

UPPER BOUNDS FOR THE FIRST EIGENVALUE OF THE OPERATOR L_r AND SOME APPLICATIONS

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ABSTRACT. We obtain upper bounds for the first eigenvalue of the linearized operator L_r of the r -mean curvature of a compact manifold immersed in a space of constant curvature δ . By the same method, we obtain an upper bound for the first eigenvalue of the stability operator associated to L_r when $\delta < 0$.

1. Introduction

Let M^n be a compact, connected, orientable Riemannian manifold, isometrically immersed in a simply connected space form $\overline{M}^{n+1}(\delta)$, with constant sectional curvature δ . We obtain upper bounds for the first eigenvalue of some elliptic operators defined on M (see below). In 1988, Heintze [H] proved that

$$\lambda_1^\Delta \leq n\delta + n \max H_1^2$$

for manifolds immersed in a hyperbolic space ($\delta < 0$) and that

$$\lambda_1^\Delta \leq n\delta + \frac{n}{\text{vol } M} \int_M H_1^2$$

for manifolds immersed in a sphere ($\delta > 0$), contained in a convex ball of radius $r \leq \frac{\pi}{4\sqrt{\delta}}$ (if $\delta > 0$). Here λ_1^Δ denotes the first eigenvalue of the Laplacian on M and H_1 denotes the mean curvature. (In fact, Heintze considered as ambient spaces Riemannian manifolds with curvature bounded above by δ .) The latter inequality was obtained by Reilly [R] in 1977 for manifolds immersed in Euclidean space ($\delta = 0$), and generalized to arbitrary δ by El Soufi and Ilias [ESI] in 1992. In all of these estimates, equality holds precisely when M is a geodesic sphere of \overline{M} . Both El Soufi and Ilias [ESI] and Heintze [H] applied these bounds to obtain the stability theorems of Barbosa and do Carmo [B-dC] for immersions in \mathbb{R}^{n+1} , and of Barbosa, do Carmo,

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and Eschenburg [B-dC-E] for immersions in S^{n+1} and H^{n+1} . (The restriction $r \leq \frac{\pi}{4\sqrt{\delta}}$ in the case of S^{n+1} in [H] is stronger than that in [B-dC-E].)

We first introduce some notation. Consider the elementary symmetric functions S_r ($r = 1, \dots, n$) of the principal curvatures and the r -mean curvatures

$$H_r = \frac{S_r}{\binom{n}{r}}.$$

Let A be the second fundamental form associated to a globally defined normal unit vector field N . We define an operator L_r by

$$L_r f = \operatorname{div}(P_r \nabla_M f),$$

where $\nabla_M f$ stands for the gradient of f in M and P_r denotes the classical Newton transformation defined inductively by

$$\begin{aligned} P_0 &= I, \\ P_r &= S_r I - A P_{r-1}. \end{aligned}$$

Each P_r is a self-adjoint operator whose trace is $c(r)H_r$, where $c(r) = (n - r) \binom{n}{r}$ (see [B-C, Lemma 2.1]). Note that $L_0 = \Delta$.

In general, the operator L_r is not elliptic and some conditions are necessary to ensure the presence of ellipticity. However, in the theorems below, the hypotheses will guarantee that L_r is elliptic; see, for instance, the remarks made at the beginning of the proofs of Theorems 1.1 and 1.2. Thus we can consider the first eigenvalue $\lambda_1^{L_r}$ of L_r . This is the object we study here.

Assume that S_{r+1} is constant. Following [B-C] we say that the immersion is r -stable if $I_r(f) \geq 0$ for any $f: M \rightarrow \mathbb{R}$ satisfying $\int_M f \, dM = 0$, where

$$I_r(f) = - \int_M f \left\{ L_r(f) + \left[\frac{n}{r+1} c(r) H_1 H_{r+1} - c(r+1) H_{r+2} + \delta c(r) H_r \right] f \right\}.$$

In 1993, Alencar, do Carmo, and Rosenberg [A-dC-R, Theorem 1.1] proved that if H_{r+1} is positive (but not necessarily constant) on M , then

$$\lambda_1^{L_r} \int_M H_r \leq c(r) \int_M H_{r+1}^2,$$

for manifolds immersed in Euclidean space, and equality holds if and only if M is a sphere. They applied this result to obtain the theorem of Barbosa and do Carmo [B-dC] and a theorem of Alencar, do Carmo, and Colares [A-dC-C]. They also proved that an immersion of a hypersurface in \mathbb{R}^{n+1} with H_{r+1} constant is r -stable if and only if M is a sphere. In 1995, Grosjean [G1] obtained sharp integral bounds for $\lambda_1^{L_r}$ of immersions in any space form $\overline{M}(\delta)$, under the additional hypothesis of convexity of the immersion.

In this paper, inspired by Heintze’s work [H], we obtain sharp upper bounds for $\lambda_1^{L_r}$ without the convexity hypothesis, in both the hyperbolic and spherical spaces. We prove the following theorems:

THEOREM 1.1. *Let M^n be a compact manifold isometrically immersed in $\overline{M}^{n+1}(\delta)$, with $\delta < 0$. If $H_{r+1} > 0$ on M , then*

$$\lambda_1^{L_r} \leq \delta c(r) \min H_r + c(r) \frac{\max H_{r+1}^2}{\min H_r},$$

and equality holds if and only if M is a geodesic sphere.

THEOREM 1.2. *Let M^n be a compact manifold isometrically immersed in $\overline{M}^{n+1}(\delta)$, with $\delta > 0$, and suppose M is contained in a convex ball of radius $\frac{\pi}{4\sqrt{\delta}}$. If $H_{r+1} > 0$ on M , then*

$$\lambda_1^{L_r} \leq \left(\delta c(r) + c(r) \frac{\max H_{r+1}^2}{\min H_r^2} \right) \frac{\int H_r}{\text{vol } M},$$

and equality holds if and only if M is a geodesic sphere.

Furthermore, using the same techniques, we obtain an upper bound for the first eigenvalue of the operator $L_r - q$, where

$$q = c(r+1)H_{r+2} - \frac{nc(r)}{r+1}H_1H_{r+1} - \delta c(r)H_r.$$

From this the r -stability theorem for manifolds immersed in a hyperbolic space, proved in 1997 by Barbosa and Colares [B-C], will follow. (In fact, in [B-C] the r -stability theorem was proved for any manifold $\overline{M}^{n+1}(\delta)$.)

To state this theorem, we need some notation. We denote by s_δ the solution of the differential equation $y'' + \delta y = 0$, with the initial conditions $y(0) = 0$, $y'(0) = 1$. Set $c_\delta = s'_\delta$; then $c'_\delta = -\delta s_\delta$ and $c_\delta^2 + \delta s_\delta^2 = 1$. We have $s_\delta(t) = \frac{1}{\sqrt{-\delta}} \sinh(\sqrt{-\delta}t)$ and $c_\delta(t) = \cosh(\sqrt{-\delta}t)$ in the case $\delta < 0$, and $s_\delta(t) = \frac{1}{\sqrt{\delta}} \sin(\sqrt{\delta}t)$, $c_\delta(t) = \cos(\sqrt{\delta}t)$ in the case $\delta > 0$. Note that if $\delta = 0$, $s_\delta(t) = t$.

We will prove the following result:

THEOREM 1.3. *If $\delta < 0$ and $H_{r+1} > 0$, there exists a point $p_0 \in \overline{M}$ such that, if $d = d(p_0, \cdot)$ denotes the distance function from p_0 in \overline{M} , then*

$$\lambda_1(L_r - q) \leq c(r) \frac{\int_M \left(\frac{\max H_{r+1}^2}{\min H_r} - H_1 H_{r+1} \right) s_\delta^2(d)}{\int_M s_\delta^2(d)},$$

and equality holds precisely when M is a geodesic sphere.

As a consequence, we obtain:

COROLLARY 1.4 ([B-C]). *The only r -stable compact immersed hypersurfaces in a hyperbolic space, with constant $H_{r+1} > 0$, are the geodesic spheres.*

REMARK 1.5. After writing this paper, we received a preprint of J.F. Grosjean (see [G2]) who obtained, independently, the results presented here.

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2. Preliminaries

Throughout this paper, we use the notations defined in the Introduction. Let M^n and $\overline{M}^{n+1}(\delta)$ be as before, let $p_0 \in \overline{M}$, and let $d = d(p_0, \cdot)$ be the distance function from p_0 in \overline{M} . Let x_i ($i = 1, \dots, n + 1$) be the normal coordinates centered in p_0 , with respect to some orthonormal basis in $T_{p_0}\overline{M}$. We denote by ∇ and ∇_M the gradients taken in \overline{M} and M , respectively.

LEMMA 2.1. *Suppose $x \in \overline{M}$. Assume $x \in B\left(p_0, \frac{\pi}{2\sqrt{\delta}}\right)$ in the case $\delta > 0$. If $u, v \in T_x\overline{M}$ and v is orthogonal to ∇d , then*

$$\frac{s_\delta^2(d)}{d^2} \sum_{i=1}^{n+1} (\langle \nabla x_i, u \rangle \langle \nabla x_i, v \rangle) = \langle u, v \rangle.$$

Proof. The map

$$L_{\tilde{x}} = (d \exp_{p_0})_{\tilde{x}} : T_{p_0}\overline{M} \rightarrow T_x\overline{M},$$

where $\exp_{p_0}(\tilde{x}) = x$, is a linear isomorphism. Using $\langle \nabla x_i, u \rangle = (L_{\tilde{x}}^{-1}u)(x_i)$, we obtain

$$\frac{s_\delta^2(d)}{d^2} \sum_{i=1}^{n+1} (\langle \nabla x_i, u \rangle \langle \nabla x_i, v \rangle) = \frac{s_\delta^2(d)}{d^2} \langle L_{\tilde{x}}^{-1}(u), L_{\tilde{x}}^{-1}(v) \rangle.$$

Since $L_{\tilde{x}}$ is a radial isometry, $L_{\tilde{x}}^{-1}(v)$ is tangent to the sphere of radius $|\tilde{x}|$ in $T_{p_0}\overline{M}$. Further, $L_{\tilde{x}}^{-1}(u) = \tilde{w} + \tilde{r}$, where \tilde{w} is tangent and \tilde{r} is orthogonal to this sphere in $T_{p_0}\overline{M}$. Hence, $\langle L_{\tilde{x}}^{-1}(u), L_{\tilde{x}}^{-1}(v) \rangle = \langle \tilde{w}, \tilde{v} \rangle$, where $\tilde{v} = L_{\tilde{x}}^{-1}(v)$.

Let $\gamma : [0, d] \rightarrow \overline{M}$ be the normalized geodesic with

$$\gamma(0) = p, \quad \gamma(d) = x, \quad \gamma'(0) = \frac{\tilde{x}}{|\tilde{x}|},$$

where $|\tilde{x}| = d$. Let $J_v(t), J_w(t)$ be Jacobi fields along γ such that

$$J_v(0) = J_w(0) = 0 \quad J'_v(0) = \frac{\tilde{v}}{|\tilde{v}|}, \quad J'_w(0) = \frac{\tilde{w}}{|\tilde{w}|}.$$

Since \overline{M} has constant sectional curvature,

$$(1) \quad \langle J_v(d), J_w(d) \rangle = \frac{s_\delta^2(d)}{|\tilde{v}||\tilde{w}|} \langle \tilde{v}, \tilde{w} \rangle.$$

Recall also that

$$J_v(t) = (d \exp_{p_0})_{t \frac{\tilde{x}}{d}} \left(t \frac{\tilde{v}}{|\tilde{v}|} \right), \quad J_w(t) = (d \exp_{p_0})_{t \frac{\tilde{x}}{d}} \left(t \frac{\tilde{w}}{|\tilde{w}|} \right).$$

Hence, $\langle J_v(d), J_w(d) \rangle = \frac{d^2}{|\tilde{v}||\tilde{w}|} \langle v, w \rangle$, where $w = L_{\tilde{x}}(\tilde{w})$, and using (1), we obtain $\langle v, w \rangle = \frac{s_\delta^2}{d^2} \langle \tilde{v}, \tilde{w} \rangle$. Thus

$$\frac{s_\delta^2}{d^2} \sum_{i=1}^{n+1} \langle \nabla x_i, u \rangle \langle \nabla x_i, v \rangle = \frac{s_\delta^2}{d^2} \langle \tilde{v}, \tilde{w} \rangle = \langle v, w \rangle = \langle v, u \rangle,$$

which concludes the proof. □

Define the position vector X of M^n in $\overline{M}^{n+1}(\delta)$ with respect to p_0 by $X = s_\delta(d)\nabla d$. Denote by X^T the component of X tangent to M ; i.e., $X^T = s_\delta(d)\nabla_M d$. Observe that

$$\nabla_M c_\delta = -\delta X^T.$$

LEMMA 2.2. *With the above notation,*

$$\sum_{i=1}^{n+1} \left\langle P_r \nabla_M \left(\frac{s_\delta}{d} x_i \right), \nabla_M \left(\frac{s_\delta}{d} x_i \right) \right\rangle + \delta \langle P_r X^T, X^T \rangle = c(r) H_r.$$

Proof. Using the fact that P_r is self-adjoint, we obtain

$$\begin{aligned} \left\langle P_r \nabla_M \frac{s_\delta}{d} x_i, \nabla_M \frac{s_\delta}{d} x_i \right\rangle &= \frac{x_i^2}{d^2} \left(c_\delta - \frac{s_\delta}{d} \right)^2 \langle P_r \nabla_M d, \nabla_M d \rangle \\ &\quad + 2 \frac{x_i s_\delta}{d^2} \left(c_\delta - \frac{s_\delta}{d} \right) \langle P_r \nabla_M d, \nabla_M x_i \rangle \\ &\quad + \frac{s_\delta^2}{d^2} \langle P_r \nabla_M x_i, \nabla_M x_i \rangle. \end{aligned}$$

Since $\sum_i x_i \nabla x_i = d \nabla d$ and

$$\langle P_r \nabla_M d, \nabla_M x_i \rangle = \langle P_r \nabla_M d, \nabla x_i \rangle,$$

because $P_r \nabla_M d$ is tangent to M , we have

$$\begin{aligned} &\sum_{i=1}^{n+1} \left\langle P_r \nabla_M \frac{s_\delta}{d} x_i, \nabla_M \frac{s_\delta}{d} x_i \right\rangle \\ &= \left(c_\delta^2 - \frac{s_\delta^2}{d^2} \right) \cdot \langle P_r \nabla_M d, \nabla_M d \rangle + \frac{s_\delta^2}{d^2} \sum_{i=1}^{n+1} \langle P_r \nabla_M x_i, \nabla_M x_i \rangle. \end{aligned}$$

Further

$$\delta \langle P_r X^T, X^T \rangle = \delta s_\delta^2 \langle P_r \nabla_M d, \nabla_M d \rangle.$$

Thus

$$\begin{aligned} & \sum_{i=1}^{n+1} \left\langle P_r \nabla_M \frac{s_\delta}{d} x_i, \nabla_M \frac{s_\delta}{d} x_i \right\rangle + \delta \langle P_r X^T, X^T \rangle \\ &= \frac{s_\delta^2}{d^2} \sum_{i=1}^{n+1} \langle P_r \nabla_M x_i, \nabla_M x_i \rangle + \left(1 - \frac{s_\delta^2}{d^2} \right) \langle P_r \nabla_M d, \nabla_M d \rangle. \end{aligned}$$

Now let $e_1, \dots, e_n \in T_p M$ be an orthonormal basis such that e_n lies in the direction of $\nabla_M d$ (if $\nabla_M d \neq 0$). Then there are numbers λ and μ satisfying $e_n = \lambda \nabla d + \mu e_n^*$, where e_n^* is a unit vector orthogonal to ∇d . From this we easily obtain $\nabla_M d = \lambda e_n$ and $(e_n^*)^T = \mu e_n$.

A simple calculation gives

$$\begin{aligned} & \sum_{i=1}^{n+1} \left\langle P_r \nabla_M \left(\frac{s_\delta}{d} x_i \right), \nabla_M \left(\frac{s_\delta}{d} x_i \right) \right\rangle + \delta \langle P_r X^T, X^T \rangle \\ &= \frac{s_\delta^2}{d^2} \sum_{j=1}^{n-1} \sum_{i=1}^{n+1} (\langle \nabla x_i, P_r e_j \rangle \langle \nabla x_i, e_j \rangle) + \frac{s_\delta^2}{d^2} \sum_{i=1}^{n+1} \langle P_r \nabla_M x_i, e_n \rangle \langle \nabla_M x_i, e_n \rangle \\ & \qquad \qquad \qquad + \left(1 - \frac{s_\delta^2}{d^2} \right) \lambda^2 \langle P_r e_n, e_n \rangle. \end{aligned}$$

Since e_j is orthogonal to ∇d for all $j = 1, \dots, n - 1$, we can apply Lemma 2.1 and obtain

$$\begin{aligned} (2) \quad & \sum_{i=1}^{n+1} \left\langle P_r \nabla_M \left(\frac{s_\delta}{d} x_i \right), \nabla_M \left(\frac{s_\delta}{d} x_i \right) \right\rangle + \delta \langle P_r X^T, X^T \rangle \\ &= \sum_{j=1}^{n-1} \langle P_r e_j, e_j \rangle + \frac{s_\delta^2}{d^2} \sum_{i=1}^{n+1} \langle P_r \nabla_M x_i, e_n \rangle \langle \nabla_M x_i, e_n \rangle \\ & \qquad \qquad \qquad + \left(1 - \frac{s_\delta^2}{d^2} \right) \lambda^2 \langle P_r e_n, e_n \rangle. \end{aligned}$$

Observing that, for any $x \in \bar{M}$ and any $u \in T_x \bar{M}$,

$$\sum_i^{n+1} \langle \nabla x_i, u \rangle \langle \nabla x_i, \nabla d \rangle = \langle u, \nabla d \rangle,$$

we obtain, after some manipulation,

$$\sum_{i=1}^{n+1} \langle P_r \nabla_M x_i, e_n \rangle \langle \nabla_M x_i, e_n \rangle = \lambda \langle P_r e_n, \nabla d \rangle + \sum_{i=1}^{n+1} \langle \nabla x_i, P_r e_n \rangle \langle \nabla x_i, \mu e_n^* \rangle.$$

Applying Lemma 2.1 again, because e_n^* is orthogonal to ∇d , the right-hand side of the last equation is equal to

$$\lambda \langle P_r e_n, \nabla d \rangle + \frac{d^2}{s_\delta^2(d)} \mu \langle P_r e_n, e_n^* \rangle.$$

Substituting this in (2), we obtain

$$\begin{aligned} \sum_{i=1}^{n+1} \left\langle P_r \nabla_M \left(\frac{s_\delta}{d} x_i \right), \nabla_M \left(\frac{s_\delta}{d} x_i \right) \right\rangle + \delta \langle P_r X^T, X^T \rangle \\ = \sum_{j=1}^n \langle P_r e_j, e_j \rangle = \text{trace}(P_r) = c(r)H_r. \quad \square \end{aligned}$$

LEMMA 2.3. *If $H_{r+1} > 0$ and $c_\delta \geq 0$, then*

$$\frac{\int_M c_\delta}{\int_M s_\delta} \leq \frac{\max H_{r+1}}{\min H_r}.$$

Proof. Recall that if $H_{r+1} > 0$, then $H_j > 0$, where $1 \leq j \leq r$ (see [B-C, Proposition 3.2]). Recall also Minkowski's formula (see [A-C])

$$\int_M [H_r c_\delta + H_{r+1} \langle X, N \rangle] = 0.$$

Then, by the Cauchy-Schwarz inequality,

$$\begin{aligned} \int_M H_r c_\delta &= - \int_M H_{r+1} \langle X, N \rangle \leq \int_M H_{r+1} |X| \\ &= \int_M H_{r+1} s_\delta \leq (\max H_{r+1}) \int_M s_\delta. \end{aligned}$$

Since $c_\delta \geq 0$, we also have $(\min H_r) \int_M c_\delta \leq \int_M H_r c_\delta$. From these inequalities we obtain

$$(\min H_r) \int_M c_\delta \leq (\max H_{r+1}) \int_M s_\delta.$$

This concludes the proof. □

3. Proofs of Theorems 1.1, 1.2, 1.3 and Corollary 1.4

We are now in a position to prove the upper bounds for the first eigenvalue of L_r . Our proofs use the Rayleigh quotient, applied with suitable test functions.

In all proofs, $p_0 \in \overline{M}$ will be a point such that

$$\int_M \frac{s_\delta(d)}{d} x_i = 0 \quad (i = 1, \dots, n + 1),$$

where $d = d(p_0, \cdot)$. The existence of such a point, assuming that M lies in a convex ball of \overline{M} , can be verified by a standard argument. Namely, if M lies in a convex ball B , then

$$Y_q = \int_M \frac{s_\delta(d(q, p))}{d(q, p)} \exp_q^{-1}(p) dp \in T_q \overline{M}$$

defines a vector field in a neighborhood of B which, at the boundary, points towards the interior of B . Thus, Y has a zero in B , and if we take p_0 as this zero, then p_0 has the required property. Note that if B has radius less than $\frac{\pi}{4\sqrt{\delta}}$, then M lies in a ball of radius $< \frac{\pi}{2\sqrt{\delta}}$ around p_0 . As a consequence, $c_\delta \geq 0$.

Proof of Theorem 1.1. Since $H_{r+1} > 0$ and M is compact, L_r is elliptic (see [B-C, Proposition 3.2]).

Using the Rayleigh quotient with the test functions $\frac{s_\delta(d)}{d}x_i$, we obtain

$$\begin{aligned} (3) \quad \lambda_1^{L_r} \int_M s_\delta^2 &= \lambda_1^{L_r} \int_M \sum_{i=1}^{n+1} \left(\frac{s_\delta}{d}x_i\right)^2 \\ &\leq \int_M \sum_{i=1}^{n+1} \left\langle P_r \nabla_M \left(\frac{s_\delta}{d}x_i\right), \nabla_M \left(\frac{s_\delta}{d}x_i\right) \right\rangle \\ &= c(r) \int_M H_r - \delta \int_M \langle P_r X^T, X^T \rangle, \end{aligned}$$

where the last equality follows from Lemma 2.2.

From Stokes' theorem it follows that

$$\int_M f L_r g + \langle P_r \nabla_M f, \nabla_M g \rangle = 0.$$

Applying this with $f = g = c_\delta$ and using the relation $\nabla_M c_\delta = -\delta X^T$, we obtain

$$\delta \int_M \langle P_r X^T, X^T \rangle = -\frac{1}{\delta} \int_M c_\delta L_r(c_\delta).$$

Hence,

$$\int_M \sum_{i=1}^{n+1} \left\langle P_r \nabla_M \left(\frac{s_\delta}{d}x_i\right), \nabla_M \left(\frac{s_\delta}{d}x_i\right) \right\rangle = c(r) \int_M H_r + \frac{1}{\delta} \int_M c_\delta L_r(c_\delta).$$

It is known that (see [A-C, Lemma 1])

$$L_r(c_\delta) = -\delta [c(r)H_r c_\delta + c(r)\langle X, N \rangle H_{r+1}].$$

From this and the inequality

$$-\langle X, N \rangle \leq |X| = s_\delta,$$

we obtain

$$\begin{aligned} \sum_{i=1}^{n+1} \left\langle P_r \nabla_M \left(\frac{s_\delta}{d} x_i \right), \nabla_M \left(\frac{s_\delta}{d} x_i \right) \right\rangle &\leq \delta c(r) \int_M s_\delta^2 H_r - c(r) \int_M c_\delta \langle X, N \rangle H_{r+1} \\ &\leq \delta c(r) \int_M s_\delta^2 H_r + c(r) \int_M c_\delta s_\delta H_{r+1} \\ &\leq \delta c(r) \int_M s_\delta^2 H_r + c(r) \max H_{r+1} \int_M c_\delta s_\delta. \end{aligned}$$

If $\delta \leq 0$, it is also known that (see Lemma 2.8 in [H])

$$\int_M s_\delta \int_M s_\delta c_\delta \leq \left(\int_M s_\delta^2 \right) \int_M c_\delta.$$

Using this inequality and Lemma 2.2, we have

$$\begin{aligned} (4) \quad \sum_{i=1}^{n+1} \left\langle P_r \nabla_M \left(\frac{s_\delta}{d} x_i \right), \nabla_M \left(\frac{s_\delta}{d} x_i \right) \right\rangle &\leq \delta c(r) \int_M s_\delta^2 H_r + c(r) \frac{(\max H_{r+1})^2}{\min H_r} \int_M s_\delta^2 \\ &\leq \delta c(r) (\min H_r) \int_M s_\delta^2 + c(r) \frac{(\max H_{r+1}^2)}{\min H_r} \int_M s_\delta^2. \end{aligned}$$

By applying (3), we obtain

$$\lambda_1^{L_r} \int_M s_\delta^2 \leq \delta c(r) (\min H_r) \int_M s_\delta^2 + c(r) \frac{\max H_{r+1}^2}{\min H_r} \int_M s_\delta^2.$$

Dividing both sides by $\int_M s_\delta^2$ gives the desired estimate.

If equality holds, then we necessarily have

$$-\langle X, N \rangle = |X||N|,$$

and this implies that ∇d is orthogonal to M . Thus, d is constant on M , and therefore M is a geodesic sphere around p_0 . □

Proof of Theorem 1.2. Since $H_{r+1} > 0$ and M is contained in a convex ball, L_r is again an elliptic operator (see [B-C, Proposition 3.2]). Put

$$c = \frac{1}{\text{vol } M} \int_M c_\delta, \quad \text{so} \quad \int_M \frac{(c_\delta - c)}{\sqrt{\delta}} = 0.$$

Recall that $s_\delta(d) = \frac{1}{\sqrt{\delta}} \sin(\sqrt{\delta}d)$ and $c_\delta(d) = \cos(\sqrt{\delta}d)$, so $|c| < 1$.

Using the Rayleigh quotient with $\frac{s_\delta(d)}{d}x_i$ and $\frac{c_\delta - c}{\sqrt{\delta}}$ as test functions, we obtain

$$\begin{aligned} & \lambda_1^{L_r} \int_M \left[s_\delta^2 + \frac{(c_\delta - c)^2}{\delta} \right] \\ & \leq \int_M \sum_{i=1}^{n+1} \left\langle P_r \nabla_M \frac{s_\delta}{d} x_i, \nabla_M \frac{s_\delta}{d} x_i \right\rangle + \int_M \left\langle P_r \nabla_M \left(\frac{c_\delta - c}{\sqrt{\delta}} \right), \nabla_M \left(\frac{c_\delta - c}{\sqrt{\delta}} \right) \right\rangle \\ & = \int_M \left[\sum_{i=1}^{n+1} \left\langle P_r \nabla_M \left(\frac{s_\delta}{d} x_i \right), \nabla_M \left(\frac{s_\delta}{d} x_i \right) \right\rangle + \delta \langle P_r X^T, X^T \rangle \right] = c(r) \int_M H_r, \end{aligned}$$

where the last equality follows from Lemma 2.2.

Further, a direct calculation gives

$$\int_M \left[s_\delta^2 + \frac{(c_\delta - c)^2}{\delta} \right] = \frac{1}{\delta} (\text{vol } M) (1 - c^2).$$

Thus,

$$(5) \quad \lambda_1^{L_r} \leq \left(\frac{1}{1 - c^2} \right) \delta \frac{c(r)}{\text{vol } M} \int_M H_r.$$

We next prove that

$$\frac{1}{1 - c^2} \leq 1 + \frac{1}{\delta} \frac{\max H_{r+1}^2}{\min H_r^2}.$$

By Lemma 2.3 we have

$$c^2 = \frac{1}{(\text{vol } M)^2} \left(\int_M c_\delta \right)^2 \leq \frac{1}{(\text{vol } M)^2} \left(\frac{\max H_{r+1}}{\min H_r} \right)^2 \left(\int_M s_\delta \right)^2,$$

and the Cauchy-Schwarz inequality gives

$$\left(\int_M s_\delta \right)^2 \leq \left(\int_M s_\delta^2 \right) \text{vol } M.$$

Therefore

$$\begin{aligned} & (1 - c^2) \left(1 + \frac{1}{\delta} \frac{\max H_{r+1}^2}{\min H_r^2} \right) \\ & \geq 1 + \frac{1}{\delta} \frac{\max H_{r+1}^2}{\min H_r^2} - \frac{1}{\text{vol } M} \frac{\max H_{r+1}^2}{\min H_r^2} \int_M s_\delta^2 - c^2 \cdot \frac{1}{\delta} \frac{\max H_{r+1}^2}{\min H_r^2} \\ & \geq 1 + \frac{1}{\delta} \frac{\max H_{r+1}^2}{\min H_r^2} - \frac{1}{\text{vol } M} \frac{\max H_{r+1}^2}{\min H_r^2} \int_M s_\delta^2 - \left(\frac{1}{\text{vol } M} \int_M c_\delta^2 \right) \frac{1}{\delta} \frac{\max H_{r+1}^2}{\min H_r^2} \\ & = 1 + \frac{\max H_{r+1}^2}{\min H_r^2} \left(\frac{1}{\delta} - \frac{1}{\delta \text{vol } M} \int_M (\delta s_\delta^2 + c_\delta^2) \right) = 1, \end{aligned}$$

where the last inequality follows from Cauchy-Schwarz inequality. Hence,

$$\frac{1}{1 - c^2} \leq 1 + \frac{1}{\delta} \frac{\max H_{r+1}^2}{\min H_r^2}.$$

From (5), we have

$$\lambda_1^{L_r} \leq \left(\delta + \frac{\max H_{r+1}^2}{\min H_r^2} \right) c(r) \frac{\int H_r}{\text{vol } M}.$$

If equality holds, we also have equality in Lemma 2.3, so $-\langle X, N \rangle = |X||N|$ and therefore ∇d is orthogonal to M . Hence, d is constant on M , and therefore M is a geodesic sphere around p_0 . \square

Proof of Theorem 1.3. Using the Rayleigh quotient for the operator $L_r - q$, with $\frac{s_\delta}{d} x_i$ as test functions, we obtain

$$\begin{aligned} (6) \quad \lambda_1(L_r - q) \int_M s_\delta^2 &\leq \int_M \sum_{i=1}^{n+1} \left(\frac{s_\delta}{d} x_i \right) \left[-L_r \left(\frac{s_\delta}{d} x_i \right) + q \left(\frac{s_\delta}{d} x_i \right) \right] \\ &= \int_M \sum_{i=1}^{n+1} \left\langle P_r \nabla_M \left(\frac{s_\delta}{d} x_i \right), \nabla_M \left(\frac{s_\delta}{d} x_i \right) \right\rangle + \int_M q s_\delta^2. \end{aligned}$$

By (4) we have

$$\begin{aligned} \int_M \sum_{i=1}^{n+1} \left\langle P_r \nabla_M \left(\frac{s_\delta}{d} x_i \right), \nabla_M \left(\frac{s_\delta}{d} x_i \right) \right\rangle \\ \leq \delta c(r) \int_M s_\delta^2 H_r + c(r) \frac{\max H_{r+1}^2}{\min H_r} \int_M s_\delta^2. \end{aligned}$$

Applying this to (6), we obtain

$$\begin{aligned} \lambda_1(L_r - q) \int_M s_\delta^2 &\leq c(r) \left(\int_M s_\delta^2 \right) \frac{\max H_{r+1}^2}{\min H_r} \\ &\quad + c(r+1) \int_M H_{r+2} s_\delta^2 - \frac{nc(r)}{r+1} \int_M H_1 H_{r+1} s_\delta^2. \end{aligned}$$

Since $H_{r+2} \leq H_1 H_{r+1}$, with equality at umbilical points (see [A-dC-R, p. 392]), we obtain

$$\lambda_1(L_r - q) \int_M s_\delta^2 \leq c(r) \left(\int_M s_\delta^2 \right) \frac{\max H_{r+1}^2}{\min H_r} - c(r) \int_M H_1 H_{r+1} s_\delta^2,$$

because $c(r+1) - \frac{nc(r)}{r+1} = -c(r)$.

Dividing both terms by $\int_M s_\delta^2$, the desired inequality follows. The case when equality holds is handled in the same way as in the previous cases. \square

Proof of Corollary 1.4. Let M^n be a compact hypersurface, immersed in $\overline{M}^{n+1}(\delta)$, where $\delta < 0$, with $H_{r+1} > 0$ and constant. Suppose M is r -stable, that is to say,

$$\int_M f(-L_r + q)(f) \geq 0 \quad \text{when} \quad \int_M f = 0.$$

Taking for f an eigenfunction of $L_r - q$ belonging to $\lambda_1^{L_r - q}$, we obtain

$$\lambda_1^{L_r - q} \int_M f^2 \geq 0,$$

and thus

$$(7) \quad \lambda_1^{L_r - q} \geq 0.$$

Also, if $H_{r+1} > 0$, then $H_j > 0$ for all $j = 1, \dots, r$ (see [B-C, Proposition 3.1]) and $H_r \geq H_{r+1}^{\frac{r}{r+1}}$ (see [M-R, Lemma 1]). Hence,

$$(8) \quad \frac{\max H_{r+1}^2}{\min H_r} \leq \frac{H_{r+1}^2}{H_{r+1}^{\frac{r}{r+1}}} = H_{r+1}^{\frac{r+2}{r+1}}.$$

Since $H_1 \geq H_{r+1}^{\frac{1}{r+1}}$, it follows that

$$(9) \quad H_{r+1}^{\frac{r+2}{r+1}} - H_1 H_{r+1} \leq 0,$$

with equality at umbilical points.

By (7), (8), (9) and Theorem 1.3, we conclude that $H_{r+1}^{\frac{r+2}{r+1}} - H_1 H_{r+1} = 0$ everywhere, and so M is a geodesic sphere. \square

REFERENCES

- [A-C] H. Alencar and A. G. Colares, *Integral formulas for the r -mean curvature linearized operator of a hypersurface*, Ann. Global Anal. Geom. **16** (1998), 203–220.
- [A-dC-C] H. Alencar, M. Do Carmo, and A. G. Colares, *Stable hypersurfaces with constant scalar curvature*, Math. Z. **213** (1993), 117–131.
- [A-dC-R] H. Alencar, M. Do Carmo, and H. Rosenberg, *On the first eigenvalue of the linearized operator of the r -th mean curvature of a hypersurface*, Ann. Global Anal. Geom. **11** (1993), 387–391.
- [B-C] L. Barbosa and A. G. Colares, *Stability of hypersurfaces with constant r -mean curvature*, Ann. Global Anal. Geom. **15** (1997), 277–297.
- [B-dC] L. Barbosa and M. Do Carmo, *Stability of hypersurfaces with constant mean curvature*, Math. Z. **195** (1984), 339–353.
- [B-dC-E] L. Barbosa, M. Do Carmo, and J. Eschenburg, *Stability of hypersurfaces of constant mean curvature in Riemannian manifolds*, Math. Z. **197** (1988), 123–138.
- [ESI] A. El-Soufi and S. Ilias, *Une inégalité du type Reilly pour les sous-varietés de l'espace hyperbolique*, Comment. Math. Helv. **67** (1992), 167–181.
- [G1] J.-F. Grosjean, *Majoration de la première valeur propre de certains opérateurs elliptiques naturels sur les hypersurfaces des espaces formes et applications*, C. R. Acad. Sci. Paris Sér. I Math. **321** (1995), 323–326.

- [G2] ———, *Estimations extrinsèques de la première valeur propre d'opérateurs elliptiques définis sur des sous-variétés*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), 807–810.
- [H] E. Heintze, *Extrinsic upper bounds for λ_1* , Math. Ann. **280** (1988), 389–402.
- [M-R] S. Montiel and A. Ros, *Compact hypersurfaces: the Alexandrov theorem for higher order mean curvatures*, Differential Geometry (B. Lawson and K. Tenenblat, eds.), Pitman Monographs, vol. 52, Longman Scientific and Technical, Essex, 1991, pp. 279–296.
- [R] R. Reilly, *On the first eigenvalue of the Laplacian for compact submanifolds of Euclidean space*, Comment. Math. Helv. **52** (1977), 525–533.

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