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# FLUX FOR BRYANT SURFACES AND APPLICATIONS TO EMBEDDED ENDS OF FINITE TOTAL CURVATURE

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ABSTRACT. We compute the flux of Killing fields through ends of constant mean curvature 1 in hyperbolic space, and we prove a result conjectured by Rossman, Umehara and Yamada: the flux matrix defined by these authors is equivalent to the flux of Killing fields. We next give a geometric description of embedded ends of finite total curvature. In particular, we show that if such an end is asymptotic to a catenoid cousin, then we can associate an axis to it. We also compute the flux of Killing fields through these ends, and we deduce some geometric properties and analogies to minimal surfaces in Euclidean space.

## 1. Introduction

Bryant surfaces are surfaces with constant mean curvature one in hyperbolic 3-space $\mathbb{H}^3$ (with the convention that the mean curvature of a surface is one half of the trace of its second fundamental form). These surfaces were first studied by Bryant [Bry87], who derived a representation in terms of holomorphic data, analogous to the Weierstrass data for minimal surfaces in $\mathbb{R}^3$.

Umehara and Yamada [UY93] defined the notion of regular ends of Bryant surfaces: These are ends that are conformally parametrized by the punctured complex disk and such that the hyperbolic Gauss map extends meromorphically to the puncture. (If the hyperbolic Gauss map has an essential singularity at the puncture, the end is said to be irregular.) Umehara and Yamada also studied the Weierstrass data of Bryant surface ends of finite total curvature.

Collin, Hauswirth and Rosenberg [CHR01] showed that properly embedded annular ends have finite total curvature and are regular. Yu [Yu01] proved that irregular ends are never embedded. Sá Earp and Toubiana [SET01] studied the geometry of embedded ends that are of finite total curvature (and hence regular). They showed that, in the upper half-space model of $\mathbb{H}^3$, such ends are, up to an isometry of $\mathbb{H}^3$, vertical Euclidean graphs and are asymptotic to

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a catenoid cousin of revolution or a horosphere as vertical Euclidean graphs. They also defined the growth of such ends. If $E$ is a half-catenoid cousin whose asymptotic boundary is $\infty$, then the image of $E$ by a Euclidean horizontal translation (which is a parabolic isometry of $\mathbb{H}^3$) is asymptotic to $E$ in the sense of Sá Earp and Toubiana (see Figure 1).

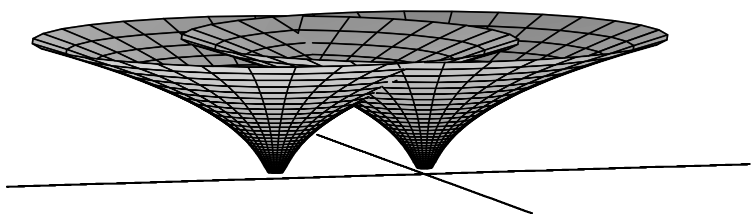

FIGURE 1. Two half-catenoid cousins asymptotic in the sense of Sá Earp and Toubiana but with different axes.

There exist two notions of flux for Bryant surfaces. The first flux is the flux of Killing fields. This flux was introduced by Korevaar, Kusner, Meeks and Solomon [KKMS92] as an analogue of the flux defined by Korevaar, Kusner and Solomon for constant mean curvature surfaces in $\mathbb{R}^3$ [KKS89]. It is the sum of an integral along a curve $\Gamma$ and an integral over a compact surface whose boundary is $\Gamma$. This flux is a homology invariant. The second flux is the residue-type flux matrix defined by Rossman, Umehara and Yamada [RUY99]. This flux can be easily computed from the Bryant representation of the surface. It is also a homology invariant. Rossman, Umehara and Yamada conjectured that these two notions of flux were equivalent.

In this paper, we prove this conjecture. We compute the flux of Killing fields associated to translations and rotations through Bryant surface ends. We show that it depends only on the residues of three meromorphic one-forms (Theorems 3.12 and 3.14). These residues are, up to constant factors, the coefficients of the flux matrix defined by Rossman, Umehara and Yamada. Moreover, we define a complex polynomial of degree at most two, called flux polynomial, whose coefficients are these residues (Theorem 3.15). This polynomial contains all the information given by the flux and satisfies a "balancing formula".

The second aim of this paper is to complete the geometric study of embedded ends of finite total curvature started in [SET01]. We show that if such an end is asymptotic to a catenoid cousin, then we can associate an axis to it (Theorem 4.5). This means that these ends are asymptotically surfaces of revolution. We call these ends catenoidal ends. An analogous result for embedded ends of finite total curvature of minimal surfaces in $\mathbb{R}^3$ was obtained by Schoen [Sch83].

We next compute the flux for embedded ends of finite total curvature. We obtain that the flux of the Killing field associated to the translation along the geodesic $(\mathcal{C}, \mathcal{D})$ through a catenoidal end is

$$\varphi = \pi(1-\mu^2)(2\,\mathrm{Re}(\mathcal{A}, \mathcal{C}, \mathcal{D}, \mathcal{B}) - 1),$$

where $(\mathcal{A}, \mathcal{B})$ is the axis of the end, $1-\mu$ its growth, and where $(\mathcal{A}, \mathcal{C}, \mathcal{D}, \mathcal{B})$ denotes the cross-ratio (Theorem 5.2). This formula is one of the simplest we could expect, since it depends only on the asymptotic behaviour of the end. We also show that the flux for a horospherical end is zero if and only if its Hopf differential is regular at the end (Theorem 5.4).

Thus, the flux for Bryant surfaces plays the same role as the flux and the torque for minimal surfaces in $\mathbb{R}^3$. (The torque is defined in [KK93]; see also [HK97] for definitions and basic properties of the flux and the torque.) Indeed, the flux and the torque for a catenoidal end depend only on the growth and the axis of the end, and the torque for a planar end (the analogue of a horospherical end) is zero if and only if the Hopf differential is regular at the end, i.e., the degree of the Gauss map at the end is at least 3 (see [Rom97]).

Finally, we give some geometric applications of the flux. If a Bryant surface has exactly two catenoidal ends (and no others) with distinct asymptotic boundaries, then the ends have the same growth and the same axis (Proposition 6.5). If a Bryant surface has exactly three catenoidal ends (and no others) with distinct asymptotic boundaries, then the axes are coplanar and concurrent (possibly in the asymptotic boundary of $\mathbb{H}^3$) (Proposition 6.9). The same results hold for minimal surfaces in $\mathbb{R}^3$.

## 2. Preliminaries and notations

In this paper, the model used for hyperbolic 3-space is the upper half-space model:

$$\mathbb{H}^3 = \{(u, v, w) \in \mathbb{R}^3 : w > 0\} = \{(\zeta, w) \in \mathbb{C} \times \mathbb{R} : w > 0\}$$

with the metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}u^2 + \mathrm{d}v^2 + \mathrm{d}w^2}{w^2} = \frac{|\mathrm{d}\zeta|^2 + \mathrm{d}w^2}{w^2}.$$

The symbols $\langle , \rangle$ and $\|.\|$ denote, respectively, the hyperbolic metric and the hyperbolic norm on $\mathbb{H}^3$. If $X_1 = (\alpha_1, \beta_1)$ and $X_2 = (\alpha_2, \beta_2)$ are two vectors in the tangent space of $\mathbb{H}^3$ at the point $(\zeta, w)$, then $\langle X_1, X_2\rangle = (\mathrm{Re}(\bar{\alpha}_1\alpha_2) + \beta_1\beta_2)/w^2$.

In the model of the unit ball of $\mathbb{R}^3$ for hyperbolic space, the asymptotic boundary of hyperbolic space is the sphere of radius 1. In the half-space model, we identify the asymptotic boundary of $\mathbb{H}^3$ with the Riemann sphere $\bar{\mathbb{C}}$ composed of the plane $\{\zeta = 0\}$ and of the point at infinity which we denote $\infty$.

The asymptotic boundary of a part of $\mathbb{H}^3$ is the set of its accumulation points in $\bar{\mathbb{C}}$.

The identification between the upper half-space model and the Minkowski model for $\mathbb{H}^3$ is the same as that described in [SET01, Remark 1.11]. Consequently, if $f$ is a constant mean curvature one immersion of a Riemann surface $M$ into the Minkowski model of the hyperbolic space, and $F = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is its Bryant representation (see [Bry87]), then we have $f = FF^*$, and the corresponding immersion $X = (\zeta, w) : M \to \mathbb{H}^3$ in the upper half-space model is given by

$$\zeta = \frac{\bar{A}C + \bar{B}D}{|A|^2 + |B|^2} \tag{2.1}$$

and

$$w = \frac{1}{|A|^2 + |B|^2}. \tag{2.2}$$

We recall that $A$, $B$, $C$ and $D$ are holomorphic functions defined on the universal cover of $M$ and satisfying $AD - BC = 1$ and $\mathrm{d}A\mathrm{d}D - \mathrm{d}B\mathrm{d}C = 0$.

If $(g, \omega)$ denote the Weierstrass data of the end (see [Bry87] or [UY93]), the 2-form $\omega\mathrm{d}g$ is called the Hopf differential of the end. It is single-valued on $M$ (in contrast to $g$ and $\omega$). It is invariant by an isometry of $\mathbb{H}^3$.

The hyperbolic Gauss map is given by $G = \mathrm{d}C/\mathrm{d}A = \mathrm{d}D/\mathrm{d}B$. It is single-valued on $M$. This expression differs slightly from that of [Bry87], [UY93] and other papers because of the chosen identification (see [SET01, Remark 1.11]). The one-form $\omega^\# = -\omega\mathrm{d}g/\mathrm{d}(1/G)$ is also single-valued on $M$. (The pair $(1/G, \omega^\#)$ gives the Weierstrass data of the dual immersion; see [UY97].) Hence the following one-forms are single-valued on $M$:

$$\begin{aligned} B\mathrm{d}A - A\mathrm{d}B &= -\frac{\omega^\#}{G^2} = \frac{\omega\mathrm{d}g}{\mathrm{d}G}, \\ C\mathrm{d}B - B\mathrm{d}A &= \frac{\omega^\#}{G} = -G\frac{\omega\mathrm{d}g}{\mathrm{d}G}, \\ D\mathrm{d}C - C\mathrm{d}D &= -\omega^\# = G^2\frac{\omega\mathrm{d}g}{\mathrm{d}G}. \end{aligned}$$

For regular ends of finite total curvature, the Hopf differential $\omega\mathrm{d}g$ has a pole of order greater than or equal to $-2$ at zero (see [UY93]). Its order does not depend on the parametrization.

In the Minkowski model, a direct isometry is a map $N \mapsto PNP^*$, where $P = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$. In the half-space model, this isometry induces on $\bar{\mathbb{C}}$ the map $\zeta \mapsto \frac{\delta\zeta+\gamma}{\beta\zeta+\alpha}$ because of the chosen identification.

If $\mathcal{A}$ and $\mathcal{B}$ are two distinct points in $\bar{\mathbb{C}}$, $(\mathcal{A}, \mathcal{B})$ denotes the oriented geodesic of $\mathbb{H}^3$ going from $\mathcal{A}$ to $\mathcal{B}$.

If $z_1$, $z_2$, $z_3$ and $z_4$ are four points in $\bar{\mathbb{C}}$ such that $z_1 \neq z_4$ and $z_2 \neq z_3$, we define their cross-ratio by

$$(z_1, z_2, z_3, z_4) = \frac{z_3 - z_1}{z_3 - z_2} \cdot \frac{z_4 - z_2}{z_4 - z_1}.$$

We recall that there exists a direct isometry (respectively an indirect isometry) of $\mathbb{H}^3$ which maps $z_1$, $z_2$, $z_3$ and $z_4$ to $z_1'$, $z_2'$, $z_3'$ and $z_4'$, respectively, where $z_1' \neq z_4'$ and $z_2' \neq z_3'$, if and only if $(z_1, z_2, z_3, z_4) = (z_1', z_2', z_3', z_4')$ (respectively $(z_1, z_2, z_3, z_4) = \overline{(z_1', z_2', z_3', z_4')}$).

In this paper, $\Omega$ will denote any neighbourhood of 0 in $\mathbb{C}$, and $\Omega^*$ will denote the set $\Omega \setminus \{0\}$.

The flux of a Killing field $Y$ through an annular Bryant surface end $E$ is defined by

$$\varphi = \int_\Gamma \langle \eta, Y \rangle - 2 \int_K \langle \nu, Y \rangle,$$

where $\Gamma$ is a generator of $\pi_1(E)$, $K$ a topological disk whose boundary is $\Gamma$, $\eta$ the conormal to $\Gamma$ in the direction of the asymptotic boundary of the end and $\nu$ the normal to $K$ chosen as follows: If $\vec{H}$ denotes the mean curvature vector of the end, we choose on $\Gamma$ the orientation such that $(\Gamma, \eta, -\vec{H})$ is the orientation of $\mathbb{H}^3$ and $\nu$ such that it induces the same orientation on $\Gamma$. The normal $\nu$ induces an orientation on $K$ and $\Gamma$. These choices have been made in order to be compatible with Stokes' formula.

This number $\varphi$ does not depend on the choices of $\Gamma$ and $K$ (see [KKS89] and [KKMS92]). We note that in [KKS89] and [KKMS92] the mean curvature is defined as the trace of the second fundamental form (and not as half of this trace), which explains the coefficient 2 in the formula.

If $\alpha$ is an $n$-form on $\mathbb{H}^3$ and $X$ a vector field, then the interior product of $\alpha$ by $X$ is denoted by $\mathrm{i}_X\alpha$ and defined by $\mathrm{i}_X\alpha(\xi_1, \ldots, \xi_{n-1}) = \alpha(X, \xi_1, \ldots, \xi_{n-1})$. The Lie derivative of $\alpha$ with respect to $X$ is denoted by $\mathrm{L}_X\alpha$. We recall Cartan's formula: $\mathrm{L}_X\alpha = \mathrm{d}(\mathrm{i}_X\alpha) + \mathrm{i}_X\mathrm{d}\alpha$.

## 3. Flux of Killing fields

### 3.1. Killing fields associated to translations.

Definition 3.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two distinct points in $\bar{\mathbb{C}}$. Let $\Phi_t$ be the translation of distance $t \in \mathbb{R}$ along the geodesic $(\mathcal{A}, \mathcal{B})$. Then the vector field $Y$ defined by

$$\frac{\mathrm{d}\Phi_t}{\mathrm{d}t} = Y(\Phi_t)$$

is called the Killing field associated to the translation along $(\mathcal{A}, \mathcal{B})$.

The Killing field associated to the translation along $(\mathcal{B}, \mathcal{A})$ is the opposite of the Killing field associated to the translation along $(\mathcal{A}, \mathcal{B})$. Elementary computations give the following lemma.

Lemma 3.2. *The Killing field associated to the translation along* $(0,\infty)$ *is*

$$Y(\zeta,w)=(\zeta,w).$$

Lemma 3.3. *Let* $\zeta_0\in\mathbb{C}^*$. *The Killing field associated to the translation along* $(\zeta_0,0)$ *is*

$$Y(\zeta,w)=\begin{pmatrix}-\frac{w^2}{\zeta_0}+\frac{\zeta^2}{\zeta_0}-\zeta\\ 2w\operatorname{Re}\frac{\zeta}{\zeta_0}-w\end{pmatrix}.$$

*Proof.* The map

$$\Phi:(u',v',w')\mapsto\frac{1}{u'^2+v'^2+w'^2}(u',v',w')$$

is an isometry of $\mathbb{H}^3$ which maps the geodesic $(\zeta_0',\infty)$ (where $\zeta_0'=\zeta_0/|\zeta_0|^2$) to the geodesic $(\zeta_0,0)$. Hence the Killing field associated to the translation along $(\zeta_0,0)$ is given by $Y(P)=\Phi_*Z(P)=\mathrm{d}_{\Phi^{-1}(P)}\Phi\cdot Z(\Phi^{-1}(P))$ for each $P=(u,v,w)=(\zeta,w)\in\mathbb{H}^3$, where $Z$ is the Killing field associated to the translation along $(\zeta_0',\infty)$.

We have $\Phi^{-1}(P)=(u',v',w')$, where $u=ru'$, $v=rv'$, $w=rw'$ and $r=u^2+v^2+w^2$. Hence we have $Z(\Phi^{-1}(P))=(u'-u_0',v'-v_0',w')$ with $\zeta_0'=u_0'+iv_0'$, so

$$\begin{aligned}
Y(P)&=\frac{1}{(u'^2+v'^2+w'^2)^2}\\
&\quad\times\begin{pmatrix}v'^2+w'^2-u'^2 & -2u'v' & -2u'w'\\ -2u'v' & u'^2+w'^2-v'^2 & -2v'w'\\ -2u'w' & -2v'w' & u'^2+v'^2-w'^2\end{pmatrix}\\
&\quad\times\begin{pmatrix}u'-u_0'\\ v'-v_0'\\ w'\end{pmatrix}\\
&=r^2\begin{pmatrix}-u_0'(v'^2+w'^2-u'^2)+2v_0'u'v'-u'v'^2-u'w'^2-u'^3\\ 2u_0'u'v'-v_0'(u'^2+w'^2-v'^2)-u'^2v'-v'w'^2-v'^3\\ 2u_0'u'w'+2v_0'v'w'-u'^2w'-v'^2w'-w'^3\end{pmatrix}\\
&=\begin{pmatrix}u_0'(u^2-v^2-w^2)+2v_0'uv-u\\ 2u_0'uv+v_0'(v^2-u^2-w^2)-v\\ 2u_0'uw+2v_0'vw-w\end{pmatrix}=\begin{pmatrix}-\frac{w^2}{\zeta_0}+\frac{\zeta^2}{\zeta_0}-\zeta\\ 2w\operatorname{Re}\frac{\zeta}{\zeta_0}-w\end{pmatrix}.
\end{aligned}$$

□

### 3.2. Killing fields associated to rotations.

Definition 3.4. Let $\mathcal{A}$ and $\mathcal{B}$ be two distinct points in $\bar{\mathbb{C}}$. Let $R_\theta$ be the rotation of angle $\theta$ (in the direct sense) about the geodesic $(\mathcal{A},\mathcal{B})$. Then the vector field $Y$ defined by

$$\frac{\mathrm{d}R_\theta}{\mathrm{d}\theta}=Y(R_\theta)$$

is called the Killing field associated to the rotation about $(\mathcal{A}, \mathcal{B})$.

The Killing field associated to the rotation about $(\mathcal{B}, \mathcal{A})$ is the opposite of the Killing field associated to the rotation about $(\mathcal{A}, \mathcal{B})$. Elementary computations give the following lemma.

LEMMA 3.5. *The Killing field associated to the rotation about* $(0, \infty)$ *is*

$$Y(\zeta, w) = (i\zeta, 0).$$

LEMMA 3.6. *Let* $\zeta_0 \in \mathbb{C}^*$. *The Killing field associated to the rotation about* $(\zeta_0, 0)$ *is*

$$Y(\zeta, w) = \begin{pmatrix} i\frac{w^2}{\zeta_0} + i\frac{\zeta^2}{\zeta_0} - i\zeta \\ -2w \operatorname{Im} \frac{\zeta}{\zeta_0} \end{pmatrix}.$$

*Proof.* We proceed as for Lemma 3.3 and we use the same notations. Since the map $\Phi$ is an indirect isometry of $\mathbb{H}^3$, we have $Y = -\Phi_* Z$, where $Z$ is the Killing field associated to the rotation about $(\zeta_0', \infty)$. □

**3.3. Flux of Killing fields associated to translations.** In this section, $\zeta_0$ and $\zeta_1$ are two complex numbers such that $\zeta_0 \neq 0$, and $E$ denotes a Bryant surface end whose Bryant representation is

$$F = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \Omega^* \to \mathrm{SL}_2(\mathbb{C}).$$

We denote by $X = (\zeta, w) : \Omega^* \to \mathbb{H}^3$ the corresponding conformal immersion in the upper half-space model.

We will denote by $(\rho, \tau)$ the polar coordinates in $\Omega$, i.e., $z = \rho e^{i\tau}$. We have the following relationships for derivation operators:

$$\frac{\partial}{\partial z} = \frac{1}{2z}\left(\rho\frac{\partial}{\partial \rho} - i\frac{\partial}{\partial \tau}\right),$$
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2\bar{z}}\left(\rho\frac{\partial}{\partial \rho} + i\frac{\partial}{\partial \tau}\right).$$

LEMMA 3.7. *Let* $Y$ *be the Killing field associated to the translation along the geodesic* $(\zeta_0 + \zeta_1, \zeta_1)$. *Then if* $\rho$ *is a sufficiently small positive number, the flux of* $Y$ *through* $E$ *is*

$$\varphi = \int_0^{2\pi} \operatorname{Re}(s(\rho, \tau))\mathrm{d}\tau,$$

*where*

$$s(\rho,\tau) = \frac{\zeta - \zeta_1}{w^2}\overline{\left(\rho\frac{\partial\zeta}{\partial\rho} + i\frac{\partial\zeta}{\partial\tau}\right)}\left(1 - \frac{\zeta - \zeta_1}{\zeta_0}\right) + \frac{\rho}{\zeta_0}\frac{\partial\zeta}{\partial\rho} - \frac{2i}{\zeta_0}\frac{\partial\zeta}{\partial\tau}\ln w - \frac{\rho}{w}\frac{\partial w}{\partial\rho}\left(2\frac{\zeta - \zeta_1}{\zeta_0} - 1\right).$$

*Proof.* Let $\rho > 0$ be such that the circle $\{z \in \mathbb{C} : |z| = \rho\}$ is contained in $\Omega$. Let $\Gamma$ be the curve on $E$ defined by $\tau \mapsto X(\rho e^{i\tau})$. Let $K$ be a disk whose boundary is $\Gamma$.

We remark that we must take on $\Gamma$ the orientation given by $-\Gamma'$. Indeed, because of the conventions for the sign of the mean curvature, positive mean curvature means that the orientation induced by the immersion $X$ is the same as the orientation induced by the mean curvature vector $\vec{H}$; consequently, the basis $(\eta, \Gamma', \vec{H})$ is indirect. We denote by $\nu$ and $\eta$ the normal to $K$ and the conormal to $\Gamma$, chosen as explained in Section 2.

The conormal $\eta$ to $\Gamma$ is a unit vector lying in the tangent plane and normal to $\Gamma'(\tau) = \frac{\partial}{\partial \tau} X(\rho e^{i\tau})$. Since the parametrization $X$ is conformal, the conormal $\eta$ is necessarily colinear to $\frac{\partial}{\partial \rho} X(\rho e^{i\tau})$. Since $\eta$ must point in the direction of $0 \in \mathbb{C}$, we have

$$\eta = -\frac{\frac{\partial}{\partial \rho} X(\rho e^{i\tau})}{\|\frac{\partial}{\partial \rho} X(\rho e^{i\tau})\|}.$$

Then we have

$$\begin{aligned}\int_\Gamma \langle \eta, Y\rangle &= \int_0^{2\pi} \langle \eta, Y\rangle \left\| \frac{\partial}{\partial \tau} X \right\| \mathrm{d}\tau \\ &= \int_0^{2\pi} \langle \eta, Y\rangle \left\| \frac{\partial}{\partial \rho} X \right\| \rho \mathrm{d}\tau \\ &= \int_0^{2\pi} -\rho \left\langle \frac{\partial}{\partial \rho} X, Y \right\rangle \mathrm{d}\tau.\end{aligned}$$

According to Lemma 3.3, we have

$$Y(\zeta, w) = \begin{pmatrix} -\frac{w^2}{\bar{\zeta}_0} + \frac{(\zeta-\zeta_1)^2}{\zeta_0} - (\zeta - \zeta_1) \\ 2w \operatorname{Re} \frac{\zeta-\zeta_1}{\zeta_0} - w \end{pmatrix}.$$

Consequently,

$$\int_\Gamma \langle \eta, Y\rangle = -\int_0^{2\pi} \operatorname{Re}(s_1(\rho, \tau)) \mathrm{d}\tau$$

with

$$s_1(\rho, \tau) = \frac{\rho}{w^2} \overline{\frac{\partial \zeta}{\partial \rho}} \left( -\frac{w^2}{\bar{\zeta}_0} + \frac{(\zeta - \zeta_1)^2}{\zeta_0} - (\zeta - \zeta_1) \right) + \frac{\rho}{w} \frac{\partial w}{\partial \rho} \left( 2 \frac{\zeta - \zeta_1}{\zeta_0} - 1 \right).$$

Let $\alpha$ be the canonical volume form of $\mathbb{H}^3$. We have $\alpha = (1/w^3) \mathrm{d}u \wedge \mathrm{d}v \wedge \mathrm{d}w$. Since $Y$ is a Killing field, we have $\mathrm{L}_Y\, \alpha = 0$, so $0 = \mathrm{d}(\mathrm{i}_Y \alpha) + \mathrm{i}_Y \mathrm{d}\alpha = \mathrm{d}(\mathrm{i}_Y \alpha)$. Hence there exists a 1-form $\beta$ such that $\mathrm{i}_Y \alpha = \mathrm{d}\beta$. The form $\beta$ is the dual form of a vector field $Z$, i.e., we have $\beta(\xi) = \langle Z, \xi \rangle$ for all vector fields $\xi$. We compute that we can take

$$Z(\zeta, w) = \begin{pmatrix} i \frac{w^2}{\bar{\zeta}_0} \ln w + \frac{i}{2} \frac{(\zeta-\zeta_1)^2}{\zeta_0} - \frac{i}{2}(\zeta - \zeta_1) \\ 0 \end{pmatrix}.$$

Let $(e_1, e_2)$ be an orthonormal basis of the tangent space of $K$ such that the basis $(e_1, e_2, \nu)$ is direct. Then we have $\mathrm{i}_Y\alpha(e_1, e_2) = \alpha(e_1, e_2, Y) = \langle \nu, Y\rangle$. Consequently, on $K$ the form $\mathrm{i}_Y\alpha$ is equal to $\langle \nu, Y\rangle$ times the canonical volume form of $K$. Hence we have

$$\int_K \langle \nu, Y\rangle = \int_K \mathrm{i}_Y\alpha.$$

On the other hand, Stokes' formula implies that

$$\int_K \mathrm{i}_Y\alpha = -\int_\Gamma \beta,$$

since we must take on $\Gamma$ the orientation given by $-\Gamma'$, as explained before.

Consequently we have

$$\int_K \langle \nu, Y\rangle = -\int_\Gamma \beta = -\int_0^{2\pi} \left\langle \frac{\partial}{\partial \tau} X, Z\right\rangle \mathrm{d}\tau = -\int_0^{2\pi} \mathrm{Re}(s_2(\rho, \tau))\mathrm{d}\tau$$

with

$$s_2(\rho,\tau) = \frac{1}{w^2}\overline{\frac{\partial \zeta}{\partial \tau}}\left(i\frac{w^2}{\bar\zeta_0}\ln w + \frac{i}{2}\frac{(\zeta-\zeta_1)^2}{\zeta_0} - \frac{i}{2}(\zeta-\zeta_1)\right).$$

So

$$\varphi = \int_\Gamma \langle \eta, Y\rangle - 2\int_K \langle \nu, Y\rangle = \int_0^{2\pi} \mathrm{Re}(-s_1(\rho,\tau) + 2s_2(\rho,\tau))\mathrm{d}\tau.$$

Since the real part does not change if we replace the first terms of $s_1$ and $s_2$ by their conjugates, we obtain the expected result. □

Lemma 3.8. *We have the following identities:*

$$\frac{1}{w^2}\overline{\frac{\partial\zeta}{\partial\bar z}} = AB' - A'B, \tag{3.1}$$

$$\frac{\rho}{w}\frac{\partial w}{\partial\rho} = -z\frac{A'\bar A + B'\bar B}{|A|^2+|B|^2} - \bar z\frac{A\bar A' + B\bar B'}{|A|^2+|B|^2}, \tag{3.2}$$

$$\frac{\partial}{\partial\tau}\ln w = -iz\frac{A'\bar A + B'\bar B}{|A|^2+|B|^2} + i\bar z\frac{A\bar A' + B\bar B'}{|A|^2+|B|^2}. \tag{3.3}$$

*Proof.* Recall that

$$\zeta = \frac{\bar A C + \bar B D}{|A|^2+|B|^2},$$

where $A$, $B$, $C$ and $D$ are multivaluated holomorphic functions. We compute that

$$\begin{aligned}\frac{\partial\zeta}{\partial\bar z} &= \frac{(\bar A' C + \bar B' D)(|A|^2+|B|^2) - (\bar A C + \bar B D)(A\bar A' + B\bar B')}{(|A|^2+|B|^2)^2}\\ &= \frac{\bar A\bar B' - \bar A'\bar B}{(|A|^2+|B|^2)^2}\end{aligned}$$

because $AD - BC = 1$. Since

$$w = \frac{1}{|A|^2 + |B|^2},$$

we obtain relation (3.1).

Relations (3.2) and (3.3) are consequences of elementary computations using the fact that we have

$$\frac{\partial}{\partial \rho} A = e^{i\tau} A', \quad \frac{\partial}{\partial \rho} \bar{A} = e^{-i\tau} \bar{A}', \quad \frac{\partial}{\partial \tau} A = i\rho e^{i\tau} A', \quad \frac{\partial}{\partial \tau} \bar{A} = -i\rho e^{-i\tau} \bar{A}',$$

and analogous identities for $B$, $C$ and $D$ (because these are multivaluated holomorphic functions). □

Lemma 3.9. *We have*

$$s(\rho, \tau) = a_1(z) + \zeta_1 a_2(z) + \frac{1}{\zeta_0} a_3(z) + 2\frac{\zeta_1}{\zeta_0} a_1(z) + \frac{\zeta_1^2}{\zeta_0} a_2(z),$$

*where*

$$\begin{aligned} a_1(z) &= 2z(B'C - A'D) + i\frac{\partial}{\partial \tau} \ln w, \\ a_2(z) &= 2z(A'B - AB'), \\ a_3(z) &= 2z(C'D - CD') - 2i\frac{\partial}{\partial \tau}(\zeta \ln w) + i\frac{\partial \zeta}{\partial \tau}. \end{aligned}$$

*Proof.* We have the above expression for $s(\rho, \tau)$ with

$$\begin{aligned} a_1(z) &= \frac{\zeta}{w^2} \overline{\left(\rho \frac{\partial \zeta}{\partial \rho} + i\frac{\partial \zeta}{\partial \tau}\right)} + \frac{\rho}{w}\frac{\partial w}{\partial \rho}, \\ a_2(z) &= -\frac{1}{w^2} \overline{\left(\rho \frac{\partial \zeta}{\partial \rho} + i\frac{\partial \zeta}{\partial \tau}\right)}, \\ a_3(z) &= -\frac{\zeta^2}{w^2} \overline{\left(\rho \frac{\partial \zeta}{\partial \rho} + i\frac{\partial \zeta}{\partial \tau}\right)} + \rho\frac{\partial \zeta}{\partial \rho} - 2i\frac{\partial \zeta}{\partial \tau} \ln w - 2\frac{\rho}{w}\frac{\partial w}{\partial \rho}\zeta. \end{aligned}$$

The claimed formula of $a_2(z)$ is a consequence of formula (3.1).

Because of formulae (3.1) and (3.2) we have

$$a_1(z) = 2z(AB' - A'B)\zeta - z\frac{A'\bar{A} + B'\bar{B}}{|A|^2 + |B|^2} - \bar{z}\frac{A\bar{A}' + B\bar{B}'}{|A|^2 + |B|^2}.$$

Then a computation shows that

$$a_1(z) = 2z(B'C - A'D) + z\frac{A'\bar{A} + B'\bar{B}}{|A|^2 + |B|^2} - \bar{z}\frac{A\bar{A}' + B\bar{B}'}{|A|^2 + |B|^2}.$$

Thus we obtain the above expression for $a_1(z)$ using formula (3.3).

Finally we have

$$\begin{aligned} a_3(z) &= -a_1(z)\zeta + \rho\frac{\partial \zeta}{\partial \rho} - 2i\frac{\partial}{\partial \tau}(\zeta \ln w) + 2i\zeta\frac{\partial}{\partial \tau}\ln w - \frac{\rho}{w}\frac{\partial w}{\partial \rho}\zeta \\ &= -2z(B'C - A'D)\zeta + \rho\frac{\partial \zeta}{\partial \rho} - 2i\frac{\partial}{\partial \tau}(\zeta \ln w) + i\zeta\frac{\partial}{\partial \tau}\ln w - \frac{\rho}{w}\frac{\partial w}{\partial \rho}\zeta \\ &= 2z(BC' - AD')\zeta + 2z\frac{\partial \zeta}{\partial z} + i\frac{\partial \zeta}{\partial \tau} - 2i\frac{\partial}{\partial \tau}(\zeta \ln w) - 2z\frac{\zeta}{w}\frac{\partial w}{\partial z} \\ &= 2z(BC' - AD')\zeta + 2zw(\bar{A}C' + \bar{B}D') + i\frac{\partial \zeta}{\partial \tau} - 2i\frac{\partial}{\partial \tau}(\zeta \ln w) \\ &= 2z(C'D - CD') + i\frac{\partial \zeta}{\partial \tau} - 2i\frac{\partial}{\partial \tau}(\zeta \ln w). \qquad \square \end{aligned}$$

As an immediate consequence of Lemmas 3.7 and 3.9 we obtain the following result.

LEMMA 3.10. *Let $Y$ be the Killing field associated to the translation along the geodesic $(\zeta_0 + \zeta_1, \zeta_1)$. Then the flux of $Y$ through $E$ is*

$$\varphi = \operatorname{Re}\left(\varphi_1 + \varphi_2\zeta_1 + \varphi_0\frac{1}{\zeta_0} + 2\varphi_1\frac{\zeta_1}{\zeta_0} + \varphi_2\frac{\zeta_1^2}{\zeta_0}\right),$$

*where $\varphi_0 = 4\pi \operatorname{Res}(D\mathrm{d}C - C\mathrm{d}D)$, $\varphi_1 = 4\pi \operatorname{Res}(C\mathrm{d}B - D\mathrm{d}A)$ and $\varphi_2 = 4\pi \operatorname{Res}(B\mathrm{d}A - A\mathrm{d}B)$.*

Now we deal with the case where one of the extremities of the geodesic is the point $\infty$.

LEMMA 3.11. *Let $Y$ be the Killing field associated to the translation along the geodesic $(\zeta_1, \infty)$. Then the flux of $Y$ through $E$ is*

$$\varphi = \operatorname{Re}(-\varphi_1 - \varphi_2\zeta_1),$$

*where $\varphi_0 = 4\pi \operatorname{Res}(D\mathrm{d}C - C\mathrm{d}D)$, $\varphi_1 = 4\pi \operatorname{Res}(C\mathrm{d}B - D\mathrm{d}A)$ and $\varphi_2 = 4\pi \operatorname{Res}(B\mathrm{d}A - A\mathrm{d}B)$.*

*Proof.* We proceed as in Lemmas 3.7, 3.9 and 3.10, replacing the expressions for $Y$ and $Z$ in Lemma 3.7 by

$$Y(\zeta, w) = \begin{pmatrix} \zeta - \zeta_1 \\ w \end{pmatrix}$$

and

$$Z(\zeta, w) = \begin{pmatrix} \frac{i}{2}(\zeta - \zeta_1) \\ 0 \end{pmatrix}. \qquad \square$$

Theorem 3.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be two distinct points in $\bar{\mathbb{C}}$. Let $Y$ be the Killing field associated to the translation along the geodesic $(\mathcal{C},\mathcal{D})$. Then the flux of $Y$ through $E$ is*

$$\varphi = \operatorname{Re}\left(\frac{\varphi_2\mathcal{C}\mathcal{D} + \varphi_1(\mathcal{C}+\mathcal{D}) + \varphi_0}{\mathcal{C}-\mathcal{D}}\right),$$

*where $\varphi_0 = 4\pi \operatorname{Res}(D\mathrm{d}C - C\mathrm{d}D)$, $\varphi_1 = 4\pi \operatorname{Res}(C\mathrm{d}B - D\mathrm{d}A)$ and $\varphi_2 = 4\pi \operatorname{Res}(B\mathrm{d}A - A\mathrm{d}B)$.*

*Proof.* If both $\mathcal{C}$ and $\mathcal{D}$ are different from $\infty$, then we set $\zeta_1 = \mathcal{D}$ and $\zeta_0 = \mathcal{C} - \mathcal{D}$, and the result comes from Lemma 3.10.

If $\mathcal{D} = \infty$ and $\mathcal{C} \neq \infty$, then we set $\zeta_1 = \mathcal{C}$, and the result comes from Lemma 3.11.

If $\mathcal{C} = \infty$ and $\mathcal{D} \neq \infty$, then the result follows from the above case and the fact that both the flux and the announced expression are antisymmetric with respect to $(\mathcal{C},\mathcal{D})$. □

### 3.4. Flux of Killing fields associated to rotations.

Lemma 3.13. *Let $E$ be a Bryant surface end given by a conformal immersion $X = (\zeta, w) : \Omega^* \to \mathbb{H}^3$ in the upper half-space model. Let $\zeta_0$ and $\zeta_1$ be two complex numbers, with $\zeta_0 \neq 0$, and let $Y$ be the Killing field associated to the rotation about the geodesic $(\zeta_0+\zeta_1, \zeta_1)$. Then if $\rho$ is a sufficiently small positive number, the flux of $Y$ through $E$ is*

$$\varphi = \int_0^{2\pi} \operatorname{Re}(is(\rho,\tau))\mathrm{d}\tau,$$

*where $s(\rho,\tau)$ has been defined in Lemma 3.7.*

*Proof.* We proceed as in Lemma 3.7, with

$$Y(\zeta,w) = \begin{pmatrix} i\frac{w^2}{\zeta_0} + i\frac{(\zeta-\zeta_1)^2}{\zeta_0} - i(\zeta-\zeta_1) \\ -2w\operatorname{Im}\frac{\zeta-\zeta_1}{\zeta_0} \end{pmatrix}$$

(see Lemma 3.6) and

$$Z(\zeta,w) = \begin{pmatrix} \frac{w^2}{\zeta_0}\ln w - \frac{1}{2}\frac{(\zeta-\zeta_1)^2}{\zeta_0} + \frac{1}{2}(\zeta-\zeta_1) \\ 0 \end{pmatrix}.$$ □

Using this lemma, we proceed as in Section 3.3 to compute the flux of Killing fields associated to rotations.

Theorem 3.14. *Let $\mathcal{C}$ and $\mathcal{D}$ be two distinct points in $\bar{\mathbb{C}}$. Let $Y$ be the Killing field associated to the rotation about the geodesic $(\mathcal{C},\mathcal{D})$. Then the flux of $Y$ through $E$ is*

$$\varphi = -\operatorname{Im}\left(\frac{\varphi_2\mathcal{C}\mathcal{D} + \varphi_1(\mathcal{C}+\mathcal{D}) + \varphi_0}{\mathcal{C}-\mathcal{D}}\right),$$

*where* $\varphi_0 = 4\pi\,\mathrm{Res}(D\mathrm{d}C - C\mathrm{d}D)$*,* $\varphi_1 = 4\pi\,\mathrm{Res}(C\mathrm{d}B - D\mathrm{d}A)$ *and* $\varphi_2 = 4\pi\,\mathrm{Res}(B\mathrm{d}A - A\mathrm{d}B)$*.*

### 3.5. Flux polynomial and equivalence with the residue-type flux matrix.

THEOREM 3.15. *Let $E$ be a Bryant surface end whose Bryant representation is*

$$F = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \Omega^* \to \mathrm{SL}_2(\mathbb{C}).$$

*Then there exists a unique polynomial $P_E(X,Y) \in \mathbb{C}[X,Y]$ such that, for all pairs $(\mathcal{C},\mathcal{D})$ of distinct points in $\bar{\mathbb{C}}$, the flux of the Killing field associated to the translation along the geodesic $(\mathcal{C},\mathcal{D})$ through $E$ is*

$$\mathrm{Re}\left(\frac{P_E(\mathcal{C},\mathcal{D})}{\mathcal{C}-\mathcal{D}}\right)$$

*and the flux of the Killing field associated to the rotation about the geodesic $(\mathcal{C},\mathcal{D})$ through $E$ is*

$$-\,\mathrm{Im}\left(\frac{P_E(\mathcal{C},\mathcal{D})}{\mathcal{C}-\mathcal{D}}\right)$$

*This polynomial $P_E$ is symmetric and we have*

$$P_E(X,Y) = \varphi_2 XY + \varphi_1(X+Y) + \varphi_0,$$

*where* $\varphi_0 = 4\pi\,\mathrm{Res}(D\mathrm{d}C - C\mathrm{d}D)$*,* $\varphi_1 = 4\pi\,\mathrm{Res}(C\mathrm{d}B - D\mathrm{d}A)$ *and* $\varphi_2 = 4\pi\,\mathrm{Res}(B\mathrm{d}A - A\mathrm{d}B)$*.*

*The polynomial*

$$\Pi_E(X) = P_E(X,X) = \varphi_2 X^2 + 2\varphi_1 X + \varphi_0$$

*is called the flux polynomial of $E$.*

*Proof.* This is a reformulation of Theorems 3.12 and 3.14. ☐

REMARK 3.16. We have

$$\Pi_E(X) = -4\pi\,\mathrm{Res}\left(\omega^{\#}\left(X - \frac{1}{G}\right)^2\right).$$

REMARK 3.17. Knowing the flux polynomial is equivalent to knowing the flux of Killing fields associated to all translations and rotations.

In [RUY99], Rossman, Umehara and Yamada defined a residue-type flux for Bryant surface ends. If an end $E$ is conformally parametrized by $\Omega^*$ and has a Bryant representation $F = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then the flux matrix of $E$ is defined by

$$\Phi = -\frac{1}{2i\pi}\int_\Gamma (\mathrm{d}F)F^{-1},$$

where $\Gamma$ is a loop around 0 with positive orientation. This matrix does not depend on the choice of $\Gamma$. It is the residue at zero of the form

$$-(\mathrm{d}F)F^{-1} = \begin{pmatrix} C\mathrm{d}B - D\mathrm{d}A & B\mathrm{d}A - A\mathrm{d}B \\ C\mathrm{d}D - D\mathrm{d}C & B\mathrm{d}C - A\mathrm{d}D \end{pmatrix},$$

which is single-valued. Hence it does not depend on the parametrization. Consequently, since $B\mathrm{d}C - A\mathrm{d}D = -(C\mathrm{d}B - D\mathrm{d}A)$, we have

$$\Phi = \frac{1}{4\pi}\begin{pmatrix} \varphi_1 & \varphi_2 \\ -\varphi_0 & -\varphi_1 \end{pmatrix}.$$

Thus the coefficients of the flux matrix $\Phi$ are, up to constants, the same as the coefficients of the flux polynomial.

This proves the conjecture of Rossman, Umehara and Yamada ([RUY99]; see the remark following Example 8): knowing the flux matrix $\Phi$ of the end $E$ is equivalent to knowing the flux through $E$ of all Killing fields associated to translations and rotations.

We considered these two notions of flux for loops $\Gamma$ generating the fundamental group of an end. We can actually define these fluxes for any loop $\Gamma$ on a Bryant surface. We consider a neighbourhood of $\Gamma$ in the surface that is conformally parametrized by $\{z \in \mathbb{C} : 1-\varepsilon < |z| < 1+\varepsilon\}$ and such that $\Gamma$ is homologous to the curve corresponding to the circle $\{|z| = 1\}$. Then the flux of a Killing field $Y$ through $\Gamma$ is equal to its flux through the curve corresponding to the circle $\{|z| = 1\}$ (since the flux is a homology invariant). Thus we obtain Theorems 3.12, 3.14 and 3.15 with $\varphi_0 = -2i\int_{\{|z|=1\}}(D\mathrm{d}C - C\mathrm{d}D)$, $\varphi_1 = -2i\int_{\{|z|=1\}}(C\mathrm{d}B - D\mathrm{d}A)$ and $\varphi_2 = -2i\int_{\{|z|=1\}}(B\mathrm{d}A - A\mathrm{d}B)$. These coefficients are, up to constants, the coefficients of the flux matrix

$$\Phi = -\frac{1}{2i\pi}\int_\Gamma(\mathrm{d}F)F^{-1} = -\frac{1}{2i\pi}\int_{\{|z|=1\}}(\mathrm{d}F)F^{-1}.$$

Hence the two notions of flux are equivalent for any loop $\Gamma$ on the surface, and consequently for any homology class on the surface.

Remark 3.18. It is easy to compute the flux matrix $\Phi$ of an end $E$ that is the image by a direct isometry of $\mathbb{H}^3$ of an end $E_0$ whose flux matrix $\Phi_0$ is known. Indeed, if $F$ and $F_0$ are the Bryant representations of $E$ and $E_0$, then there exists a matrix $P \in \mathrm{SL}_2(\mathbb{C})$ such that $F_0 = PF$. Then $\Phi = P^{-1}\Phi_0 P$.

## 4. Embedded Bryant surface ends of finite total curvature

Let us first recall and complete the results of Sá Earp and Toubiana [SET01].

Let $E$ be an embedded Bryant surface end of finite total curvature which is not part of a horosphere. We recall that $E$ is necessarily regular (see [Yu01]).

Then, according to [Bry87], the associated Weierstrass data have the form

$$\begin{cases} g(z) = z^\mu f(z), \\ \omega = z^\nu h(z)\mathrm{d}z, \end{cases}$$

in $\Omega^*$, where $f$ and $h$ are holomorphic functions in a neighbourhood of zero such that $f(0) \neq 0$ and $h(0) \neq 0$, and $\mu$ and $\nu$ are real numbers such that $\mu > 0$, $\nu \leqslant -1$, $\mu + \nu \in \mathbb{Z}$ and $\mu + \nu \geqslant -1$.

Since $f(0) \neq 0$, we can define a function $z \mapsto f(z)^{1/\mu}$ in a neighbourhood of zero. Consequently, we can replace $z$ by $zf(z)^{1/\mu}$ and assume that the Weierstrass data have the form

$$\begin{cases} g(z) = z^\mu, \\ \omega = z^\nu h(z)\mathrm{d}z. \end{cases} \tag{4.1}$$

We distinguish two cases: the case where $\mu + \nu = -1$ will be dealt with in Section 4.1, and the case where $\mu + \nu \geqslant 0$ will be dealt with in Section 4.2.

### 4.1. Catenoidal ends.

**4.1.1.** *General representation.* In this section we assume that $\mu + \nu = -1$. In this case the Hopf differential $\omega\mathrm{d}g$ is of degree $-2$. Then, according to [SET01], we have $\mu \neq 1$ and, after replacing $f(z)$ by 1,

$$h(0) = \frac{1-\mu^2}{4\mu} \tag{4.2}$$

and

$$\frac{4\mu}{1-\mu}h'(0) = 2\mu h'(0).$$

This second equation implies that

$$h'(0) = 0. \tag{4.3}$$

The Bryant representation of $E$ is given by

$$F = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a_1 z^{\lambda_1} f_1 + a_2 z^{\lambda_2} f_2 & b_1 z^{r_1} r + b_2 z^{r_2} g_2 \\ c_1 z^{\lambda_1} f_1 + c_2 z^{\lambda_2} f_2 & d_1 z^{r_1} r + d_2 z^{r_2} g_2 \end{pmatrix}, \tag{4.4}$$

where $f_1$, $f_2$, $r$ and $g_2$ are holomorphic functions near 0 satisfying $f_1(0) = f_2(0) = r(0) = g_2(0) = 1$, $\lambda_1 = \frac{-1-\mu}{2}$, $\lambda_2 = \frac{1-\mu}{2}$, $r_1 = \frac{\mu-1}{2}$ and $r_2 = \frac{1+\mu}{2}$, and where $a_1$, $a_2$, $b_1$, $b_2$, $c_1$, $c_2$, $d_1$ and $d_2$ are complex numbers satisfying $a_1 d_1 - b_1 c_1 = 0$, $a_1 c_2 - a_2 c_1 \neq 0$ and $b_1 d_2 - b_2 d_1 \neq 0$.

The functions $f_1$ and $f_2$ are such that $(z \mapsto z^{\lambda_1} f_1(z), z \mapsto z^{\lambda_2} f_2(z))$ is a basis of the vector space of the solutions of the equation

$$X'' - \frac{(z^{-1-\mu}h)'}{z^{-1-\mu}h}X' - \mu h z^{-2} X = 0.$$

The functions $r$ and $g_2$ are such that $(z \mapsto z^{r_1} r(z), z \mapsto z^{r_2} g_2(z))$ is a basis of the vector space of the solutions of the equation

$$X'' - \frac{(z^{-1+\mu}h)'}{z^{-1+\mu}h} X' - \mu h z^{-2} X = 0.$$

REMARK 4.1. Since $\lambda_2 = \lambda_1 + 1$, the function $f_2$ is uniquely defined, and the function $f_1$ is uniquely defined if we fix the value of its derivative at zero. In the same way, since $r_2 = r_1 + 1$, the function $g_2$ is uniquely defined, and the function $r$ is uniquely defined if we fix the value of its derivative at zero.

From the identity $\omega = A\mathrm{d}C - C\mathrm{d}A$ (see [UY93] or [Ros02]) we obtain that

$$h = (a_1 c_2 - a_2 c_1)(f_1 f_2 - z f_1' f_2 + z f_1 f_2'). \tag{4.5}$$

Taking the order 1 terms, we get

$$f_2'(0) = 0. \tag{4.6}$$

In the same way, from the identity $g^2\omega = B\mathrm{d}D - D\mathrm{d}B$ (see [UY93] or [Ros02]) we obtain that

$$h = (b_1 d_2 - b_2 d_1)(r g_2 - z r' g_2 + z r g_2'). \tag{4.7}$$

Taking the order 1 terms, we get

$$g_2'(0) = 0. \tag{4.8}$$

**4.1.2.** *Canonical representation.* Sá Earp and Toubiana [SET01] showed that we can reduce ourselves to a simpler Bryant representation up to an isometry of $\mathbb{H}^3$. More precisely, we can choose complex numbers $\alpha$, $\beta$, $\gamma$ and $\delta$ satisfying $\alpha\delta - \beta\gamma = 1$, $\alpha a_1 + \beta c_1 = \alpha b_1 + \beta d_1 = \gamma a_2 + \delta c_2 = 0$, and $\alpha a_2 + \beta c_2 = 1$. If we replace $F = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ by $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we obtain an end which is the image of $E$ by a direct isometry $\Psi$ of $\mathbb{H}^3$, which has the same Weierstrass data as $E$, and whose Bryant representation is given by

$$\begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} = \begin{pmatrix} z^{\lambda_2} f_2(z) & \frac{\mu-1}{\mu+1} z^{r_2} g_2(z) \\ \frac{\mu^2-1}{4\mu} z^{\lambda_1} f_1(z) & \frac{(1+\mu)^2}{4\mu} z^{r_1} g_1(z) \end{pmatrix}, \tag{4.9}$$

where $g_1$ is a holomorphic function near 0 satisfying $g_1(0) = 1$. The isometry $\Psi$ induces on $\bar{\mathbb{C}}$ the map $\zeta \mapsto \frac{\delta\zeta+\gamma}{\beta\zeta+\alpha}$, which we also denote by $\Psi$.

DEFINITION 4.2. Let $\mu \in (0,1) \cup (1,\infty)$ and $\mathcal{Z} \in \mathbb{C}$. An end which has Weierstrass data given by (4.1) and a Bryant representation $F$ given by (4.9), where $f_1$ has been chosen such that

$$\mathcal{Z} = \frac{\mu^2 - 1}{4\mu} f_1'(0)$$

(see Remark 4.1), is called a canonical catenoidal end of growth $1 - \mu$, of asymptotic boundary $\infty$ and of axis $(\mathcal{Z}, \infty)$, and the Weierstrass data given by

(4.1) and the Bryant representation $F$ given by (4.9) are called, respectively, its canonical Weierstrass data and its canonical Bryant representation.

We now explain this terminology by giving a geometric description of such an end.

Proposition 4.3. *Let $\mu \in (0,1) \cup (1,\infty)$, $\mathcal{Z} \in \mathbb{C}$ and $E$ be a canonical catenoidal end of growth $1-\mu$, of asymptotic boundary $\infty$ and of axis $(\mathcal{Z},\infty)$. Then there exists a parametrization $(w,\tau) \mapsto (\zeta(w,\tau), w)$ of $E$ in the upper half-space model of $\mathbb{H}^3$, and a parametrization $(w,\tau) \mapsto (\tilde{\zeta}(w,\tau), w)$ of a half-catenoid cousin of growth $1-\mu$, of asymptotic boundary $\infty$ and of axis $(0,\infty)$, such that*

$$\zeta(w,\tau) = \tilde{\zeta}(w,\tau) + \mathcal{Z} + \mathrm{o}(1)$$

*when $w$ tends to $\infty$ if $\mu < 1$ and to $0$ if $\mu > 1$.*

*Proof.* We assume that the Weierstrass data of $E$ are given by (4.1) and its Bryant representation $F$ by (4.9), with $\mathcal{Z} = \frac{\mu^2-1}{4\mu} f_1'(0)$ (see Remark 4.1).

Because of formulae (2.1) and (2.2), in the upper half-space model the end $E$ is given by

$$\zeta(z) = (u+iv)(z) = \frac{\mu^2-1}{4\mu z} \frac{f_1\bar{f}_2 + g_1\bar{g}_2|z|^{2\mu}}{|f_2|^2 + (\frac{\mu-1}{\mu+1})^2|g_2|^2|z|^{2\mu}},$$

$$w(z) = \frac{|z|^{\mu-1}}{|f_2|^2 + (\frac{\mu-1}{\mu+1})^2|g_2|^2|z|^{2\mu}}.$$

From this we deduce that the asymptotic boundary of $E$ is actually $\infty$.

Define

$$\tilde{\zeta}(z) = \frac{\mu^2-1}{4\mu z} \frac{1+|z|^{2\mu}}{1+(\frac{\mu-1}{\mu+1})^2|z|^{2\mu}}$$

and

$$\tilde{w}(z) = \frac{|z|^{\mu-1}}{1+(\frac{\mu-1}{\mu+1})^2|z|^{2\mu}}.$$

These functions $\tilde{\zeta}$ and $\tilde{w}$ are the coordinates of the catenoid cousin of growth $1-\mu$ and of axis of revolution $(0,\infty)$, such that the end at $\infty \in \bar{\mathbb{C}}$ corresponds to $z=0$. The height $\tilde{w}$ depends only on $|z|$.

We have $\zeta(z) = \tilde{\zeta}(z) + \mathcal{Z} + \mathrm{o}(1)$ since $\mathcal{Z} = \frac{\mu^2-1}{4\mu} f_1'(0)$, and $w(z) = \tilde{w}(z)(1 + \mathrm{O}(z^2))$ since $f_2'(0) = g_2'(0) = 0$.

Let $(\rho,\tau)$ denote the polar coordinates in $\Omega$ (i.e., $z = \rho e^{i\tau}$). Since $\partial w/\partial\rho \neq 0$ for $\rho$ sufficiently small, we can make the change of parameters $(\rho,\tau) \mapsto (w,\tau)$.

Since $f_2'(0) = g_2'(0) = 0$, we have the following asymptotic expansion:

$$w = \rho^{\mu-1}\left(1 + \sum_{j=1}^{p} \alpha_j \rho^{2j\mu} + \mathrm{O}(\rho^2)\right),$$

where $p$ is the largest integer such that $2p\mu < 2$ and the $\alpha_j$ are real constants which depend only on $\mu$.

Consequently, we have the following asymptotic expansion for the inverse function:

$$\rho = w^{\frac{1}{\mu-1}}\left(1 + \sum_{j=1}^{p} \beta_j w^{\frac{2j\mu}{\mu-1}} + \mathrm{O}(w^{\frac{2}{\mu-1}})\right)$$

when $w$ tends to $\infty$ if $\mu < 1$ and to $0$ if $\mu > 1$, and where the $\beta_j$ are real constants which depend only on $\mu$.

We also have

$$\zeta = \frac{\mu^2 - 1}{4\mu\rho e^{i\tau}}\left(1 + \sum_{j=1}^{q} \gamma_j \rho^{2j\mu}\right) + \mathcal{Z} + \mathrm{o}(1),$$

where $q$ is the largest integer such that $2q\mu \leqslant 1$ and the $\gamma_j$ are real constants which depend only on $\mu$.

Using the asymptotic expansion of $w$, we get

$$\begin{aligned}\zeta(w,\tau) &= \frac{\mu^2 - 1}{4\mu e^{i\tau}} w^{-\frac{1}{\mu-1}}\left(1 + \sum_{j=1}^{p} \delta_j w^{\frac{2j\mu}{\mu-1}} + \mathrm{O}(w^{\frac{2}{\mu-1}})\right) + \mathcal{Z} + \mathrm{o}(1)\\ &= \frac{\mu^2 - 1}{4\mu e^{i\tau}} w^{-\frac{1}{\mu-1}}\left(1 + \sum_{j=1}^{q} \delta_j w^{\frac{2j\mu}{\mu-1}}\right) + \mathcal{Z} + \mathrm{o}(1),\end{aligned}$$

where the $\delta_j$ are real constants which depend only on $\mu$.

The same arguments hold for the canonical catenoid of axis $(0,\infty)$ parametrized by $(\tilde{\zeta}, \tilde{w})$. Consequently we get

$$\zeta(w,\tau) = \tilde{\zeta}(w,\tau) + \mathcal{Z} + \mathrm{o}(1). \qquad \square$$

This means that the end $E$ is asymptotic, in the neighbourhood of $\infty \in \bar{\mathbb{C}}$, to a half-catenoid cousin of growth $1-\mu$ and of axis of revolution $(\mathcal{Z}, \infty)$, in a stronger sense than that used in [SET01] (where two half-catenoid cousins whose asymptotic boundary is $\infty$ and which have the same growth are considered asymptotic to each other, up to a Euclidean homothety, independently of the location of their axes). The complex number $\mathcal{Z}$ is the only number with this property.

Definition 4.4. Let $\mu \in (0,1)\cup(1,\infty)$ and $\mathcal{A}$, $\mathcal{B}$ be two distinct points in $\bar{\mathbb{C}}$. Let $E$ be an embedded Bryant surface end of finite total curvature which is not part of a horosphere. We say that $E$ is a catenoidal end of growth $1-\mu$, of asymptotic boundary $\mathcal{B}$ and of axis $(\mathcal{A},\mathcal{B})$ if there exists an isometry of $\mathbb{H}^3$ (direct or indirect) which maps $\mathcal{A}$ to 0, $\mathcal{B}$ to $\infty$ and $E$ to a canonical catenoidal end of growth $1-\mu$, of asymptotic boundary $\infty$ and of axis $(0,\infty)$.

A half-catenoid cousin of growth $1-\mu$, of asymptotic boundary $\mathcal{B}$ and of axis of revolution $(\mathcal{A},\mathcal{B})$ is of course a catenoidal end of growth $1-\mu$, of asymptotic boundary $\mathcal{B}$ and of axis $(\mathcal{A},\mathcal{B})$.

A canonical catenoidal end of growth $1-\mu$, of asymptotic boundary $\infty$ and of axis $(\mathcal{Z},\infty)$ is a catenoidal end of growth $1-\mu$, of asymptotic boundary $\infty$ and of axis $(\mathcal{Z},\infty)$: it suffices to consider the isometry $(\zeta,w)\mapsto(\zeta-\mathcal{Z},w)$.

We can now prove the following theorem.

Theorem 4.5. *Let $E$ be an embedded Bryant surface end of finite total curvature which is not part of a horosphere. Assume that its Weierstrass data are given by* (4.1) *with $\mu+\nu=-1$. Then there exist a unique real $\chi$ and a unique pair of distinct points $(\mathcal{A},\mathcal{B})$ such that $E$ is a catenoidal end of growth $\chi$, of asymptotic boundary $\mathcal{B}$ and of axis $(\mathcal{A},\mathcal{B})$.*

*Moreover, we have $\chi=1-\mu$ and, if the Bryant representation of $E$ is given by*

$$F(z)=\begin{pmatrix} z^{\lambda_1}a(z) & z^{r_1}b(z)\\ z^{\lambda_1}c(z) & z^{r_1}d(z)\end{pmatrix},$$

*we have $\mathcal{A}=c'(0)/a'(0)$ and $\mathcal{B}=c(0)/a(0)$.*

*Proof.* The existence has already been shown in Section 4.1.1 and in the beginning of Section 4.1.2 (upon choosing $f_1'(0)=0$; see Remark 4.1).

The uniqueness of $\mathcal{B}$ is clear, since the asymptotic boundary of $E$ is the set of its accumulation points in $\bar{\mathbb{C}}$.

Assume that there exist two points $\mathcal{A}_1$ and $\mathcal{A}_2$ and two numbers $\mu_1$ and $\mu_2$ such that $E$ is both a catenoidal end of growth $1-\mu_1$ and of axis $(\mathcal{A}_1,\mathcal{B})$ and a catenoidal end of growth $1-\mu_2$ and of axis $(\mathcal{A}_2,\mathcal{B})$. Then there exists an isometry $\Psi_1$ of $\mathbb{H}^3$ which maps $\mathcal{A}_1$ to 0, $\mathcal{B}$ to $\infty$ and $E$ to a canonical catenoidal end of growth $1-\mu_1$ and of axis $(0,\infty)$, and there exists an isometry $\Psi_2$ of $\mathbb{H}^3$ which maps $\mathcal{A}_2$ to 0, $\mathcal{B}$ to $\infty$ and $E$ to a canonical catenoidal end of growth $1-\mu_2$ and of axis $(0,\infty)$.

Consequently there exists a parametrization $(w,\tau)\mapsto(\zeta_2(w,\tau),w)$ of the end $\Psi_2(E)$ such that $\zeta_2(w,\tau)=\tilde{\zeta}_{\mu_2}(w,\tau)+\mathrm{o}(1)$ when $w$ tends to $\infty$ if $\mu_2<1$ and to 0 if $\mu_2>1$, where $\tilde{\zeta}_{\mu_2}$ corresponds to the canonical catenoid of growth $1-\mu_2$ and of axis $(0,\infty)$.

The isometry $\Psi_1\circ\Psi_2^{-1}$ fixes $\infty$ and maps 0 to $\mathcal{Z}=\Psi_1(\mathcal{A}_2)$. Assume that this isometry is direct. Then it is the composition of a twist about $(0,\infty)$ and of the Euclidean translation by the vector $\mathcal{Z}$. Consequently, the end $\Psi_1(E)$ has a parametrization of the form $(w,\tau)\mapsto(\lambda\zeta_2(w/|\lambda|,\tau)+\mathcal{Z},w)$ with $\lambda\in\mathbb{C}^*$.

On the other hand, there exists a parametrization $(w,\tau') \mapsto (\zeta_1(w,\tau'),w)$ of $\Psi_1(E)$ such that $\zeta_1(w,\tau') = \tilde\zeta_{\mu_1}(w,\tau') + \mathrm{o}(1)$ when $w$ tends to $\infty$ if $\mu_1 < 1$ and to $0$ if $\mu_1 > 1$, where $\tilde\zeta_{\mu_1}$ corresponds to the canonical catenoid of growth $1-\mu_1$ and of axis $(0,\infty)$.

The numbers $1-\mu_1$ and $1-\mu_2$ must have the same sign, since $w$ cannot tend to both $\infty$ and $0$. Also, there exists $\tau$ such that $\mathcal{Z} = c(w)\tilde\zeta_{\mu_1}(w,\tau)$ with $c(w) \geqslant 0$, and for each $w$ there exists $\tau'(w)$ such that $\zeta_1(w,\tau) = \lambda\zeta_2(w/|\lambda|,\tau'(w))$. We deduce that

$$\lambda\tilde\zeta_{\mu_2}(w/|\lambda|,\tau'(w)) + \mathcal{Z} - \tilde\zeta_{\mu_1}(w,\tau) = \mathrm{o}(1).$$

Using the above expressions for $\tilde\zeta_{\mu_1}$ and $\tilde\zeta_{\mu_2}$, we obtain that $\mu_1 = \mu_2$ and $e^{i(\tau'(w)-\tau)} \to \lambda$, and hence $|\lambda| = 1$. Writing $\lambda = e^{i\theta}$, we have

$$(e^{i(\theta+\tau-\tau'(w))} + c(w) - 1)\tilde\zeta_{\mu_1}(w,\tau) = \mathrm{o}(1),$$

and so

$$e^{i(\theta+\tau-\tau'(w))} + c(w) - 1 = \mathrm{o}(\tilde\zeta_{\mu_1}(w,\tau)^{-1}). \tag{4.10}$$

Taking the imaginary part in (4.10) we get

$$\sin(\theta+\tau-\tau'(w)) = \mathrm{o}(\tilde\zeta_{\mu_1}(w,\tau)^{-1}),$$

and consequently

$$\cos(\theta+\tau-\tau'(w)) - 1 = \mathrm{o}(\tilde\zeta_{\mu_1}(w,\tau)^{-1}).$$

On the other hand, taking the real part in (4.10) we get

$$\cos(\theta+\tau-\tau'(w)) + c(w) - 1 = \mathrm{o}(\tilde\zeta_{\mu_1}(w,\tau)^{-1}),$$

so

$$c(w) = \mathrm{o}(\tilde\zeta_{\mu_1}(w,\tau)^{-1}),$$

and finally

$$\mathcal{Z} = c(w)\tilde\zeta_{\mu_1}(w,\tau) = \mathrm{o}(1).$$

This means that $\mathcal{Z} = 0$. We conclude that $\mathcal{A}_2 = \mathcal{A}_1$.

If the isometry $\Psi_1 \circ \Psi_2^{-1}$ is indirect, then it is the composition of the symmetry about the plane $\{\mathrm{Re}\,\zeta = 0\}$ and of the two aforementioned isometries, so the same arguments hold, with $\tilde\zeta_{\mu_2}$ replaced by its conjugate.

To complete the proof, it now suffices to compute the values of $\mathcal{A}$ and $\mathcal{B}$. Using the notations of the beginning of Section 4.1.2 with $f_1'(0) = 0$ (see Remark 4.1), we have $a = a_1 f_1 + a_2 z f_2$ and $c = c_1 f_1 + c_2 z f_2$. Hence we get $a(0) = a_1$, $a'(0) = a_2$, $c(0) = c_1$ and $c'(0) = c_2$. The expression for $\mathcal{B}$ follows from formulae (2.1) and (2.2), and since $\gamma a_2 + \delta c_2 = 0$, we have $\Psi(c_2/a_2) = 0$ (even if $a_2 = 0$), so $\mathcal{A} = c_2/a_2 = c'(0)/a'(0)$. □

REMARK 4.6. The fact that $|\lambda| = 1$ means that, among all the half-catenoid cousins of growth $1-\mu$, of asymptotic boundary $\mathcal{B}$ and of axis $(\mathcal{A},\mathcal{B})$, there exists a unique one to which $E$ is strongly asymptotic.

The following fact is now clear.

PROPOSITION 4.7. *Let $E$ be a catenoidal end of growth $1-\mu$, of asymptotic boundary $\mathcal{B}$ and of axis $(\mathcal{A},\mathcal{B})$. Let $\Psi$ be an isometry of $\mathbb{H}^3$ (direct or indirect). Then $\Psi(E)$ is a catenoidal end of growth $1-\mu$, of asymptotic boundary $\Psi(\mathcal{B})$ and of axis $(\Psi(\mathcal{A}),\Psi(\mathcal{B}))$.*

REMARK 4.8. In Definition 4.4 we can require the isometry to be direct.

REMARK 4.9. The notion of a canonical end has no geometrical meaning, but it will be more convenient to use this terminology to compute the flux (see Section 5.1). Any catenoidal end of axis $(\mathcal{Z},\infty)$ is the image of a canonical one by a twist about $(\mathcal{Z},\infty)$.

### 4.2. Horospherical ends.

**4.2.1.** *General representation.* In this section, we assume that $E$ is an end whose Weierstrass data are given by (4.1) with $\mu+\nu\geqslant 0$. Since we have a single-valued embedding, according to [SET01] we have $\nu=-2$, $\mu\in\mathbb{N}$, $\mu\geqslant 2$, and

$$\begin{cases} h'(0)=2h(0)^2 & \text{if } \mu=2,\\ h'(0)=0 & \text{if } \mu\geqslant 3.\end{cases} \tag{4.11}$$

The Bryant representation of $E$ is given by

$$F=\begin{pmatrix} A & B\\ C & D\end{pmatrix}=\begin{pmatrix} a_1z^{-1}f_1+a_2f_2 & b_1r+b_2z^{2\mu-1}g_2\\ c_1z^{-1}f_1+c_2f_2 & d_1r+d_2z^{2\mu-1}g_2\end{pmatrix}, \tag{4.12}$$

where $f_1$, $f_2$, $r$ and $g_2$ are holomorphic functions near 0 satisfying $f_1(0)=f_2(0)=r(0)=g_2(0)=1$, and where $a_1$, $a_2$, $b_1$, $b_2$, $c_1$, $c_2$, $d_1$ and $d_2$ are complex numbers satisfying $a_1d_1-b_1c_1=0$, $a_1c_2-a_2c_1\neq 0$ and $b_1d_2-b_2d_1\neq 0$.

The functions $f_1$ and $f_2$ are such that $(z\mapsto z^{-1}f_1(z), z\mapsto f_2(z))$ is a basis of the vector space of the solutions of the equation

$$X''-\frac{(z^{-2}h)'}{(z^{-2}h)}X'-\mu hz^{\mu-3}X=0.$$

The functions $r$ and $g_2$ are such that $(z\mapsto r(z), z\mapsto z^{2\mu-1}g_2(z))$ is a basis of the vector space of the solutions of the equation

$$X''-\frac{(z^{2\mu-2}h)'}{(z^{2\mu-2}h)}X'-\mu hz^{\mu-3}X=0.$$

REMARK 4.10. The function $f_2$ is uniquely defined, and the function $f_1$ is uniquely defined if we fix the value of its derivative at zero.

**4.2.2.** *Canonical representation.* In the same way as for the case of catenoidal ends, Sá Earp and Toubiana [SET01] showed that we can reduce ourselves to a simpler Bryant representation up to an isometry of $\mathbb{H}^3$. More precisely, we can choose complex numbers $\alpha$, $\beta$, $\gamma$ and $\delta$ satisfying $\alpha\delta-\beta\gamma=1$, $\alpha a_1+\beta c_1=\alpha b_1+\beta d_1=\gamma a_2+\delta c_2=0$, and $\alpha a_2+\beta c_2=1$. If we replace $F=\begin{pmatrix} A & B \\ C & D\end{pmatrix}$ by $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix}\begin{pmatrix} A & B \\ C & D\end{pmatrix}$, we obtain an end which is the image of $E$ by a direct isometry $\Psi$ of $\mathbb{H}^3$, which has the same Weierstrass data as $E$, and whose Bryant representation is given by

$$F(z)=\begin{pmatrix} A(z) & B(z) \\ C(z) & D(z)\end{pmatrix}=\begin{pmatrix} f_2(z) & bz^{2\mu-1}g_2(z) \\ cz^{-1}f_1(z) & g_1(z)\end{pmatrix}, \tag{4.13}$$

where $g_1$ is a holomorphic function near 0 satisfying $g_1(0)=1$, $b\in\mathbb{C}^*$ and $c\in\mathbb{C}^*$. The isometry $\Psi$ induces on $\bar{\mathbb{C}}$ the map $\zeta\mapsto\frac{\delta\zeta+\gamma}{\beta\zeta+\alpha}$, which we also denote by $\Psi$.

Definition 4.11. An end which has Weierstrass data given by (4.1) and a Bryant representation $F$ given by (4.13) is called a canonical horospherical end of asymptotic boundary $\infty$, and the Weierstrass data given by (4.1) and the Bryant representation $F$ given by (4.9) are called respectively its canonical Weierstrass data and its canonical Bryant representation.

We now assume that the end $E$ has Weierstrass data given by (4.1) and a Bryant representation $F$ given by (4.13).

Because of formulae (2.1) and (2.2), in the upper half-space model the end $E$ is given by

$$\zeta(z)=(u+iv)(z)=\frac{c}{z}\frac{f_1\bar{f}_2+\frac{b}{c}z\bar{z}^{2\mu-1}g_1\bar{g}_2}{|f_2|^2+|b|^2|z|^{2\mu}|g_2|^2},$$

$$w(z)=\frac{1}{|f_2|^2+|b|^2|z|^{2\mu}|g_2|^2}.$$

From the identity $\omega=A\mathrm{d}C-C\mathrm{d}A$ we obtain that

$$c(-f_1f_2+zf_1'f_2-zf_1f_2')=h.$$

Taking the order zero term, we get

$$c=-h(0). \tag{4.14}$$

Taking the order one term, we get

$$h'(0)=-2cf_2'(0). \tag{4.15}$$

Taking the order one term in the identity $AD-BC=1$, we get

$$f_2'(0)+g_1'(0)=0. \tag{4.16}$$

Since the end has finite total curvature and is regular, we can write

$$\omega \mathrm{d}g = \sum_{j=-2}^{\infty} q_j z^j \mathrm{d}z^2.$$

We compute that

$$\omega \mathrm{d}g = \mu z^{\mu-3} h(z) \mathrm{d}z^2.$$

Hence we have

$$q_{-2} = 0$$

and

$$\begin{cases} q_{-1} = 2h(0) & \text{if } \mu = 2, \\ q_{-1} = 0 & \text{if } \mu \geqslant 3. \end{cases} \tag{4.17}$$

**4.3. Classification.** Here we summarize the results we have obtained.

THEOREM 4.12. *Let $E$ be an embedded Bryant surface end of finite total curvature. Then we are in one of the following cases:*

- *$E$ is part of a horosphere.*
- *$E$ is not part of a horosphere and there exists a point $\mathcal{B} \in \bar{\mathbb{C}}$ such that $E$ is a horospherical end of asymptotic boundary $\mathcal{B}$.*
- *$E$ is not part of a horosphere and there exist a real $\mu \in (0,1) \cup (1,\infty)$ and two distinct points $\mathcal{A}, \mathcal{B} \in \bar{\mathbb{C}}$ such that $E$ is a catenoidal end of growth $1-\mu$, of asymptotic boundary $\mathcal{B}$, and of axis $(\mathcal{A}, \mathcal{B})$.*

*Proof.* It suffices to show that we cannot be in two cases at the same time. This is a consequence of the fact that the Hopf differential is zero for horospheres, is non-zero and has a degree greater than or equal to $-1$ for horospherical ends, and has a degree equal to $-2$ for catenoidal ends. □

## 5. Flux for embedded ends of finite total curvature

### 5.1. Flux for catenoidal ends.

LEMMA 5.1. *Let $\mu \in (0,1) \cup (1,\infty)$ and $\mathcal{Z} \in \mathbb{C}$. Let $E$ be a canonical catenoidal end of growth $1-\mu$, of asymptotic boundary $\infty$ and of axis $(\mathcal{Z}, \infty)$. Let $\zeta_0$ and $\zeta_1$ be two complex numbers, with $\zeta_0 \neq 0$. Then the flux polynomial of $E$ is*

$$\Pi_E(X) = 2\pi(\mu^2 - 1)(X - \mathcal{Z}),$$

*the flux of the Killing field associated to the translation along the geodesic $(\zeta_0 + \zeta_1, \zeta_1)$ through $E$ is*

$$\pi(\mu^2 - 1)\left(2\,\mathrm{Re}\left(\frac{\zeta_1 - \mathcal{Z}}{\zeta_0}\right) + 1\right),$$

*the flux of the Killing field associated to the rotation about the geodesic* $(\zeta_0 + \zeta_1, \zeta_1)$ *through* $E$ *is*

$$2\pi(1-\mu^2)\operatorname{Im}\left(\frac{\zeta_1 - \mathcal{Z}}{\zeta_0}\right),$$

*the flux of the Killing field associated to the translation along the geodesic* $(\zeta_1, \infty)$ *through* $E$ *is*

$$\pi(1-\mu^2),$$

*and the flux of the Killing field associated to the rotation about the geodesic* $(\zeta_1, \infty)$ *through* $E$ *is zero.*

*Proof.* Using the canonical Bryant representation (4.9), we compute that the coefficients of the flux polynomial are

$$\begin{aligned}
\varphi_0 &= 4\pi \operatorname{Res}(D\mathrm{d}C - C\mathrm{d}D) = 2\pi(1-\mu^2)\mathcal{Z},\\
\varphi_1 &= 4\pi \operatorname{Res}(C\mathrm{d}B - D\mathrm{d}A) = \pi(\mu^2-1),\\
\varphi_2 &= 4\pi \operatorname{Res}(B\mathrm{d}A - A\mathrm{d}B) = 0.
\end{aligned}$$

Applying Theorems 3.12, 3.14 and 3.15, we obtain the announced results. □

THEOREM 5.2. *Let* $\mu \in (0,1) \cup (1,\infty)$. *Let* $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ *and* $\mathcal{D}$ *be four points in* $\bar{\mathbb{C}}$ *such that* $\mathcal{A} \neq \mathcal{B}$ *and* $\mathcal{C} \neq \mathcal{D}$. *Let* $E$ *be a catenoidal end of growth* $1-\mu$, *of asymptotic boundary* $\mathcal{B}$ *and of axis* $(\mathcal{A}, \mathcal{B})$. *Then the flux of the Killing field associated to the translation along the geodesic* $(\mathcal{C}, \mathcal{D})$ *through* $E$ *is*

$$\pi(1-\mu^2)(2\operatorname{Re}(\mathcal{A}, \mathcal{C}, \mathcal{D}, \mathcal{B}) - 1),$$

*the flux of the Killing field associated to the rotation about the geodesic* $(\mathcal{C}, \mathcal{D})$ *through* $E$ *is*

$$-2\pi(1-\mu^2)\operatorname{Im}(\mathcal{A}, \mathcal{C}, \mathcal{D}, \mathcal{B}),$$

*and the flux polynomial of* $E$ *is*

$$\Pi_E(X) = 2\pi(1-\mu^2)\frac{(X-\mathcal{A})(X-\mathcal{B})}{\mathcal{B}-\mathcal{A}}.$$

*In the case where* $\mathcal{A} = \infty$, *respectively* $\mathcal{B} = \infty$, *the above formula is to be interpreted as*

$$\Pi_E(X) = 2\pi(1-\mu^2)(X-\mathcal{B}),$$

*respectively*

$$\Pi_E(X) = -2\pi(1-\mu^2)(X-\mathcal{A}).$$

*Proof.* We first compute the flux $\varphi$ of the Killing field $Y$ associated to the translation along the geodesic $(\mathcal{C}, \mathcal{D})$.

As was shown in Section 4.1, the end $E$ has Weierstrass data given by (4.1), a Bryant representation $F$ given by (4.4), and, given a complex number $\mathcal{Z}$, there exists a direct isometry $\Psi$ of $\mathbb{H}^3$ which maps $E$ to a canonical catenoidal end of growth $1-\mu$, of asymptotic boundary $\infty$ and of axis $(\mathcal{Z}, \infty)$.

Assume that neither $\mathcal{C}$ nor $\mathcal{D}$ is equal to $\mathcal{B}$. Set $\zeta_0 = \Psi(\mathcal{C}) - \Psi(\mathcal{D})$ and $\zeta_1 = \Psi(\mathcal{D})$. Then $\zeta_0$ and $\zeta_1$ are different from $\infty$, and $\Psi$ maps $\mathcal{A}$ to $\mathcal{Z}$, $\mathcal{B}$ to $\infty$, $\mathcal{C}$ to $\zeta_0 + \zeta_1$ and $\mathcal{D}$ to $\zeta_1$. Hence the flux $\varphi$ of $Y$ through $E$ is equal to the flux of the Killing field associated to the translation along the geodesic $(\zeta_0 + \zeta_1, \zeta_1)$ through $\Psi(E)$. This flux has been calculated in Lemma 5.1: we have

$$\varphi = \pi(\mu^2 - 1)\left(2\operatorname{Re}\left(\frac{\zeta_1 - \mathcal{Z}}{\zeta_0}\right) + 1\right).$$

We compute that

$$(\mathcal{Z}, \zeta_0 + \zeta_1, \infty, \zeta_1) = -\frac{\zeta_1 - \mathcal{Z}}{\zeta_0},$$

and since the map $\Psi$ conserves the cross-ratio, we have $(\mathcal{Z}, \zeta_0 + \zeta_1, \infty, \zeta_1) = (\mathcal{A}, \mathcal{C}, \mathcal{D}, \mathcal{B})$.

Next, assume that $\mathcal{D} = \mathcal{B}$. Set $\zeta_1 = \Psi(\mathcal{C})$. Then $\zeta_1 \neq \infty$ (since $\mathcal{C} \neq \mathcal{D}$), and $\Psi$ maps $\mathcal{A}$ to $\mathcal{Z}$, $\mathcal{B}$ to $\infty$, $\mathcal{C}$ to $\zeta_1$ and $\mathcal{D}$ to $\infty$. Hence the flux $\varphi$ of $Y$ through $E$ is equal to the flux of the Killing field associated to the translation along the geodesic $(\zeta_1, \infty)$ through $\Psi(E)$. This flux has been calculated in Lemma 5.1: we have $\varphi = \pi(1 - \mu^2)$, and since $(\mathcal{A}, \mathcal{C}, \mathcal{D}, \mathcal{B}) = 1$ in this case, the result is still true.

Finally, assume that $\mathcal{C} = \mathcal{B}$. The flux with respect to the geodesic $(\mathcal{C}, \mathcal{D})$ is the opposite of the flux with respect to $(\mathcal{D}, \mathcal{C})$. Hence we have $\varphi = -\pi(1-\mu^2)$ according to what has just been done. Consequently, since $(\mathcal{A}, \mathcal{C}, \mathcal{D}, \mathcal{B}) = 0$ in this case, the result is still true.

We proceed in the same way for the flux of Killing fields associated to rotations. Then the expression for the flux polynomial follows from Theorem 3.15. □

### 5.2. Flux for horospherical ends.

LEMMA 5.3. *Let $E$ be a canonical horospherical end of asymptotic boundary $\infty$. Let $\zeta_0$ and $\zeta_1$ be two complex numbers, with $\zeta_0 \neq 0$. Let $q_{-1}$ be the coefficient of the term of order $-1$ in the canonical Hopf differential of the end. Then the flux polynomial of $E$ is*

$$\Pi_E(X) = -2\pi q_{-1}^2,$$

*the flux of the Killing field associated to the translation along the geodesic $(\zeta_0 + \zeta_1, \zeta_1)$ through $E$ is*

$$-2\pi\operatorname{Re}\left(\frac{q_{-1}^2}{\zeta_0}\right),$$

*the flux of the Killing field associated to the rotation about the geodesic $(\zeta_0 + \zeta_1, \zeta_1)$ through $E$ is*

$$2\pi\operatorname{Im}\left(\frac{q_{-1}^2}{\zeta_0}\right),$$

*the flux of the Killing field associated to the translation along the geodesic $(\zeta_1, \infty)$ through $E$ is zero, and the flux of the Killing field associated to the rotation about the geodesic $(\zeta_1, \infty)$ through $E$ is zero.*

*Proof.* Using the canonical Bryant representation (4.13), we compute that the coefficients of the flux polynomial are

$$\begin{aligned}\varphi_0 &= 4\pi \operatorname{Res}(D\mathrm{d}C - C\mathrm{d}D) = -8\pi c g_1'(0),\\ \varphi_1 &= 4\pi \operatorname{Res}(C\mathrm{d}B - D\mathrm{d}A) = 0,\\ \varphi_2 &= 4\pi \operatorname{Res}(B\mathrm{d}A - A\mathrm{d}B) = 0.\end{aligned}$$

Using equations (4.15) and (4.16), we obtain that $\varphi_0 = -4\pi h'(0)$. Then we deduce from equations (4.11) and (4.17) that $\varphi_0 = -2\pi q_{-1}^2$. Applying Theorems 3.12, 3.14 and 3.15, we obtain the announced results. □

Theorem 5.4. *Let $\mathcal{B} \in \bar{\mathbb{C}}$ and $E$ be a horospherical end of asymptotic boundary $\mathcal{B}$.*

*If $\mathcal{B} \in \mathbb{C}$, then there exists a complex number $\kappa$ such that, for all pairs $(\mathcal{C}, \mathcal{D})$ of distinct points in $\bar{\mathbb{C}}$, the flux of the Killing field associated to the translation along the geodesic $(\mathcal{C}, \mathcal{D})$ through $E$ is*

$$\varphi = -2\pi \operatorname{Re}\left(\kappa \frac{(\mathcal{C}-\mathcal{B})(\mathcal{D}-\mathcal{B})}{\mathcal{C}-\mathcal{D}}\right),$$

*the flux of the Killing field associated to the rotation about the geodesic $(\mathcal{C}, \mathcal{D})$ through $E$ is*

$$\varphi = 2\pi \operatorname{Im}\left(\kappa \frac{(\mathcal{C}-\mathcal{B})(\mathcal{D}-\mathcal{B})}{\mathcal{C}-\mathcal{D}}\right),$$

*and the flux polynomial of $E$ is*

$$\Pi_E(X) = -2\pi\kappa(X-\mathcal{B})^2.$$

*If $\mathcal{B} = \infty$, then there exists a complex number $\kappa$ such that, for all pairs $(\mathcal{C}, \mathcal{D})$ of distinct points in $\bar{\mathbb{C}}$, the flux of the Killing field $Y$ associated to the translation along the geodesic $(\mathcal{C}, \mathcal{D})$ through $E$ is*

$$\varphi = -2\pi \operatorname{Re}\left(\frac{\kappa}{\mathcal{C}-\mathcal{D}}\right),$$

*the flux of the Killing field $Y$ associated to the rotation about the geodesic $(\mathcal{C}, \mathcal{D})$ through $E$ is*

$$\varphi = 2\pi \operatorname{Im}\left(\frac{\kappa}{\mathcal{C}-\mathcal{D}}\right),$$

*and the flux polynomial of $E$ is*

$$\Pi_E(X) = -2\pi\kappa.$$

*The number $\kappa$ is called the flux coefficient of $E$. We have $\kappa = 0$ (or, equivalently, $\Pi_E(X) = 0$) if and only if the Hopf differential $\omega\mathrm{d}g$ of the end $E$ is* *holomorphic at zero, i.e., the degree $\mu$ of the secondary Gauss map $g$ at zero is at least* 3.

*Proof.* We first compute the flux $\varphi$ of the Killing field $Y$ associated to the translation along the geodesic $(\mathcal{C}, \mathcal{D})$.

As was shown in Section 4.2, there exists a direct isometry $\Psi$ of $\mathbb{H}^3$ which maps $E$ to a canonical horospherical end of asymptotic boundary $\infty$. We use the notations of the beginning of Section 4.2.2, with $f_1'(0) = 0$ (see Remark 4.10).

Assume first that neither $\mathcal{C}$ nor $\mathcal{D}$ is equal to $\mathcal{B}$. Set $\zeta_0 = \Psi(\mathcal{C}) - \Psi(\mathcal{D})$ and $\zeta_1 = \Psi(\mathcal{D})$. Then $\zeta_0$ and $\zeta_1$ are different from $\infty$, and $\Psi$ maps $\mathcal{B}$ to $\infty$, $\mathcal{C}$ to $\zeta_0 + \zeta_1$ and $\mathcal{D}$ to $\zeta_1$. Hence the flux $\varphi$ of $Y$ through $E$ is equal to the flux of the Killing field associated to the translation about the geodesic $(\zeta_0 + \zeta_1, \zeta_1)$ through $\Psi(E)$. This flux has been calculated in Lemma 5.3: we have

$$\varphi = -2\pi \operatorname{Re}\left(\frac{q_{-1}^2}{\zeta_0}\right).$$

We have $a = a_1 f_1 + a_2 z f_2$ and $c = c_1 f_1 + c_2 z f_2$. Hence we get $a(0) = a_1$, $a'(0) = a_2$, $c(0) = c_1$ and $c'(0) = c_2$.

Following the argument given at the beginning of Section 4.2.2, we compute that, if $\mathcal{B} \in \mathbb{C}$, then

$$\zeta_0 = \frac{(a'(0)\mathcal{B} - c'(0))^2(\mathcal{C} - \mathcal{D})}{(\mathcal{C} - \mathcal{B})(\mathcal{D} - \mathcal{B})}$$

and if $\mathcal{B} = \infty$, then

$$\zeta_0 = a'(0)^2(\mathcal{C} - \mathcal{D}).$$

We deal with the cases where $\mathcal{C}$ or $\mathcal{D}$ is equal to $\mathcal{B}$ as for Theorem 5.2, using Lemma 5.3. We proceed in the same way for the flux of Killing fields associated to rotations. Then the expression for the flux polynomial follows from Theorem 3.15.

Moreover, the vanishing of $\kappa$ is equivalent to that of $q_{-1}$. □

REMARK 5.5. Corollary 5 in [RUY99] states that an embedded end has a vanishing flux matrix if and only if its Hopf differential is holomorphic at the end. This only occurs for horospherical ends with vanishing flux coefficient $\kappa$.

### 5.3. Flux for horospheres.

THEOREM 5.6. *Let $E$ be an end which is part of a horosphere. Then the flux of the Killing field associated to the translation along any geodesic or to the rotation about any geodesic is zero, and the flux polynomial of $E$ is zero.*

*Proof.* Let $\Gamma$ be a generator of $\pi_1(E)$. Since a horosphere is simply connected, $\Gamma$ is homotopic to zero in the horosphere. Consequently, the fluxes are zero. Thus the flux polynomial is also zero. □

## 6. Geometric applications

Definition 6.1. Let $n$ be a positive integer. Let $\Sigma$ be a complete immersed Bryant surface. We say that $\Sigma$ is an $n$-catenoidal surface if $\Sigma$ has exactly $n$ ends and each end is an embedded end of finite total curvature.

Proposition 6.2. *Let $\Sigma$ be an $n$-catenoidal surface. Then the sum of the fluxes of any Killing field through its ends is zero.*

*Proof.* Let $W$ be a compact set in $\mathbb{H}^3$ such that $\Sigma \setminus W$ is the disjoint union of the ends $E_j$ of $\Sigma$ and such that $\partial W$ is a regular surface. Let $U_j$ be the part of $\partial W$ that is in the interior of $E_j$. Let $\Sigma'$ be the union of $\Sigma \cap W$ and the $U_j$. We can calculate the flux of $E_j$ using the curve $\partial U_j$ and the surface $U_j$. Since $\Sigma'$ is homologous to 0, the result follows from [KKMS92]. $\square$

Remark 6.3. The corresponding statement for the flux matrix is Theorem 1 in [RUY99].

Corollary 6.4. *Let $\Sigma$ be an $n$-catenoidal surface. Then the sum of the flux polynomials of its ends is zero.*

Proposition 6.5. *Let $\Sigma$ be a 2-catenoidal surface. Assume that its ends $E_1$ and $E_2$ are catenoidal ends of growths $1-\mu_1$ and $1-\mu_2$, of asymptotic boundaries $\mathcal{B}_1$ and $\mathcal{B}_2$, and of axes $(\mathcal{A}_1, \mathcal{B}_1)$ and $(\mathcal{A}_2, \mathcal{B}_2)$. Assume that $\mathcal{B}_1 \neq \mathcal{B}_2$. Then we have $\mu_1 = \mu_2$, $\mathcal{A}_1 = \mathcal{B}_2$ and $\mathcal{A}_2 = \mathcal{B}_1$ (that is to say, the two ends have the same growth, the same axis, but two different asymptotic boundaries).*

*Proof.* Without loss of generality, we can assume that $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}_1$ and $\mathcal{B}_2$ are different from $\infty$.

The sum of the flux polynomials of the two ends is zero. In particular, these polynomials have the same roots. Since $\mathcal{B}_1 \neq \mathcal{B}_2$, we have $(\mathcal{A}_1, \mathcal{B}_1) = (\mathcal{B}_2, \mathcal{A}_2)$. Finally we obtain $1-\mu_1^2 = 1-\mu_2^2 = 0$, i.e., $\mu_1 = \mu_2$. $\square$

Remark 6.6. Levitt and Rosenberg [LR85] showed that if, in addition, $\Sigma$ is properly embedded, then $\Sigma$ is a surface of revolution, and hence a catenoid cousin. It is essential that the end be properly embedded; indeed, Rossman and Sato [RS98] constructed a one-parameter family of genus one 2-catenoidal surfaces with catenoidal ends.

Remark 6.7. The flux polynomial does not allow us to eliminate the case of a 2-catenoidal surface with two catenoidal ends having the same asymptotic boundary. We do not know if such a surface exists. If it exists, its ends have the same axis.

Remark 6.8. Genus zero 2-catenoidal surfaces have been classified in [UY93].

PROPOSITION 6.9. *Let $\Sigma$ be a 3-catenoidal surface. Assume that its three ends are catenoidal and that their asymptotic boundaries are distinct. Then, given the growths, the axes of the three ends are uniquely determined, lie in the same plane and are concurrent (possibly in the asymptotic boundary of $\mathbb{H}^3$).*

*Proof.* We use obvious notations. Up to an isometry of $\mathbb{H}^3$, we can assume that $\mathcal{B}_1 = -1$, $\mathcal{B}_2 = 0$ and $\mathcal{B}_3 = 1$. We set $\sigma_j = 1 - \mu_j^2$. Considering the coefficients of the sum of the flux polynomials of the ends, we get

$$\frac{\sigma_1}{\mathcal{A}_1 + 1} + \frac{\sigma_2}{\mathcal{A}_2} + \frac{\sigma_3}{\mathcal{A}_3 - 1} = 0,$$
$$\sigma_1 \frac{\mathcal{A}_1 - 1}{\mathcal{A}_1 + 1} + \sigma_2 + \sigma_3 \frac{\mathcal{A}_3 + 1}{\mathcal{A}_3 - 1} = 0,$$
$$-\sigma_1 \frac{\mathcal{A}_1}{\mathcal{A}_1 + 1} + \sigma_3 \frac{\mathcal{A}_3}{\mathcal{A}_3 - 1} = 0.$$

A computation gives

$$\mathcal{A}_1 = \frac{\sigma_1 - \sigma_2 + \sigma_3}{3\sigma_1 + \sigma_2 - \sigma_3},$$
$$\mathcal{A}_2 = \frac{\sigma_2}{\sigma_3 - \sigma_1},$$
$$\mathcal{A}_3 = \frac{\sigma_1 - \sigma_2 + \sigma_3}{\sigma_1 - \sigma_2 - 3\sigma_3}.$$

Consequently the points $\mathcal{A}_j$ are uniquely determined. Moreover, all the $\mathcal{A}_j$ and $\mathcal{B}_j$ are real. This means they lie in the same plane.

All the geodesics $(\mathcal{A}_j, \mathcal{B}_j)$ lie in the plane $\{v = \operatorname{Im}\zeta = 0\}$. Assume that $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ are all different from $\infty$. Then the equations of the geodesics $(\mathcal{A}_1, -1)$, $(\mathcal{A}_2, 0)$ and $(\mathcal{A}_3, 1)$ are, respectively,

$$u^2 - u(\mathcal{A}_1 - 1) + w^2 - \mathcal{A}_1 = 0,$$
$$u^2 - u\mathcal{A}_2 + w^2 = 0,$$
$$u^2 - u(\mathcal{A}_3 + 1) + w^2 + \mathcal{A}_3 = 0.$$

Thus the abscissa of the intersection point of the first and the second axes is

$$u = \frac{\mathcal{A}_1}{1 - \mathcal{A}_1 + \mathcal{A}_2},$$

and the abscissa of the intersection point of the second and the third axes is

$$u = \frac{\mathcal{A}_3}{1 - \mathcal{A}_2 + \mathcal{A}_3}.$$

Hence the three axes are concurrent if and only if these two numbers are equal. The expressions for the $\mathcal{A}_j$ computed above show that this is the case.

We proceed in the same manner if exactly one of the $\mathcal{A}_j$ is equal to $\infty$. If two of the $\mathcal{A}_j$ are equal to $\infty$, then we deduce from the expressions for the

$\mathcal{A}_j$ that the third one is also equal to $\infty$; in this case the axes are concurrent at $\infty$. $\square$

REMARK 6.10. Levitt and Rosenberg [LR85] showed that if, in addition, $\Sigma$ is properly embedded, then the plane $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ is a plane of symmetry of $\Sigma$. We can deduce from this that the axes lie in this plane.

REMARK 6.11. A classification of irreducible genus zero 3-catenoidal surfaces is given in [UY00].

There is an analogue of Proposition 6.9 for minimal surfaces in Euclidean space $\mathbb{R}^3$ which was first noticed by Kusner in his thesis.

PROPOSITION 6.12. *Let $\Sigma$ be a minimal surface in $\mathbb{R}^3$. Assume that $\Sigma$ has finite total curvature, three ends and that all the ends are asymptotic to catenoids. Then the axes of the ends lie in the same plane and they are either parallel or concurrent.*

*Proof.* Let $E_1$, $E_2$, $E_3$ be the ends of $\Sigma$, $F_j$ the flux of $E_j$ and $T_j(P)$ its torque at the point $P$. We recall that the axis of $E_j$ is the set of points where $T_j$ is zero, and that we have the formula $T_j(Q) = T_j(P) + F_j \times \overrightarrow{PQ}$. Moreover, the two following "balancing formulae" hold:

$$F_1 + F_2 + F_3 = 0,$$
$$\forall P \in \mathbb{R}^3, \quad T_1(P) + T_2(P) + T_3(P) = 0.$$

We can assume that the three axes are not all identical (for otherwise the result is clear). For each $j \in 1, 2, 3$, let $P_j$ be a point of the axis of $E_j$. Then the three axes are the straight lines $P_j + \mathbb{R}F_j$. We can assume that $P_1$, $P_2$ and $P_3$ do not lie on the same straight line, since the axes are distinct.

We have $0 = T_1(P_1) = -T_2(P_1) - T_3(P_1) = F_2 \times \overrightarrow{P_1P_2} + F_3 \times \overrightarrow{P_1P_3}$. Hence we get

$$0 = \langle T_1(P_1), \overrightarrow{P_1P_3}\rangle = \langle F_2 \times \overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}\rangle = \det(F_2, \overrightarrow{P_1P_2}, \overrightarrow{P_1P_3}).$$

This means that the axis of $E_2$ lies in the plane containing $P_1$, $P_2$ and $P_3$. We obtain the same result for $E_1$ and $E_3$. Hence the three axes are coplanar.

If the axes of $E_1$ and $E_2$ are parallel, then the axis of $E_3$ is also parallel to them since $F_3 = -F_1 - F_2$.

If the axes of $E_1$ and $E_2$ are not parallel, then they meet at a point $P_0$, and we get $T_3(P_0) = -T_1(P_0) - T_2(P_0) = 0$. So the three axes are concurrent at $P_0$. $\square$

REMARK 6.13. There are no genus zero 3-catenoidal minimal surfaces in $\mathbb{R}^3$ with parallel ends. This follows from the classification of genus zero 3-catenoidal minimal surfaces in $\mathbb{R}^3$ given by Barbanel and Lopez. The flux for genus zero $n$-catenoidal minimal surfaces in $\mathbb{R}^3$ is treated in [KUY97].

Proposition 6.14. *There is no* 2*-catenoidal surface with one catenoidal end and one horospherical end.*

*Proof.* Assume that such a surface exists. Without loss of generality, we can assume that $\infty$ is not in the asymptotic boundary of the surface and that $\infty$ is not an extremity of the axis of the catenoidal end. Then the flux polynomials of its ends have the same roots. This is impossible since the flux polynomial of a catenoidal end has two simple roots and the flux polynomial of a horospherical is either zero or has a double root. □

Remark 6.15. According to [CHR01, Theorem 12], if a catenoidal surface is properly embedded, then either it is a horosphere or all its ends are catenoidal.

Example 6.16. In [dSN99], de Sousa Neto has constructed Costa-type Bryant surfaces. Let $\Sigma$ be such a surface. It is a 3-catenoidal surface of positive genus. It has two catenoidal ends $E_1$ and $E_2$, which have the same asymptotic boundary $\mathcal{B}$, and one horospherical end $E_3$, whose asymptotic boundary $\mathcal{B}_3$ is different from $\mathcal{B}$. Let $(\mathcal{A}_1, \mathcal{B})$ and $(\mathcal{A}_2, \mathcal{B})$ be the axes of $E_1$ and $E_2$, and $1 - \mu_1$ and $1 - \mu_2$ their respective growths. Since the sum of the flux polynomials of the ends is zero, and since $\mathcal{B} \neq \mathcal{B}_3$, we obtain that the flux coefficient of $E_3$ is zero, and therefore that $\mathcal{A}_1 = \mathcal{A}_2$ and $\mu_1^2 + \mu_2^2 = 2$.

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