

CONTRACTIVE AND COMPLETELY CONTRACTIVE HOMOMORPHISMS OF PLANAR ALGEBRAS

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ABSTRACT. We consider contractive homomorphisms of a planar algebra $\mathcal{A}(\Omega)$ over a finitely connected bounded domain $\Omega \subseteq \mathbb{C}$ and ask if they are necessarily completely contractive. We show that a homomorphism $\rho : \mathcal{A}(\Omega) \rightarrow \mathcal{B}(\mathcal{H})$ for which $\dim(\mathcal{A}(\Omega)/\ker \rho) = 2$ is the direct integral of homomorphisms ρ_T induced by operators on two-dimensional Hilbert spaces via a suitable functional calculus $\rho_T : f \mapsto f(T)$, $f \in \mathcal{A}(\Omega)$. It is well known that contractive homomorphisms ρ_T induced by a linear transformation $T : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ are necessarily completely contractive. Consequently, using Arveson's dilation theorem for completely contractive homomorphisms, one concludes that such a homomorphism ρ_T possesses a dilation. In this paper, we construct this dilation explicitly. In view of recent examples discovered by Dritschel and McCullough, we know that not all contractive homomorphisms ρ_T are completely contractive even if T is a linear transformation on a finite-dimensional Hilbert space. We show that one may be able to produce an example of a contractive homomorphism ρ_T of $\mathcal{A}(\Omega)$ which is not completely contractive if an operator space which is naturally associated with the problem is not the MAX space. Finally, within a certain special class of contractive homomorphisms ρ_T of the planar algebra $\mathcal{A}(\Omega)$, we construct a dilation.

1. Introduction

All our Hilbert spaces are over complex numbers and are assumed to be separable. Let $T \in \mathcal{B}(\mathcal{H})$, the algebra of bounded operators on \mathcal{H} . The operator T induces a homomorphism $\rho_T : p \mapsto p(T)$, where p is a polynomial. Equip the polynomial ring with the supremum norm on the unit disc, that is, $\|p\| = \sup\{|p(z)| : z \in \mathbb{D}\}$. A well-known inequality due to von Neumann (cf. [18]) asserts that ρ_T is contractive, that is, $\|\rho_T\| \leq 1$, if and only if the operator

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T is a contraction. Thus in this case, contractivity of the homomorphism ρ_T is equivalent to the operator T being a contraction. As is well known, Sz.-Nagy [24] showed that a contraction T on a Hilbert space \mathcal{H} dilates to a unitary operator U on a Hilbert space \mathcal{K} containing \mathcal{H} , that is, $Pp(U)h = p(T)h$ for all $h \in \mathcal{H}$ and any polynomial p , where $P : \mathcal{K} \rightarrow \mathcal{H}$ is the projection of \mathcal{K} onto \mathcal{H} . The unitary operator U has a continuous functional calculus and hence induces a $*$ -homomorphism $\varphi_U : C(\sigma(U)) \rightarrow \mathcal{B}(\mathcal{K})$. It is easy to check that $P[(\varphi_U)|_{\mathcal{A}(\mathbb{D})}(f)]|_{\mathcal{H}} = \rho_T(f)$, for f in $\mathcal{A}(\mathbb{D})$, where $\mathcal{A}(\mathbb{D})$ is the closure of the polynomials with respect to the supremum norm on the disc \mathbb{D} .

Let Ω be a finitely connected bounded domain in \mathbb{C} . We make the standing assumption that the boundary of Ω is the disjoint union of simple analytic closed curves. Let T be a bounded linear operator on the Hilbert space \mathcal{H} with spectrum $\sigma(T) \subseteq \Omega$. Given a rational function $r = p/q$ with no poles in the spectrum $\sigma(T)$, there is the natural functional calculus $r(T) = p(T)q(T)^{-1}$. Thus T induces a unital homomorphism $\rho_T = r(T)$ on the algebra of rational functions $\text{Rat}(\Omega)$ with poles off Ω . Let $\mathcal{A}(\Omega)$ be the closure of $\text{Rat}(\Omega)$ with respect to the norm $\|r\| := \sup\{|r(z)| : z \in \Omega\}$. Since functions holomorphic in a neighborhood of $\bar{\Omega}$ can be approximated by rational functions with poles off $\bar{\Omega}$, it follows that they belong to $\mathcal{A}(\Omega)$.

The homomorphism ρ_T is said to be *dilatable* if there exists a normal operator N on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ with $\sigma(N) \subseteq \partial\Omega$ such that the induced homomorphism $\varphi_N : C(\sigma(N)) \rightarrow \mathcal{B}(\mathcal{K})$, via the functional calculus for the normal operator N , satisfies the relation

$$(1.1) \quad P(\varphi_N)|_{\mathcal{A}(\Omega)}(f)h = \rho_T(f)h,$$

for h in \mathcal{H} and f in $\mathcal{A}(\Omega)$. Here $P : \mathcal{K} \rightarrow \mathcal{H}$ is the projection of \mathcal{K} onto \mathcal{H} .

The observations about the disk prompt two basic questions:

- (i) When is ρ_T contractive?
- (ii) Do contractive homomorphisms ρ_T necessarily dilate?

For the disc algebra, the answer to the first question is given by von Neumann's inequality while the answer to the second question is affirmative by Sz.-Nagy's dilation theorem. If the domain Ω is simply connected these questions can be reduced to that of the disc (cf. [23]).

If the domain Ω is the annulus, while no satisfactory answer to the first question is known, the answer to the second question was shown to be affirmative by Agler (cf. [4]).

If $\rho_T : \mathcal{A}(\Omega) \rightarrow \mathcal{M}_2$ is a homomorphism induced by an operator $T : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ then it is possible to obtain a characterization of contractivity and then use it to show that the second question has an affirmative answer. We do this in Section 3.2. In Section 2, we show that a larger class of contractive homomorphisms, we call them contractive homomorphisms of *rank 2*, dilate. This is done by proving that the rank 2 homomorphisms are direct integrals of homomorphisms induced by two-dimensional operators.

Arveson (cf. [5] and [6]) has shown that the existence of a dilation of a contractive homomorphism ρ of the algebra $\mathcal{A}(\Omega)$ is equivalent to complete contractivity of the homomorphism ρ . We recall some of these notions in greater detail in Section 4. We then show how one may proceed to possibly construct an example of a contractive homomorphism of the algebra $\mathcal{A}(\Omega)$ which does not dilate.

In the final section of the paper, we obtain a general criterion for contractivity. This involves a factorization of a certain positive definite kernel. More importantly, we outline a scheme for constructing the dilation of a homomorphism $\rho_T : \mathcal{A}(\Omega) \rightarrow \mathcal{M}_n$ induced by an operator T with distinct eigenvalues. This scheme is a generalization of the construction of the dilation in Section 3.2.

2. Homomorphisms of rank two

A homomorphism $\rho : \mathcal{A}(\Omega) \rightarrow \mathcal{B}(\mathcal{H})$ is said to be of *rank n* if it has the property $\dim(\mathcal{A}(\Omega)/\ker \rho) = n$. In this section, we shall begin construction of dilation for homomorphisms of rank 2. Nakazi and Takahashi showed that contractive homomorphisms $\rho : \mathcal{A}(\Omega) \rightarrow \mathcal{B}(\mathcal{H})$ of rank 2 are completely contractive for any uniform sub-algebra of the algebra of continuous functions $C(\bar{\Omega})$ (see [17]). We would like to mention here that a generalization of this result was obtained by Meyer in Theorem 4.1 of [12]. He showed that given a commutative unital closed subalgebra \mathcal{A} of $\mathcal{B}(\mathcal{K})$ (for some Hilbert space \mathcal{K}) and a positive integer d , any $d - 1$ contractive unital homomorphism $\rho : \mathcal{A} \rightarrow \mathcal{M}_d$ is completely contractive. In what follows, we construct explicit dilations for homomorphisms from $\mathcal{A}(\Omega)$ to $\mathcal{B}(\mathcal{H})$ of rank two.

We first show that any homomorphism ρ of rank 2 is the direct integral of homomorphisms of the form ρ_T as defined in the introduction, where $T \in \mathcal{M}_2$. The existence of dilation of a contractive homomorphism ρ_T induced by a two-dimensional operator T is established in [13] by showing that the homomorphism ρ_T must be completely contractive. It then follows that every contractive homomorphism ρ of rank 2 must be completely contractive. This implies by Arveson's theorem that they possess a dilation. However, it is not always easy to construct the dilation whose existence is guaranteed by the theorem of Arveson. In this case, we shall explicitly construct the dilation of a homomorphism of rank 2. This is achieved by constructing the dilation of a contractive homomorphism of the form ρ_T for a two-dimensional operator T .

LEMMA 1. *If $\rho_T : \mathcal{A}(\Omega) \rightarrow \mathcal{B}(\mathcal{L})$ is a homomorphism of rank two, then up to unitary equivalence, the Hilbert space \mathcal{L} is a direct integral*

$$\mathcal{L} = \int_{\Lambda}^{\oplus} \mathcal{L}_{\lambda} d\nu(\lambda),$$

where each \mathcal{L}_λ is two-dimensional. In this decomposition, the operator T is of the form

$$T = \int_{\Lambda}^{\oplus} \begin{pmatrix} z_1(\lambda) & c(\lambda) \\ 0 & z_2(\lambda) \end{pmatrix} d\nu(\lambda).$$

Proof. To begin with, it is easy to see (see Lemma 1 of [17]) that \mathcal{L} is a direct sum of two Hilbert spaces \mathcal{H} and \mathcal{K} and the operator $T : \mathcal{H} \oplus \mathcal{K} \rightarrow \mathcal{H} \oplus \mathcal{K}$ is of the form

$$\begin{pmatrix} z_1 I_{\mathcal{H}} & C \\ 0 & z_2 I_{\mathcal{K}} \end{pmatrix}, \text{ with } z_1, z_2 \in \Omega \text{ or } \begin{pmatrix} z I_{\mathcal{H}} & C \\ 0 & z I_{\mathcal{K}} \end{pmatrix}, \text{ with } z \in \Omega,$$

where C is a bounded operator from \mathcal{K} to \mathcal{H} . Now if we put $\mathcal{K}_0 = (\ker C)^\perp$, $\mathcal{K}_1 = \ker C$, $\mathcal{H}_0 = \overline{\text{Ran } C}$ and $\mathcal{H}_1 = (\text{Ran } C)^\perp$, then with respect to the decomposition $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1$ and $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ we have

$$C = \begin{pmatrix} \tilde{C} & 0 \\ 0 & 0 \end{pmatrix}$$

where the operator \tilde{C} is from \mathcal{K}_0 to \mathcal{H}_0 . The polar decomposition of \tilde{C} then yields $\tilde{C} = VP$, where the operator V is unitary and P is positive. We apply the spectral theorem to the positive operator P and conclude that there exists a unitary operator $\Gamma : \int_{\Lambda}^{\oplus} \mathcal{H}_\lambda d\nu(\lambda) \rightarrow \mathcal{K}_0$ which intertwines the multiplication operator M on the Hilbert space $\int_{\Lambda}^{\oplus} \mathcal{H}_\lambda d\nu(\lambda)$ and P .

Now notice that the operator $T : \mathcal{H} \oplus \mathcal{K} \rightarrow \mathcal{H} \oplus \mathcal{K}$ can be rewritten as

$$\begin{pmatrix} z_1 I_{\mathcal{H}_1} & 0 & \tilde{C}_{\mathcal{K}_0 \rightarrow \mathcal{H}_0} & 0 \\ 0 & z_1 I_{\mathcal{H}_0} & 0 & 0 \\ 0 & 0 & z_2 I_{\mathcal{K}_0} & 0 \\ 0 & 0 & 0 & z_2 I_{\mathcal{K}_1} \end{pmatrix}.$$

Interchanging the third and the second column and then the second and third row, which can be effected by a unitary operator, we see that the operator T is unitarily equivalent to the direct sum of a diagonal operator D and an operator \tilde{T} of the form

$$\begin{pmatrix} z_1 I_{\mathcal{H}_0} & \tilde{C}_{\mathcal{K}_0 \rightarrow \mathcal{H}_0} \\ 0 & z_2 I_{\mathcal{K}_0} \end{pmatrix},$$

where \tilde{C} has dense range. It is clear that if we conjugate the operator \tilde{T} by the operator $I_{\mathcal{H}_0} \oplus U_{\mathcal{H}_0 \rightarrow \mathcal{K}_0}$, where U is any unitary operator identifying \mathcal{H}_0 and \mathcal{K}_0 , then we obtain a unitarily equivalent copy of \tilde{T} (again, denoted by \tilde{T}), which is of the form

$$\begin{pmatrix} z_1 I_{\mathcal{H}_0} & \tilde{C}_{\mathcal{K}_0 \rightarrow \mathcal{H}_0} U_{\mathcal{H}_0 \rightarrow \mathcal{K}_0} \\ 0 & z_2 I_{\mathcal{H}_0} \end{pmatrix}.$$

Now, if we apply the polar decomposition to \tilde{C} then we see that the off diagonal entry is a positive operator on \mathcal{H}_0 . One then sees that \tilde{T} is unitarily

equivalent to

$$\begin{pmatrix} z_1 I_{\int_{\Lambda}^{\oplus} \mathcal{H}_{\lambda} d\nu(\lambda)} & M \\ 0 & z_2 I_{\int_{\Lambda}^{\oplus} \mathcal{H}_{\lambda} d\nu(\lambda)} \end{pmatrix}$$

via conjugation using the operator $\Gamma \oplus \Gamma$. We need to conjugate this operator one more time using the unitary W that identifies $\int_{\Lambda}^{\oplus} \mathcal{H}_{\lambda} d\nu(\lambda) \oplus \int_{\Lambda}^{\oplus} \mathcal{H}_{\lambda} d\nu(\lambda)$ and $\int_{\Lambda}^{\oplus} \mathcal{H}_{\lambda} \oplus \mathcal{H}_{\lambda} d\nu(\lambda)$, where $W(s_1 \oplus s_2)(\lambda) = s_1(\lambda) \oplus s_2(\lambda)$ for $s_1 \oplus s_2 \in \int_{\Lambda}^{\oplus} \mathcal{H}_{\lambda} d\nu(\lambda)$. It is easy to calculate $W\tilde{T}W^*$ and verify the claim. \square

In view of the above lemma, it is now enough to consider dilations of homomorphisms ρ_T where T is a linear transformation on \mathbb{C}^2 .

3. Dilations and Abrahamse-Nevanlinna-Pick interpolation

3.1. Consider any reproducing kernel Hilbert space \mathcal{H}_K of holomorphic functions on Ω with $K : \Omega \times \Omega \rightarrow \mathbb{C}$ as the kernel. Assume that the multiplication operator M by the independent variable z is bounded. Then $M^*(K(\cdot, z)) = \bar{z}K(\cdot, z)$ and it is clear by differentiation that $M^*\bar{\partial}_z K(\cdot, z) = K(\cdot, z) + \bar{z}\bar{\partial}_z K(\cdot, z)$.

The matrix representation of the operator M^* restricted to the subspace \mathcal{M} spanned by the two vectors $K(\cdot, z_1)$ and $K(\cdot, z_2)$ has two distinct eigenvalues \bar{z}_1 and \bar{z}_2 . Similarly, the operator M^* restricted to the subspace \mathcal{N} spanned by the two vectors $K(\cdot, z)$ and $\bar{\partial}_z K(\cdot, z)$ has only one eigenvalue \bar{z} of multiplicity 2. In the lemma below, we identify certain 2-dimensional subspaces of $\mathcal{H}_K \oplus \mathcal{H}_K$ which are invariant under the multiplication operator M^* and then find out the form of the matrix. The reproducing kernel K satisfies

(3.1a) $\quad K(z_1, z_2) = \langle K(\cdot, z_2), K(\cdot, z_1) \rangle, \quad z_1, z_2 \in \Omega,$

(3.1b) $\quad (\partial_z K)(z, u) = \langle K(\cdot, u), \bar{\partial}_z K(\cdot, z) \rangle, \quad u, z \in \Omega.$

Using (3.1) and applying the Gram-Schmidt orthogonalization process to the set $\{K(\cdot, z_1), K(\cdot, z_2)\}$, we get the orthonormal pair of vectors

$$e(z_1) = \frac{K(\cdot, z_1)}{K(z_1, z_1)^{1/2}}$$

and

$$f(z_1, z_2) = \frac{K(z_1, z_1)K(\cdot, z_2) - K(z_2, z_2)K(\cdot, z_1)}{K(z_1, z_1)^{1/2} (K(z_1, z_1)K(z_2, z_2) - |K(z_1, z_2)|^2)^{1/2}}.$$

Now for any $\mu \in \mathbb{D}$, the pair of vectors

$$h_1(z_1, z_2) = \begin{pmatrix} 0 \\ e(z_1) \end{pmatrix} \text{ and } h_2(z_1, z_2) = \begin{pmatrix} (1 - |\mu|^2)^{1/2} e(z_2) \\ \mu f(z_1, z_2) \end{pmatrix}$$

are orthonormal in $\mathcal{H}_K \oplus \mathcal{H}_K$. Similarly, orthonormalizing the pair of vectors $\{K(\cdot, z), \bar{\partial}_z K(\cdot, z)\}$, using (3.1b), we see that the pair $\{e(z), f(z)\}$, where

$$e(z) = \frac{K(\cdot, z)}{K(z, z)^{1/2}}$$

and

$$f(z) = \frac{K(z, z)\bar{\partial}_z K(\cdot, z) - \langle \bar{\partial}_z K(\cdot, z), K(\cdot, z) \rangle K(\cdot, z)}{K(z, z)^{1/2} (K(z, z) \|\bar{\partial}_z K(\cdot, z)\|^2 - |\langle \bar{\partial}_z K(\cdot, z), K(\cdot, z) \rangle|^2)^{1/2}}$$

are orthonormal. Now, for any $\lambda \in \mathbb{D}$,

$$k_1(z) = \begin{pmatrix} 0 \\ e(z) \end{pmatrix} \text{ and } k_2(z) = \begin{pmatrix} (1 - |\lambda|^2)^{1/2} e(z) \\ \lambda f(z) \end{pmatrix}$$

form a set of two orthonormal vectors in $\mathcal{H}_K \oplus \mathcal{H}_K$.

Note that from the definition of M^* it follows that $M^*e(z_1) = \bar{z}_1 e(z_1)$ for all $z_1 \in \Omega$. Therefore we have $(M^* \oplus M^*)h_1(z_1, z_2) = \bar{z}_1 h_1(z_1, z_2)$. Now,

$$\begin{aligned} M^*f(z_1, z_2) &= \frac{K(z_1, z_1)\bar{z}_2 K(\cdot, z_2) - K(z_2, z_2)\bar{z}_1 K(\cdot, z_1)}{K(z_1, z_1)^{1/2}(K(z_1, z_1)K(z_2, z_2) - |K(z_1, z_2)|^2)^{1/2}} \\ &= \bar{z}_2 f(z_1, z_2) + \frac{(\bar{z}_2 - \bar{z}_1)K(z_1, z_2)}{(K(z_1, z_1)K(z_2, z_2) - |K(z_1, z_2)|^2)^{1/2}} e(z_1). \end{aligned}$$

It follows that \mathcal{M} is invariant under $M^* \oplus M^*$. In particular, we have

$$\begin{aligned} (M^* \oplus M^*)h_2(z_1, z_2) &= \begin{pmatrix} (1 - |\mu|^2)^{1/2} M^*e(z_2) \\ \mu M^*f(z_1, z_2) \end{pmatrix} \\ &= \begin{pmatrix} (1 - |\mu|^2)^{1/2} \bar{z}_2 e(z_2) \\ \mu \left(\bar{z}_2 f(z_1, z_2) + \frac{(\bar{z}_2 - \bar{z}_1)K(z_1, z_2)}{(K(z_1, z_1)K(z_2, z_2) - |K(z_1, z_2)|^2)^{1/2}} e(z_1) \right) \end{pmatrix} \\ &= \bar{z}_2 \begin{pmatrix} (1 - |\mu|^2)^{1/2} e(z_2) \\ \mu f(z_1, z_2) \end{pmatrix} + \begin{pmatrix} 0 \\ \mu \frac{(\bar{z}_2 - \bar{z}_1)K(z_1, z_2)}{(K(z_1, z_1)K(z_2, z_2) - |K(z_1, z_2)|^2)^{1/2}} e(z_1) \end{pmatrix} \\ &= \bar{z}_2 h_2(z_1, z_2) + \mu \frac{(\bar{z}_2 - \bar{z}_1)|K(z_1, z_2)|}{(K(z_1, z_1)K(z_2, z_2) - |K(z_1, z_2)|^2)^{1/2}} h_1(z_1, z_2), \end{aligned}$$

where we have absorbed the argument of $K(z_1, z_2)$ in μ .

Now recall that $(M^* - \bar{z})K(\cdot, z) = 0$. Differentiating with respect to \bar{z} , we obtain $M^*\bar{\partial}_z K(\cdot, z) = K(\cdot, z) + \bar{z}\bar{\partial}_z K(\cdot, z)$. Thus the subspace \mathcal{N} spanned by the vectors $k_1(z), k_2(z)$ is invariant under M^* . A little more computation, similar to the one above, gives us the matrix representation of the restriction of the operator $M^* \oplus M^*$ to the subspace \mathcal{N} .

So, we have proved the following lemma.

LEMMA 2. *The two-dimensional space \mathcal{M} spanned by the vectors $h_1(z_1, z_2)$ and $h_2(z_1, z_2)$ is an invariant subspace for the operator $M^* \oplus M^*$ on $\mathcal{H}_K \oplus$*

\mathcal{H}_K and the restriction of this operator to the subspace \mathcal{M} has the matrix representation

$$\begin{pmatrix} \bar{z}_1 & \frac{\mu(\bar{z}_2 - \bar{z}_1)|K(z_1, z_2)|}{(K(z_1, z_1)K(z_2, z_2) - |K(z_1, z_2)|^2)^{1/2}} \\ 0 & \bar{z}_2 \end{pmatrix}.$$

Similarly, the two-dimensional space \mathcal{N} spanned by the two vectors $k_1(z), k_2(z)$ is an invariant subspace for the operator $M^* \oplus M^*$ on $\mathcal{H} \oplus \mathcal{H}$ and the restriction of this operator to the subspace \mathcal{N} has the matrix representation

$$\begin{pmatrix} \bar{z} & \frac{\lambda K(z, z)}{(K(z, z)\|\bar{\partial}_z K(\cdot, z)\|^2 - |\langle \bar{\partial}_z K(\cdot, z), K(\cdot, z) \rangle|^2)^{1/2}} \\ 0 & \bar{z} \end{pmatrix}.$$

Let μ, λ be a pair of complex numbers and fix a pair of 2×2 matrices A_s and B_t defined by

$$(3.2) \quad A_s = \begin{pmatrix} z_1 & 0 \\ s\mu(z_1 - z_2) & z_2 \end{pmatrix}, \quad z_1, z_2 \in \Omega \quad \text{and} \quad B_t = \begin{pmatrix} z & 0 \\ t\lambda & z \end{pmatrix}, \quad z \in \Omega,$$

where s, t are a pair of positive real numbers. Let

$$(3.3a) \quad s_K = \frac{|K(z_1, z_2)|}{(K(z_1, z_1)K(z_2, z_2) - |K(z_1, z_2)|^2)^{1/2}},$$

$$(3.3b) \quad t_K = \frac{K(z, z)}{(K(z, z)\|\bar{\partial}_z K(\cdot, z)\|^2 - |\langle \bar{\partial}_z K(\cdot, z), K(\cdot, z) \rangle|^2)^{1/2}}.$$

It follows from the lemma that the matrix A_s (respectively, B_t) is the compression of the operator $M \oplus M$ on the Hilbert space $\mathcal{H}_K \oplus \mathcal{H}_K$ to the two-dimensional subspaces \mathcal{M} (respectively, \mathcal{N}) if and only if $s = s_K$ and $|\mu| \leq 1$ (respectively, $t = t_K$ and $|\lambda| \leq 1$).

A natural family of Hilbert spaces $H_\alpha^2(\Omega)$ consisting of modulus automorphic holomorphic functions on Ω was studied in the paper [2]. This family is indexed by $\alpha \in \mathbb{T}^m$, where m is the number of bounded connected components in $\mathbb{C} \setminus \Omega$ and \mathbb{T} is the unit circle. Each $H_\alpha^2(\Omega)$ has a reproducing kernel $K_\alpha(z, w)$. It was shown in [2] that every pure subnormal operator with spectrum $\bar{\Omega}$ and the spectrum of the normal extension contained in $\partial\bar{\Omega}$ is unitarily equivalent to M on one of these Hilbert spaces.

In the following subsection, we will show that any contractive homomorphism of the algebra $\text{Rat}(\Omega)$ is of the form ρ_{A_s} or ρ_{B_t} with $K = K_\alpha$ and $|\mu| \leq 1$ and $|\lambda| \leq 1$, respectively. Since the operator $M \oplus M$ is subnormal, we will have exhibited the dilation.

3.2. Construction of dilations. The generalization of the Nevanlinna-Pick theorem due to Abrahamse states that given n points w_1, w_2, \dots, w_n in the open unit disk, there is a holomorphic function $f : \Omega \rightarrow \mathbb{C}$ with $f(z_i) = w_i$

for $i = 1, 2, \dots, n$ if and only if the matrix

$$(3.4) \quad M(\underline{w}, \alpha) \stackrel{\text{def}}{=} (((1 - w_i \bar{w}_j) K_\alpha(z_i, z_j)))$$

is positive semidefinite. A deep result due to Widom (cf. [11, page 140]) shows that the map $\alpha \mapsto K_\alpha(z, w)$ is continuous for any fixed pair (z, w) in $\Omega \times \Omega$.

In what follows, we shall first show that a homomorphism $\rho : \text{Rat}(\Omega) \rightarrow \mathcal{M}_2$ is contractive if and only if it is of the form ρ_{A_s} or ρ_{B_t} with $|\mu| \leq 1$ and $|\lambda| \leq 1$, respectively and

$$(3.5a) \quad s = s_\Omega(z_1, z_2) := \sup \{ |r(z_1)|^2 : r \in \text{Rat}(\Omega), \|r\| \leq 1 \text{ and } r(z_2) = 0 \}$$

for any fixed but arbitrary pair $z_1, z_2 \in \Omega$, and

$$(3.5b) \quad t = t_\Omega(z) := \sup \{ |r'(z)| : r \in \text{Rat}(\Omega), \|r\| \leq 1 \text{ and } r(z) = 0 \}$$

for $z \in \Omega$.

We wish to point out that the extremal quantities $s_\Omega(z_1, z_2)$ and $t_\Omega(z)$ would remain the same even if we were to replace the space $\text{Rat}(\Omega)$ by the holomorphic function on Ω . The solution to the first extremal problem, with holomorphic functions in place of $\text{Rat}(\Omega)$, exists by a normal family argument. Let $F : \Omega \rightarrow \mathbb{D}$ be a holomorphic function with $F(z_2) = 0$ and $F(z_1) = a$, where we have set $a = s_\Omega(z_1, z_2)$, temporarily. It then follows that $M((0, a), \alpha)$ must be non-negative definite for all $\alpha \in \mathbb{T}^m$. Consequently, we have

$$\det \begin{pmatrix} K_\alpha(z_1, z_1) & K_\alpha(z_1, z_2) \\ K_\alpha(z_2, z_1) & (1 - a^2) K_\alpha(z_2, z_2) \end{pmatrix} \geq 0$$

for all $\alpha \in \mathbb{T}^m$. This condition is equivalent to requiring

$$(3.6) \quad |a|^2 \leq 1 - \frac{|K_\alpha(z_1, z_2)|^2}{K_\alpha(z_1, z_1) K_\alpha(z_2, z_2)} \leq 1 - \sup \left\{ \frac{|K_\alpha(z_1, z_2)|^2}{K_\alpha(z_1, z_1) K_\alpha(z_2, z_2)} : \alpha \in \mathbb{T}^m \right\}.$$

As we have pointed out earlier, since $\alpha \rightarrow K_\alpha(z_i, z_j)$ is continuous for any pair of fixed indices i and j , there exists a single α_0 depending only on z_1, z_2 for which the supremum in the above inequality is attained. For this choice of α_0 and a , clearly the determinant of $M((0, a), \alpha_0)$ is zero. It follows from [11, Theorem 4.4, p. 135] that the solution is unique and hence is a Blaschke product [11, Theorem 4.1, p. 130].

Similarly, the solution to the second extremal problem, with holomorphic functions in place of $\text{Rat}(\Omega)$, is a function which is holomorphic in a neighborhood of $\bar{\Omega}$ [11, Theorem 1.6, p. 114]. Hence it is the limit of functions from $\text{Rat}(\Omega)$. The following Lemma first appeared as [13, Remark 2, p. 308].

LEMMA 3. *The homomorphism ρ_{A_s} is contractive if and only if $\|r(A_s)\| \leq 1$ for all r in $\text{Rat}(\Omega)$ with $\|r\| \leq 1$ and $r(z_1) = 0$.*

The homomorphism ρ_{B_t} is contractive if and only if $\|r(B_t)\| \leq 1$ for all r in $\text{Rat}(\Omega)$ with $\|r\| \leq 1$ and $r(z) = 0$.

Proof. The two proofs are similar, so we shall prove only (1). Suppose $r(A)$ is a contraction for all $r \in \text{Rat}(\Omega)$ with $\|r\| \leq 1$ and $r(z_1) = 0$. We have to prove that $r(A)$ is a contraction for all $r \in \text{Rat}(\Omega)$ with $\|r\| \leq 1$. For any such rational function r , let $r(z) = u$. Put $\varphi_u(z) = \frac{z-u}{1-\bar{u}z}$ and $\psi(z) = \varphi_u(r(z))$. Then ψ is in $\text{Rat}(\Omega)$, $\|\psi\| \leq 1$ and $\psi(z) = 0$. By hypothesis, $\psi(A)$ is a contraction. Now note that $\varphi_u^{-1}(z) = \frac{z+u}{1+\bar{u}z}$. Since φ_u^{-1} maps \mathbb{D} into \mathbb{D} , by von Neumann's inequality, $\|r(A)\| = \|\varphi_u^{-1}\psi(A)\| \leq 1$. \square

This lemma makes it somewhat simple to derive the contractivity conditions for the homomorphisms induced by A_s and B_t .

LEMMA 4. The homomorphism ρ_{A_s} is contractive if and only if

$$s^2 = s_\Omega(z_1, z_2)^{-1} - 1 \text{ and } |\mu| \leq 1.$$

Similarly, the homomorphism ρ_{B_t} is contractive if and only if

$$t = t_\Omega(z)^{-1} \text{ and } |\lambda| \leq 1.$$

Proof. First, using the functional calculus for A_s , we see that

$$r \begin{pmatrix} z_1 & 0 \\ s\mu(z_1 - z_2) & z_2 \end{pmatrix} = \begin{pmatrix} r(z_1) & 0 \\ s\mu(r(z_1) - r(z_2)) & r(z_2) \end{pmatrix} = \begin{pmatrix} r(z_1) & 0 \\ s\mu r(z_1) & 0 \end{pmatrix},$$

assuming $r(z_2) = 0$. Therefore, contractivity of ρ_{A_s} would imply

$$\begin{aligned} s^2|\mu|^2 + 1 &\leq (\sup \{|r(z_2)|^2 : r \in \text{Rat}(\Omega), \|r\| \leq 1 \text{ and } r(z_2) = 0\})^{-1} \\ &= s_\Omega(z_1, z_2)^{-1}. \end{aligned}$$

Or, equivalently, if we put $s = s_\Omega(z_1, z_2)^{-1} - 1$, then we must have $|\mu| \leq 1$. Now an application of Lemma 3 completes the proof.

To obtain the contractivity condition for ρ_{B_t} , using the functional calculus, we see that

$$r \begin{pmatrix} z & 0 \\ t\lambda & z \end{pmatrix} = \begin{pmatrix} r(z) & 0 \\ t\lambda r'(z) & r(z) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a\lambda r'(z) & 0 \end{pmatrix}$$

assuming $r(z) = 0$. Therefore, contractivity of ρ_{B_t} would imply that

$$t|\lambda| \leq (\sup \{|r'(w)| : r \in \text{Rat}(\Omega), \|r\| \leq 1 \text{ and } r(w) = 0\})^{-1} = t_\Omega(z)^{-1}.$$

Or equivalently, if we put $t = t_\Omega(z)^{-1}$, then we must have $|\lambda| \leq 1$. \square

We now have enough material to construct the dilation for a homomorphism $\rho_T : \mathcal{A}(\Omega) \rightarrow \mathcal{M}_2$. In this case, T is a 2×2 matrix with spectrum in Ω . Since we can apply a unitary conjugation to make T upper-triangular, it is enough to exhibit the dilation for the two matrices $T = A_s$ and $T = B_t$.

3.3. Dilation for A_s . Recall that there exists an α_0 depending only on z_1 and z_2 such that $\det M((0, a), \alpha_0) = 0$. For now, set $\alpha_0 = \alpha$. Let the subspace \mathcal{M} of $H_\alpha^2 \oplus H_\alpha^2$ be as in the first part of Lemma 2. For brevity, let

$$m^2 = 1 - \frac{|K_\alpha(z_1, z_2)|^2}{K_\alpha(z_1, z_1)K_\alpha(z_2, z_2)} > 0.$$

Then

$$\det M((0, m), \alpha) = \begin{pmatrix} K_\alpha(z_1, z_1) & K_\alpha(z_1, z_2) \\ K_\alpha(z_2, z_1) & (1 - m^2)K_\alpha(z_2, z_2) \end{pmatrix} = 0$$

by the definition of m . As we have pointed out earlier, there is a holomorphic function $f : \Omega \rightarrow \mathbb{D}$ such that $f(z_1) = 0$ and $f(z_2) = m$. Moreover, if g is any holomorphic function from Ω to \mathbb{D} such that $g(z_1) = 0$, then the matrix $M((0, g(z_2)), \alpha)$ is positive semidefinite, which implies that

$$|g(z_2)|^2 \leq 1 - \frac{|K_\alpha(z_1, z_2)|^2}{K_\alpha(z_1, z_1)K_\alpha(z_2, z_2)}.$$

Thus

$$m = \sup \left\{ |g(z_2)| : g \text{ is a holomorphic function from } \Omega \text{ to } \mathbb{D} \text{ and } g(z_1) = 0 \right\}.$$

Hence

$$s_\Omega(z_1, z_2)^{-1} - 1 = \frac{1}{m^2} - 1 = \frac{|K_\alpha(z_1, z_2)|^2}{K_\alpha(z_1, z_1)K_\alpha(z_2, z_2) - |K_\alpha(z_1, z_2)|^2} = s_K^2.$$

So by the first part of Lemma 2, we have that the restriction of the operator $M^* \oplus M^*$ to the subspace \mathcal{M} in the orthonormal basis $\{h_1(z_1, z_2), h_2(z_1, z_2)\}$ has the matrix representation A_s^* with $s^2 = s_\Omega(z_1, z_2)^{-1} - 1$ whenever $|\mu| \leq 1$.

3.4. Having constructed the dilation, it is natural to calculate the characteristic function, in the sense of Sz.-Nagy and Foias, when $\Omega = \mathbb{D}$. In this case, the general form of the matrix T discussed above is

$$(3.7) \quad T := \begin{pmatrix} z_1 & 0 \\ \mu(1 - |z_1|^2)^{1/2}(1 - |z_2|^2)^{1/2} & z_2 \end{pmatrix},$$

where z_1 and z_2 are two points in the open unit disk \mathbb{D} and $\mu \in \mathbb{C}$, $|\mu| \leq 1$. We are using the explicit value of $s_{\mathbb{D}}(z_1, z_2)$ for the unit disc.

LEMMA 5. For $i = 1, 2$, let $\varphi_i(z) = (z - z_i)/(1 - \bar{z}_i z)$. The characteristic function of T is

$$\theta_T(z) = \begin{pmatrix} (1 - |\mu|^2)^{1/2} \varphi_2(z) & -\mu \\ \bar{\mu} \varphi_1(z) \varphi_2(z) & (1 - |\mu|^2)^{1/2} \varphi_1(z) \end{pmatrix}.$$

Proof. Recall that \mathcal{M} is the subspace spanned by the orthonormal vectors $h_1(z_1, z_2)$ and $h_2(z_1, z_2)$. Since the compression of $M \oplus M$ to the co-invariant subspace \mathcal{M} is T , by Beurling-Lax-Halmos theorem, we need to only find up to unitary coincidence (see [25], page 192, for definitions) the inner function whose range is \mathcal{M}^\perp . So let $\begin{pmatrix} f \\ g \end{pmatrix}$ be a vector in the orthogonal complement of \mathcal{M} . The condition of orthogonality to h_1 implies that $g(z_1) = 0$, which is equivalent to $g = \varphi_1 \xi$ for arbitrary $\xi \in H^2(\mathbb{D})$. Now the orthogonality condition to h_2 implies that $(1 - |\mu|^2)^{1/2} f(z_2) + \mu \xi(z_2) = 0$, which is the same as

$$(3.8) \quad (1 - |\mu|^2)^{1/2} \varphi_1(z_2) f(z_2) + \mu g(z_2) = 0.$$

This implies that there is an $\eta_1 \in H^2(\mathbb{D})$ such that

$$(1 - |\mu|^2)^{1/2} f + \mu g' = \varphi_2 \eta_1.$$

It is obvious that, conversely, if $\begin{pmatrix} f \\ g \end{pmatrix}$ is a function from $H^2(\mathbb{D}) \oplus H^2(\mathbb{D})$ such that g is in range of φ and satisfies (3.8), then it is in the orthogonal complement of \mathcal{M} .

Now let $\eta_2 = (1 - |\mu|^2)^{1/2} \xi - \bar{\mu} f$. Then

$$\theta \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} (1 - |\mu|^2)^{1/2} \varphi_2 \eta_1 - \mu \eta_2 \\ \bar{\mu} \varphi_1 \varphi_2 \eta_1 + (1 - |\mu|^2)^{1/2} \varphi_1 \eta_2 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

Thus if $\begin{pmatrix} f \\ g \end{pmatrix}$ satisfies (3.8), then it is in the range of θ . Conversely, it is easy to see that any element in the range of θ will satisfy (3.8). Thus the orthogonal complement of \mathcal{M} in \mathcal{H} is the range of θ . So θ is the characteristic function of the given matrix. \square

We would like to remark here that for $z_1 = z_2$ the characteristic function $\theta_T(u)$ for $T := \begin{pmatrix} z & 0 \\ \lambda(1 - |z|^2) & z \end{pmatrix}$, $|\lambda| \leq 1$, can be obtained directly from the definition in case $z = 0$. A little computation, using the transformation rule for the characteristic function under a biholomorphic automorphism of the unit disk [25, pp. 239–240], produces the formula

$$\theta_T(u) = \begin{pmatrix} (1 - |\lambda|^2)^{1/2} \varphi(u) & \lambda \\ \bar{\lambda} \varphi^2(u) & (1 - |\lambda|^2)^{1/2} \varphi(u) \end{pmatrix}, \quad u \in \mathbb{D}$$

in the general case.

Let T_μ be the matrix defined in (3.7). Note that if $T_{\mu'}$ and T_μ are two such matrices with $|\mu'| = |\mu|$, then

$$\begin{aligned} \theta_{T_{\mu'}}(z) &= \begin{pmatrix} (1 - |\mu'|^2)^{1/2}\varphi_2 & -\mu' \\ \bar{\mu}'\varphi_1\varphi_2 & (1 - |\mu'|^2)^{1/2}\varphi_1 \end{pmatrix} \\ &= \begin{pmatrix} (1 - |\mu|^2)^{1/2}\varphi_2 & -e^{i\psi}\mu \\ e^{-i\psi}\bar{\mu}\varphi_1\varphi_2 & (1 - |\mu|^2)^{1/2}\varphi_1 \end{pmatrix} \text{ for some } \psi \in [0, 2\pi] \\ &= \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{i\psi/2} \end{pmatrix} \begin{pmatrix} (1 - |\mu|^2)^{1/2}\varphi_2 & -\mu \\ \bar{\mu}\varphi_1\varphi_2 & (1 - |\mu|^2)^{1/2}\varphi_1 \end{pmatrix} \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{i\psi/2} \end{pmatrix} \\ &= \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{i\psi/2} \end{pmatrix} \theta_{T_\mu}(z) \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{i\psi/2} \end{pmatrix}, \end{aligned}$$

and hence their characteristic functions coincide. So they are unitarily equivalent. Conversely, if $T_{\mu'}$ and T_μ are unitarily equivalent, then their characteristic functions coincide and hence the singular values of the characteristic functions are same. Note that when $z_1 \neq z_2$, we have

$$\theta_{T_{\mu'}}(z_1)\theta_{T_{\mu'}}(z_1)^* = \begin{pmatrix} (1 - |\mu'|^2)|\omega|^2 + |\mu'|^2 & 0 \\ 0 & 0 \end{pmatrix}$$

for some $\omega \in \mathbb{C}$ (independent of μ'). When $z_1 = z_2$, then

$$\theta_{T_{\mu'}}(z_1)\theta_{T_{\mu'}}(z_1)^* = \begin{pmatrix} 0 & |\mu'|^2 \\ 0 & 0 \end{pmatrix}.$$

In either case, coincidence of $\theta_{T_{\mu'}}$ and θ_{T_μ} means that $|\mu'| = |\mu|$. Thus using the explicit characteristic function we have proved the following.

THEOREM 6. *Two matrices $T_{\mu'}$ and T_μ as defined in (3.7) are unitarily equivalent if and only if $|\mu'| = |\mu|$.*

3.5. Dilation for B_t . We now shift our attention to the construction of dilation when the homomorphism ρ_T is induced by a 2×2 matrix T with equal eigenvalues. So $\sigma(T) = \{z\}$. The domain Ω has its associated Szego kernel, which is denoted by $\hat{K}_\Omega(z, w)$. Recall that a generalization due to Ahlfors to multiply connected domains of the Schwarz lemma says that

$$t_\Omega(z) := (\sup \{|r'(z)| : r \in \text{Rat}(\Omega), \|r\| \leq 1 \text{ and } r(z) = 0\})^{-1} = \hat{K}_\Omega(z, z)^{-1}.$$

Let $\partial\Omega$ be the topological boundary of Ω and let $|d\nu|$ be the arc-length measure on $\partial\Omega$. Consider the measure $dm = |\hat{K}_\Omega(\nu, z)|^2|d\nu|$, and let the associated Hardy space $H^2(\Omega, dm)$ be denoted by \mathcal{H} . The measure dm is mutually absolutely continuous with respect to the arc-length measure. Thus the evaluation functionals on \mathcal{H} are bounded and hence \mathcal{H} possesses a reproducing kernel K . Then it is known that K satisfies the property

$$\hat{K}_\Omega(z, z)^{-1} = \frac{K(z, z)}{(K(z, z)\|\partial_{\bar{z}}K(\cdot, z)\|^2 - |\langle \partial_{\bar{z}}K(\cdot, z), K(\cdot, z) \rangle|^2)^{1/2}} = t_K;$$

see [13, Theorem 2.2]. Now a (subnormal) dilation for $B_t := \begin{pmatrix} \lambda_{t\Omega}(z) & 0 \\ 0 & z \end{pmatrix}$, where $|\lambda| \leq 1$, is the operator $M \oplus M$ on the Hilbert space $\mathcal{H} \oplus \mathcal{H}$. This is easily verified since the restriction of $M^* \oplus M^*$ to the subspace \mathcal{N} , which was described in the second part of Lemma 2, is B_t^* .

REMARK 7. If we choose $|\mu| = 1$ then A_s^* is the restriction of M^* to the two-dimensional subspace spanned by the vectors $K_\alpha(\cdot, z_1)$ and $K_\alpha(\cdot, z_2)$ in the Hardy space $H_\alpha^2(\Omega)$ by our construction. Except in this case, the dilation of the homomorphism ρ_{A_s} we have constructed is a minimal subnormal dilation. (This dilation then may be extended to a minimal normal dilation.) While it is known that a minimal dilation is not unique when Ω is finitely connected, our construction gives a measure of this non-uniqueness. More explicitly, for each $\alpha_0 \in \mathbb{T}^m$ for which

$$\sup \left\{ \frac{|K_\alpha(z_1, z_2)|^2}{K_\alpha(z_1, z_1)K_\alpha(z_2, z_2)} : \alpha \in \mathbb{T}^m \right\} = \frac{|K_{\alpha_0}(z_1, z_2)|^2}{K_{\alpha_0}(z_1, z_1)K_{\alpha_0}(z_2, z_2)},$$

the matrix representation of the operator $M^* \oplus M^*$ restricted to the 2-dimensional subspace \mathcal{M} of the Hilbert space $H_{\alpha_0}^2 \oplus H_{\alpha_0}^2$ equals A_s .

4. The operator space

The problem that we are considering naturally gives rise to an operator space structure. In this section, we show that. We begin by recalling basic definitions.

A vector space X is called an operator space if for each $k \in \mathbb{N}$ there are norms $\|\cdot\|_k$ on $X \otimes \mathcal{M}_k$ satisfying:

- (1) Whenever $A = ((a_{ij})) \in \mathcal{M}_k, ((x_{ij})) \in X \otimes \mathcal{M}_k$ and $B = ((b_{ij})) \in \mathcal{M}_k$, then

$$\|A \cdot ((x_{ij})) \cdot B\|_k \leq \|A\| \|((x_{ij}))\|_k \|B\|$$

where

$$A \cdot ((x_{ij})) \cdot B = \left(\left(\sum_{p=1}^m \sum_{l=1}^k a_{ip} x_{pl} b_{lj} \right) \right) \in X \otimes \mathcal{M}_k$$

and $\|A\|$ and $\|B\|$ are operator norms on $\mathcal{M}_k = \mathcal{B}(\mathbb{C}^k)$.

- (2) For all positive integers m, k and for all $R \in X \otimes \mathcal{M}_k$ and $S \in X \otimes \mathcal{M}_m$, we have

$$\left\| \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} \right\|_{m+k} = \max\{\|R\|_m, \|S\|_k\}.$$

Two such operator spaces $(X, \|\cdot\|_{X,k})$ and $(Y, \|\cdot\|_{Y,k})$ are said to be completely isometric if there is a linear bijection $\tau : X \rightarrow Y$ such that $\tau \otimes I_k : (X, \|\cdot\|_{X,k}) \rightarrow (Y, \|\cdot\|_{Y,k})$ is an isometry for every $k \in \mathbb{N}$.

Let X be an operator space and let $\rho : X \rightarrow \mathcal{B}(\mathcal{H})$ be a linear map, where \mathcal{H} is a Hilbert space. If for each $k \in \mathbb{N}$ the map $\rho \otimes I_k : (X, \|\cdot\|_k) \rightarrow \mathcal{B}(\mathcal{H} \otimes \mathcal{M}_k)$

is contractive then ρ is said to be *completely contractive*. Let \mathcal{H} be finite-dimensional, let $T \in \mathcal{B}(\mathcal{H})$, let $X = \mathcal{A}(\Omega)$ and let $\rho = \rho_T$ be as defined earlier. We assume that the eigenvalues z_1, z_2, \dots, z_n of T are distinct.

To begin with, we introduce a notation. We denote by $I_{\underline{z}}^k$ the subset of $\mathbb{C}^n \otimes \mathcal{M}_k$ defined as

$$I_{\underline{z}}^k = \{(R(z_1), R(z_2), \dots, R(z_n)) : R \in \mathcal{A}(\Omega) \otimes \mathcal{M}_k \text{ and } \|R\| \leq 1\},$$

where $\|R\| = \sup_{z \in \bar{\Omega}} \|R(z)\|$. When $k = 1$, we denote it by $I_{\underline{z}}$ rather than $I_{\underline{z}}^1$.

LEMMA 8. *The set $I_{\underline{z}}$ defined above is a compact set.*

Proof. Clearly, $I_{\underline{z}}$ is a subset of \bar{D}^n . So it is enough to show that $I_{\underline{z}}$ is a closed set. Recall from Section 3 that the generalization of the Nevanlinna-Pick theorem due to Abrahamse states that given n points w_1, w_2, \dots, w_n in the open unit disk, there is a holomorphic function $f : \Omega \rightarrow \mathbb{C}$ with $f(z_i) = w_i$ for $i = 1, 2, \dots, n$ if and only if the matrix

$$(4.1) \quad M(\underline{w}, \alpha) \stackrel{\text{def}}{=} ((1 - w_i \bar{w}_j) K_\alpha(z_i, z_j))$$

is positive semidefinite for all $\alpha \in \mathbb{T}^m$. So

$$\begin{aligned} I_{\underline{z}} &= \{(w_1, w_2, \dots, w_n) \in \bar{\mathbb{D}}^n : \text{the matrix } M(\underline{w}, \alpha) \\ &\quad \text{is positive semidefinite for all } \alpha \in \mathbb{T}^m\} \\ &= \{(w_1, w_2, \dots, w_n) \in \bar{\mathbb{D}}^n : \lambda_{\min}(M(\underline{w}, \alpha)) \geq 0 \text{ for all } \alpha \in \mathbb{T}^m\} \\ &= \bigcap_{\alpha \in \mathbb{T}^m} \{(w_1, w_2, \dots, w_n) \in \bar{\mathbb{D}}^n : \lambda_{\min}(M(\underline{w}, \alpha)) \geq 0\} \\ &= \bigcap_{\alpha \in \mathbb{T}^m} (\lambda_{\min}(M(\underline{w}, \alpha)))^{-1}([0, \infty)), \end{aligned}$$

where $\lambda_{\min}(A)$ for a hermitian matrix A denotes its smallest eigenvalue. It is a continuous function on the set of hermitian matrices (see, for example, [7, Corollary III.2.6]). Thus $\underline{w} \rightarrow \lambda_{\min}(M(\underline{w}, \alpha))$ is a continuous function on \mathbb{C}^n . Since an arbitrary intersection of closed sets is closed, $I_{\underline{z}}$ is a closed set. \square

It is easy to see that the set $I_{\underline{z}}^k$ is convex and balanced, so it is the closed unit ball of some norm on $\mathbb{C}^n \otimes \mathcal{M}_k$. The sets of the form $I_{\underline{z}}^k$ were first studied, in the case $k = 1$, by Cole and Wermer [8]. The sets $I_{\underline{z}}^k$ are examples of *matrixially hyperconvex* sets studied by Paulsen in [21]. Paulsen points out that the sequence of sets $I_{\underline{z}}^k \subseteq \mathbb{C}^n \otimes \mathcal{M}_k$ determines an operator space structure on \mathbb{C}^m , that is, the set $I_{\underline{z}}^k$ determines a norm $\|\cdot\|_{\underline{z},k}$ in $\mathbb{C}^n \otimes \mathcal{M}_k$ such that $I_{\underline{z}}^k$ is the closed unit ball in this norm and the sequence $\{\mathbb{C}^n \otimes \mathcal{M}_k, \|\cdot\|_{\underline{z},k}\}$ satisfies the conditions (1) and (2) above. We denote this operator space by $\text{HC}_{\Omega, \underline{z}}(\mathbb{C}^n)$. Paulsen also notes that this operator space is completely isometric to a quotient of a function algebra. Indeed, it is not difficult to see

that $\text{HC}_{\Omega, \underline{z}}(\mathbb{C}^n)$ is completely isometrically isomorphic to the quotient of the operator algebra $\mathcal{A}(\Omega)$ by \mathcal{Z} , where

$$\mathcal{Z} = \{f \in \mathcal{A}(\Omega) : f(z_1) = f(z_2) = \dots = f(z_n) = 0\}.$$

If $k = 1$, we will write $\|\cdot\|_{\underline{z}}$ rather than $\|\cdot\|_{z,1}$.

LEMMA 9. *There are n matrices $V_1, V_2, \dots, V_n \in \mathcal{M}_n$ such that the map $\rho_T \otimes I_k : \mathcal{A}(\Omega) \otimes \mathcal{M}_k \rightarrow \mathcal{M}_n \otimes \mathcal{M}_k$ is of the form*

$$(\rho_T \otimes I_k)R = V_1 \otimes R(z_1) + V_2 \otimes R(z_2) + \dots + V_n \otimes R(z_n)$$

for any $R \in \mathcal{A}(\Omega) \otimes \mathcal{M}_k$ and any $k \in \mathbb{N}$. The matrices V_i depend on the set $\{z_1, z_2, \dots, z_n\}$.

Proof. If F and G are two elements of $\mathcal{A}(\Omega) \otimes \mathcal{M}_k$ which agree on the set $\{z_1, z_2, \dots, z_n\}$, then define $H \in \mathcal{A}(\Omega) \otimes \mathcal{M}_k$ by $H = F - G$. Then H vanishes at the points z_1, z_2, \dots, z_n and hence $H(z) = (z - z_1)(z - z_2) \dots (z - z_n)W(z)$ for some W in $\mathcal{A}(\Omega) \otimes \mathcal{M}_k$. By the functional calculus,

$$(\rho_T \otimes I_k)H = (T - z_1)(T - z_2) \dots (T - z_n)W(T).$$

Note that $(z - z_1)(z - z_2) \dots (z - z_n)$ is the characteristic polynomial of T and by Cayley-Hamilton theorem, $(T - z_1)(T - z_2) \dots (T - z_n) = 0$. Thus $(\rho_T \otimes I_k)H = 0$. So if $F, G \in \mathcal{A}(\Omega) \otimes \mathcal{M}_k$ are such that $F(z_i) = G(z_i)$ for all $i = 1, 2, \dots, n$, then $(\rho_T \otimes I_k)F = (\rho_T \otimes I_k)G$. Now define V_1, V_2, \dots, V_n by

$$V_i = \rho_T \left(\frac{(z - z_1) \dots (z - z_{i-1})(z - z_{i+1}) \dots (z - z_n)}{(z_i - z_1) \dots (z_i - z_{i-1})(z_i - z_{i+1}) \dots (z_i - z_n)} \right)$$

for $i = 1, 2, \dots, n$. Given $R \in \mathcal{A}(\Omega) \otimes \mathcal{M}_k$, it agrees with the function

$$\tilde{R}(z) = \sum_{i=1}^n \frac{(z - z_1) \dots (z - z_{i-1})(z - z_{i+1}) \dots (z - z_n)}{(z_i - z_1) \dots (z_i - z_{i-1})(z_i - z_{i+1}) \dots (z_i - z_n)} R(z_i)$$

on the set $\{z_1, z_2, \dots, z_n\}$ and hence

$$\begin{aligned} (\rho_T \otimes I_k)R &= (\rho_T \otimes I_k)\tilde{R} \\ &= \sum_{i=1}^n \rho_T \left(\frac{(z - z_1) \dots (z - z_{i-1})(z - z_{i+1}) \dots (z - z_n)}{(z_i - z_1) \dots (z_i - z_{i-1})(z_i - z_{i+1}) \dots (z_i - z_n)} \right) \otimes R(z_i) \\ &= \sum_{i=1}^n V_i \otimes R(z_i) \end{aligned}$$

completing the proof of the lemma. □

The referee points out that $V_i^2 = V_i$, $1 \leq i \leq n$, and $V_1 + \dots + V_n = I_n$. In particular, V_1, \dots, V_n of the preceding lemma cannot be arbitrary.

At this point, we note that $\mathcal{A}(\Omega)$ being a closed sub-algebra of the commutative C^* -algebra of all continuous functions on the boundary of Ω inherits a natural operator space structure, denoted by $\text{MIN}(\mathcal{A}(\Omega))$. Recall

that a celebrated theorem of Arveson says that a contractive homomorphism $\rho_T : \mathcal{A}(\Omega) \rightarrow \mathcal{L}(\mathcal{H})$ dilates if and only if it is completely contractive when $\mathcal{A}(\Omega)$ is equipped with the MIN operator space structure. The contractivity and complete contractivity of the homomorphism ρ_T amount to, respectively,

$$(4.2) \quad \sup \{ \|w_1 V_1 + \cdots + w_n V_n\| : \underline{w} = (w_1, \dots, w_n) \in I_{\underline{z}} \} \leq 1,$$

where $\|\cdot\|$ is the operator norm on \mathcal{M}_n , and

$$(4.3) \quad \sup \left\{ \left\| \sum_{i=1}^n V_i \otimes W_i \right\| : W_i \in \mathcal{M}_k \text{ and } W = (W_1, \dots, W_n) \in I_{\underline{z}}^k \text{ for } k \geq 1 \right\} \leq 1,$$

where $\|\cdot\|$ is the operator norm on $\mathcal{M}_n \otimes \mathcal{M}_k$. Now, we state the following theorem whose proof is evident from the discussion above.

THEOREM 10. *The contractive homomorphism $\rho_T : \mathcal{A}(\Omega) \rightarrow \mathcal{M}_n$ is completely contractive with respect to the MIN operator space structure on $\mathcal{A}(\Omega)$ if and only if the contractive linear map $L_T : (\mathbb{C}^n, \|\cdot\|_{\underline{z}}) \rightarrow \mathcal{M}_n$ defined by $L_T(\underline{w}) = w_1 V_1 + w_2 V_2 + \cdots + w_n V_n$ is completely contractive on the operator space $\text{HC}_{\Omega, \underline{z}}(\mathbb{C}^n)$.*

The theorem above brings us to our concluding remarks of this section. Given a Banach space, there are two extremal operator space structures on it, denoted by $\text{MAX}(X)$ and $\text{MIN}(X)$. We refer the reader to [20] for definitions and basic details. However, this theorem shows that if $\text{HC}_{\Omega, \underline{z}}(\mathbb{C}^n)$ is completely isometric to $\text{MAX}(\mathbb{C}^n, \|\cdot\|_{\underline{z}})$, then every contractive homomorphism ρ_T of the algebra $\mathcal{A}(\Omega)$, induced by an n -dimensional linear transformation T with distinct eigenvalues in Ω , will necessarily dilate. This gives rise to the question of determining when $\text{HC}_{\Omega, \underline{z}}(\mathbb{C}^n)$ is the same as $\text{MAX}(\mathbb{C}^n, \|\cdot\|_{\underline{z}})$, which is an interesting question in its own right.

In [20], Paulsen related a problem similar to the one that we are considering to certain questions in the setting of operator spaces and thereby solved it. For $n \geq 1$, let G be a closed unit ball in \mathbb{C}^n corresponding to a norm $\|\cdot\|_G$ on \mathbb{C}^n . Let $\mathcal{A}(G)$ denote the closure of polynomials in $C(G)$, the algebra of all continuous functions on G equipped with the sup norm. He showed that if $\text{MIN}(\mathbb{C}^n, \|\cdot\|_G)$ is not completely isometric to $\text{MAX}(\mathbb{C}^n, \|\cdot\|_G)$, then there exists a unital contractive homomorphism $\rho : \mathcal{A}(G) \rightarrow \mathcal{B}(\mathcal{H})$, for some Hilbert space \mathcal{H} , which is not completely contractive. Paulsen proved the remarkable result that for $n \geq 5$,

$$(4.4) \quad \text{MIN}(\mathbb{C}^n, \|\cdot\|_G) \text{ is not completely isometric to } \text{MAX}(\mathbb{C}^n, \|\cdot\|_G),$$

for any closed unit ball G . For $n = 2$ and $G = \mathbb{D}^2$, Ando's theorem implies that $\text{MIN}(\mathbb{C}^2, \|\cdot\|_{\mathbb{D}^2})$ is completely isometric to $\text{MAX}(\mathbb{C}^2, \|\cdot\|_{\mathbb{D}^2})$. The fact that (4.4) holds for $n \geq 3$ and any closed unit ball G is pointed out in [22, Exercise 3.7].

In the same spirit, a similar question about a class of homomorphisms, first introduced by Parrott [19] (see also [14], [15] and [16]), led Paulsen to define a natural operator space, which he called COT. Let G be a unit ball and let \underline{w} be a point in the interior of G . Let X be the Banach space $X = (\mathbb{C}^n, \|\cdot\|_{G, \underline{w}})$, where $\|\cdot\|_{G, \underline{w}}$ is the Caratheodory norm of G at the point \underline{w} . The question of whether $\text{COT}_{\underline{w}}(X)$ is completely isometric to $\text{MIN}(X^*)$ for $\underline{w} \in G$ was first raised in [20]. Paulsen showed that the answer is affirmative when $\underline{w} = 0$. Later in an unpublished note, it was shown by Dash [9] that $\text{COT}_{\underline{w}}(G)$ and $\text{MIN}(X^*)$ are not necessarily completely isometric. The question of deciding whether a contractive homomorphism $\rho_T : \mathcal{A}(\Omega) \rightarrow \mathcal{B}(\mathcal{H})$ is completely contractive or not is similar in nature. It amounts to deciding if $\text{HC}_{\Omega, \underline{z}}(\mathbb{C}^n)$ is completely isometric to $\text{MIN}(\mathbb{C}^n, \|\cdot\|_z)$ or $\text{MAX}(\mathbb{C}^n, \|\cdot\|_z)$. It is likely that the operator space $\text{HC}_{\Omega, \underline{z}}(\mathbb{C}^n)$ is completely isometric to $\text{MIN}(\mathbb{C}^n, \|\cdot\|_z)$ for every $n \geq 3$. We pose this as an open problem whose solution defies us at the moment.

5. A factorization condition

Let T be a linear transformation on an n -dimensional Hilbert space V with distinct eigenvalues z_1, z_2, \dots, z_n in Ω . Let v_1, v_2, \dots, v_n be the n linearly independent eigenvectors of T^* . If $\sigma = \{z_1, z_2, \dots, z_n\}$, then define a positive definite function $K : \sigma \times \sigma \rightarrow \mathbb{C}$ by setting

$$(5.1) \quad \left((K(z_j, z_i))_{i,j} \right)^n := \left(\langle v_i, v_j \rangle \right)_{i,j=1}^n.$$

As before, let $\rho_T : \mathcal{A}(\Omega) \rightarrow \mathcal{L}(V)$ be the homomorphism induced by T . Suppose there exists a dilation of the homomorphism ρ_T . Then it follows from [1, Theorem 2] that there is a flat unitary vector bundle \mathcal{E} of rank n (see [2] for definitions and complete results on the model theory in multiply connected domains) such that $\rho_T(f)$ is the compression of the subnormal operator M_f on the Hardy space $H^2_{\mathcal{E}}(\Omega)$ to a semi-invariant subspace in it. Consequently, for some choice of a flat unitary vector bundle \mathcal{E} , the homomorphism

$$(5.2) \quad \rho_M : \mathcal{A}(\Omega) \rightarrow \mathcal{B}(H^2_{\mathcal{E}}(\Omega))$$

dilates ρ_T . The homomorphism ρ_M is induced by the multiplication operator M on $H^2_{\mathcal{E}}(\Omega)$, which is subnormal. Thus the homomorphism $\rho_N : C(\partial\Omega) \rightarrow \mathcal{B}(\mathcal{H})$ induced by the normal extension N on the Hilbert space $\mathcal{H} \supseteq H^2_{\mathcal{E}}(\Omega)$ of the operator M is a dilation of the homomorphism ρ_T in the sense of (1.1). The multiplication operator M on $H^2_{\mathcal{E}}$ is called a bundle shift. We recall [2, Theorem 3] that $\dim \ker(M - z)^* = n$. Let $K^{\mathcal{E}}_z(i), i = 1, 2, \dots, n$, be a basis (not necessarily orthogonal) of $\ker(M - z)^*$. We set

$$(5.3) \quad K^{\mathcal{E}}(z_j, z_i) := \left(\langle K^{\mathcal{E}}_{z_i}(\ell), K^{\mathcal{E}}_{z_j}(p) \rangle \right)_{\ell, p=1}^n, \text{ for } 1 \leq i, j \leq n.$$

If ρ_T dilates, then the linear transformation T can be realized as the compression of the operator M on $H^2_{\mathcal{E}}(\Omega)$ to an n -dimensional co-invariant subspace, say $\mathfrak{M} \subseteq H^2_{\mathcal{E}}(\Omega)$. The subspace \mathfrak{M} must consist of eigenvectors of the bundle shift M . Let $x_i, 1 \leq i \leq n$, be a set of n vectors in \mathbb{C}^n and

$$\mathfrak{M} = \left\{ \sum_{\ell=1}^n x_i(\ell) K_{z_i}^{\mathcal{E}}(\ell) : 1 \leq i \leq n \right\}.$$

The map which sends v_i to $\sum_{\ell=1}^n x_i(\ell) K_{z_i}^{\mathcal{E}}(\ell), 1 \leq i \leq n$, intertwines T^* and the restriction of M^* to \mathfrak{M} . For this map to be an isometry as well, we must have

$$(5.4) \quad \langle v_i, v_j \rangle = \langle K^{\mathcal{E}}(z_j, z_i) x_i, x_j \rangle, \quad x_i \in \mathbb{C}^n, \quad 1 \leq i, j \leq n.$$

Conversely, if there is a flat unitary vector bundle \mathcal{E} and n vectors x_1, \dots, x_n in \mathbb{C}^n satisfying (5.4), then ρ_T obviously dilates. So we have proved the following theorem.

THEOREM 11. *The homomorphism ρ_T is dilatable in the sense of (1.1) if and only if the kernel K , as defined in (5.1), can be written as*

$$K(z_j, z_i) = \langle K^{\mathcal{E}}(z_j, z_i) x_i, x_j \rangle, \quad \text{for some choice of } x_1, \dots, x_n \in \mathbb{C}^n,$$

and some choice of a flat unitary vector bundle \mathcal{E} of rank n .

It is interesting to see how contractivity of ρ_T is related to the above theorem. Note that ρ_T is contractive if and only if $\|f(T)^*\| \leq \|f\|$ by the definition of ρ_T . Since $T^*v_i = \bar{z}_i v_i$, we note that $f(T)^*v_i = \overline{f(z_i)}v_i$, for $1 \leq i \leq n$ and $f \in \text{Rat}(\Omega)$. It then follows that

$$\begin{aligned} \|\rho_T(f)^*\|^2 &= \sup \left\{ \left\| f(T)^* \left(\sum_{i=1}^n \alpha_i v_i \right) \right\|^2 : \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C} \right\} \\ &= \sup \left\{ \left\| \sum_{i=1}^n \alpha_i \overline{f(z_i)} v_i \right\|^2 : \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C} \right\} \\ &= \sup \left\{ \sum_{i,j=1}^n \alpha_i \bar{\alpha}_j \overline{f(z_i)} f(z_j) \langle v_i, v_j \rangle : \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C} \right\}. \end{aligned}$$

Therefore, $\|f(T)^*\| \leq \|f\|$ if and only if

$$\sum_{i,j=1}^n \alpha_i \bar{\alpha}_j \overline{f(z_i)} f(z_j) \langle v_i, v_j \rangle \leq \sum_{i,j=1}^n \alpha_i \bar{\alpha}_j \langle v_i, v_j \rangle,$$

for all $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$ and all $f \in \text{Rat}(\Omega)$ with $\|f\| \leq 1$. Thus contractivity of ρ_T is equivalent to non-negative definiteness of the matrix

$$(5.5) \quad \left((1 - \overline{f(z_i)} f(z_j)) K(z_j, z_i) \right)_{i,j=0}^n,$$

for all $f \in \text{Rat}(\Omega)$, $\|f\| \leq 1$. If ρ_T is dilatable then the theorem above tells us that

$$(5.6) \quad \left((1 - \overline{f(z_i)}f(z_j))K(z_j, z_i) \right)_{i,j=0}^n = \left((1 - \overline{f(z_i)}f(z_j))\langle K_{\mathcal{E}}(z_j, z_i)x_i, x_j \rangle \right)_{i,j=0}^n.$$

The last matrix is non-negative definite because M on $H_{\mathcal{E}}^2(\Omega)$ induces a contractive homomorphism. We therefore see, in this concrete fashion, that if the homomorphism ρ_T were dilatable then it would be contractive.

The interesting point to note here is that our construction of the dilation of ρ_T when T is a 2×2 matrix proves that the general dilation in that case is of the form $H_{\alpha}^2(\Omega) \otimes \mathbb{C}^2$.

Suppose that the homomorphism ρ_T admits a dilation of the form

$$(5.7) \quad \rho_{M \otimes I} : \mathcal{A}(\Omega) \rightarrow \mathcal{B}(H_{\alpha}^2(\Omega) \otimes \mathbb{C}^n)$$

for some $\alpha \in \mathbb{T}^m$, that is, the multiplication operator $M \otimes I$ on $H_{\alpha}^2(\Omega) \otimes \mathbb{C}^n$ is a dilation of T . Since the eigenvectors $\{v_1, v_2, \dots, v_n\}$ for T^* span V and the set of eigenvectors of $M^* \otimes I : H_{\alpha}^2(\Omega) \otimes \mathbb{C}^n \rightarrow H_{\alpha}^2(\Omega) \otimes \mathbb{C}^n$ at z_i is the set of vectors $\{K_{\alpha}(\cdot, z_i) \otimes a_j : a_j \in \mathbb{C}^n, 1 \leq j \leq n\}$ for $1 \leq i \leq n$, it follows that any map $\Gamma : V \rightarrow H_{\alpha}^2(\Omega) \otimes \mathbb{C}^n$ that intertwines T^* and M^* must be defined by $\Gamma(v_i) = K_{\alpha}(\cdot, z_i) \otimes a_i$ for some choice of a set of n vectors a_1, a_2, \dots, a_n in \mathbb{C}^n . Now Γ is isometric if and only if

$$(5.8) \quad \left(\left(K(z_j, z_i) \right) \right) = \left(\left(v_i, v_j \right) \right) = \left(\left(K_{\alpha}(z_j, z_i) \langle a_i, a_j \rangle \right) \right).$$

Clearly, this means that $\left(\left(K(z_j, z_i) \right) \right)$ admits $\left(\left(K_{\alpha}(z_j, z_i) \right) \right)$ as a factor in the sense that $\left(\left(K(z_j, z_i) \right) \right)$ is the Schur product of $\left(\left(K_{\alpha}(z_j, z_i) \right) \right)$ and a positive definite matrix, namely, the matrix $A = \left(\left(a_i, a_j \right) \right)$.

Conversely, the contractivity assumption on ρ_T does not necessarily guarantee that K_{α} is a factor of K . However, if we make this stronger assumption, that is, we assume there exists a positive definite matrix A such that $\left(\left(K(z_j, z_i) \right) \right) = \left(\left(K_{\alpha}(z_j, z_i) a_{ij} \right) \right)$, where $A = \left(\left(a_{ij} \right) \right)$. Since A is positive, it follows that $A = \left(\left(a_i, a_j \right) \right)$ for some set of n vectors a_1, \dots, a_n in \mathbb{C}^n . Therefore if we define the map $\Gamma : V \rightarrow H_{\alpha}^2(\Omega) \otimes \mathbb{C}^n$ to be $\Gamma(v_i) = K_{\alpha}(\cdot, z_i) \otimes a_i$ for $1 \leq i \leq n$ then Γ is clearly unitary and is an intertwiner between T and M^* . Thus the theorem above has the following corollary:

COROLLARY 12. *The homomorphism ρ_T is dilatable to a homomorphism $\tilde{\rho}$ of the form (5.7) if the kernel K , as defined in (5.1), is the Schur product of a positive definite matrix A and the restriction of K_{α} to the set $\sigma \times \sigma$ for some $\alpha \in \mathbb{T}^m$.*

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