90x^9 -$ $112x^8 + 118x^7 - 108x^6 + 88x^5$ $-48x^4 + 16x^3 + 8x^2 - 32x + 32$
3	$1.6423396218\ldots$	$1.4315049900\ldots$	$x^{11} - 2x^{10} + 3x^9 - 5x^8 + 6x^7 - 9x^6$ $+10x^5 - 14x^4 + 12x^3 - 14x^2 + 8x - 8$
4	$1.4381104594\ldots$	$1.0483489832\ldots$	$x^{14} - 2x^{10} - 5x^6 - 6x^4 - 4x^2 - 8$
5	$1.3206389565\ldots$	$0.8024722174\ldots$	$x^8 - x^7 + x^6 - 2x^5 + x^4 - x^3 + x^2 - 2$
6	$2^{1/4}$	0.5	$x^4 - 2$
7	1	0	$x - 1$

TABLE 1. Dimensions of V_n for the Lévy dragon, together with the spectral radius of the weighted incidence matrix M_n and the factor of the characteristic polynomial of M_n for which ρ_n is a maximal root.

References

[AT1]   S. Akiyama and J. M. Thuswaldner, *A survey on topological properties of tiles related to number systems*, Geom. Dedicata **109** (2004), 89–105. MR 2113188 (2005h:37035)

[AT2]   _____ , *The topological structure of fractal tilings generated by quadratic number systems*, Comput. Math. Appl. **49** (2005), 1439–1485. MR 2149493

[AT3]   _____ , *Topological properties of two-dimensional number systems*, J. Théor. Nombres Bordeaux **12** (2000), 69–79. MR 1827838 (2002g:11013)

[BKS]   S. Bailey, T. Kim, and R. S. Strichartz, *Inside the Lévy dragon*, Amer. Math. Monthly **109** (2002), 689–703. MR 1927621 (2003f:28011)

[B]   C. Bandt, *Self-similar sets. V. Integer matrices and fractal tilings of $\mathbf{R}^n$*, Proc. Amer. Math. Soc. **112** (1991), 549–562. MR 1036982 (92d:58093)

[BW]   C. Bandt and Y. Wang, *Disk-like self-affine tiles in* $\mathbb{R}^2$, Discrete Comput. Geom.
        26 (2001), 591–601. MR 1863811 (2002h:52028)

[BP]   A. Benedek and R. Panzone, *Un sistema numérico de base real para los números
        complejos*, Notas de Álgebra y Análisis [Notes on Algebra and Analysis], 18, Univer-
        sidad Nacional del Sur Instituto de Matemática, Bahía Blanca, 2000. MR 1799736
        (2001h:11102)

[DPPS] S. Drenning, J. Palagallo, T. Price, and R. S. Strichartz, *Outer boundaries of
        self-similar tiles*, Experiment. Math. **14** (2005), 199–209. MR 2169523

[DK]   P. Duvall and J. Keesling, *The Hausdorff dimension of the boundary of the Lévy
        dragon*, Geometry and topology in dynamics (Winston-Salem, NC, 1998/San An-
        tonio, TX, 1999), Contemp. Math., vol. 246, Amer. Math. Soc., Providence, RI,
        1999, pp. 87–97. MR 1732373 (2000m:54040)

[E]   G. A. Edgar, ed., *Classics on fractals*, Westview Press, Boulder, CO, 2004.
        MR 2049443 (2005f:28002)

[FW]   T. Flaherty and Y. Wang, *Haar-type multiwavelet bases and self-affine multi-tiles*,
        Asian J. Math. **3** (1999), 387–400. MR 1796511 (2001j:42030)

[G1]   W. J. Gilbert, *Complex numbers with three radix expansions*, Canad. J. Math. **34**
        (1982), 1335–1348. MR 678674 (85c:11013)

[G2]   _____, *Complex bases and fractal similarity*, Ann. Sci. Math. Québec **11** (1987),
        65–77. MR 912163 (89a:11017)

[KS]   I. Kátai and J. Szabó, *Canonical number systems for complex integers*, Acta Sci.
        Math. (Szeged) **37** (1975), 255–260. MR 0389759 (52 #10590)

[KL]   I. Kirat and K.-S. Lau, *On the connectedness of self-affine tiles*, J. London Math.
        Soc. (2) **62** (2000), 291–304. MR 1772188 (2001i:52027)

[LW]   J. C. Lagarias and Y. Wang, *Self-affine tiles in* $\mathbf{R}^n$, Adv. Math. **121** (1996), 21–49.
        MR 1399601 (97d:52034)

[L]   K.-S. Leung, *The radix expansions and the disklikeness of self-affine tiles*, Ph.D.
        thesis, The Chinese University of Hong Kong, Hong Kong, 2004.

[Lé]   P. Lévy, *Les courbes planes ou gauches et les surfaces composée de parties sem-
        blables au tout*, Journal de l'École Polytechnique (1938), 227–247, 249–291.

[LAT]   J. Luo, S. Akiyama, and J. M. Thuswaldner, *On the boundary connectedness of con-
        nected tiles*, Math. Proc. Cambridge Philos. Soc. **137** (2004), 397–410. MR 2092067
        (2005g:37032)

[LRT]   J. Luo, H. Rao, and B. Tan, *Topological structure of self-similar sets*, Fractals **10**
        (2002), 223–227. MR 1910665 (2003d:28014)

[MW]   R. D. Mauldin and S. C. Williams, *Hausdorff dimension in graph directed con-
        structions*, Trans. Amer. Math. Soc. **309** (1988), 811–829. MR 961615 (89i:28003)

[MTT]   W. Müller, J. M. Thuswaldner, and R. F. Tichy, *Fractal properties of number
        systems*, Period. Math. Hungar. **42** (2001), 51–68. MR 1832694 (2002k:11130)

[NN]   S.-M. Ngai and N. Nguyen, *The Heighway dragon revisited*, Discrete Comput.
        Geom. **29** (2003), 603–623. MR 1976609 (2004d:28023)

[NT1]   S.-M. Ngai and T.-M. Tang, *A technique in the topology of connected self-similar
        tiles*, Fractals **12** (2004), 389–403. MR 2109984 (2006b:52018)

[NT2]   _____, *Topology of connected self-similar tiles in the plane with disconnected
        interiors*, Topology Appl. **150** (2005), 139–155. MR 2133675 (2006b:52019)

[ST]   K. Scheicher and J. M. Thuswaldner, *Canonical number systems, counting au-
        tomata and fractals*, Math. Proc. Cambridge Philos. Soc. **133** (2002), 163–182.
        MR 1900260 (2003b:11070)

[SW]   R. S. Strichartz and Y. Wang, *Geometry of self-affine tiles. I*, Indiana Univ. Math.
        J. **48** (1999), 1–23. MR 1722192 (2000k:52017)

[T]   J. M. Thuswaldner, *Fractals and number systems in real quadratic number fields*, Acta Math. Hungar. **90** (2001), 253–269. MR 1910772 (2003c:11091)

[W1]   G. T. Whyburn, *Topological analysis*, Princeton Mathematical Series. No. 23, Princeton University Press, Princeton, N. J., 1958. MR 0099642 (20 #6081)

[W2]   ———, *Analytic Topology*, American Mathematical Society Colloquium Publications, v. 28, American Mathematical Society, New York, 1942. MR 0007095 (4,86b)

D.-W. Deng, School of Mathematics and Computational Science, Xiangtan University, Hunan 411105, PR China

E-mail address: `tmtang@xtu.edu.cn`

S.-M. Ngai, Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA 30460-8093, USA

E-mail address: `ngai@gsu.mat.georgiasouthern.edu`