Illinois Journal of Mathematics
Volume 47, Number 4, Winter 2003, Pages 1189–1206
S 0019-2082

OPTIMAL CONTROL SYSTEMS GOVERNED BY SECOND-ORDER ODES WITH DIRICHLET BOUNDARY DATA AND VARIABLE PARAMETERS

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Abstract. Optimal control systems governed by second-order ODEs with boundary data and variable parameters are considered. Using variational methods a theorem on existence of optimal processes is proven and a sufficient condition for continuous (or semicontinuous) dependence of optimal trajectories and controls on parameters is given.

1. Introduction

In this paper we consider stability results for solutions to optimal control problems governed by second-order ODEs with boundary data and variable parameters. Specifically, we are interested in results which guarantee the convergence of optimal controls and corresponding solutions in a suitable topology if initial and terminal data and some time varying parameter in the model converge.

Let I be the interval $[0,\pi]$ and denote by H^1 the Sobolev space of absolutely continuous vector-valued functions $x : I \to \mathbb{R}^n$ on I with values in $\mathbb{R}^n$ that have a square-integrable derivative $\dot{x}$. This space is endowed with the norm

$$\|x\|_{H^1}^2 = \int_I |x(t)|^2 + |\dot{x}(t)|^2\,dt, \tag{1.1}$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^n$. Furthermore, let H_0^1 denote the subspace of all functions which vanish at the endpoints of I. It follows from the Poincaré inequality that H_0^1 can be equipped with the norm

$$\|x\|_{H_0^1}^2 = \int_I |\dot{x}(t)|^2\,dt. \tag{1.2}$$

Received November 20, 2002; received in final form May 7, 2003.
2000 *Mathematics Subject Classification.* Primary 49J15, 34H05. Secondary 49J45.
The authors were supported in part by NATO Collaboration Grant PST.CLG.977282. The first and second author were supported in part by NSF Grant 0205093, the first author by an SIUE Hoppe Research Professor Award, and the third author by Grant 7T11A00421 of the State Committee for Scientific Research.

Also let L^p, $1 \leq p < \infty$, denote the standard space of Lebesgue integrable vector valued functions on the interval I with values in some space $\mathbb{R}^k$.

We consider the following optimal control problem $\mathcal{P} = \mathcal{P}^{\omega,a,b}$: minimize the objective

$$\mathcal{J}_{\omega,a,b}(x,u) = \int_I \Phi(t,\omega(t),x(t),\dot{x}(t),u(t))dt + l(a,b) \tag{1.3}$$

over all solutions $x \in H^1$ of the two-point boundary value problem

$$\ddot{x}(t) = \varphi(t,\omega(t),x(t),u(t)), \quad x(0) = a, \quad x(\pi) = b, \tag{1.4}$$

with controls

$$u \in \mathcal{U} = \left\{ u \in L^2 : u(t) \in M \text{ a.e.} \right\}, \tag{1.5}$$

where M is a non-empty, compact and convex set in $\mathbb{R}^m$. In the formulation $\omega : I \to Q \subset \mathbb{R}^r$ is a time-varying parameter in L^p with values in some set Q. We make the following **assumptions** on the objective and dynamics:

(A) The function $\varphi : I \times \mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is affine in the control,

$$\varphi(t,\omega,x,u) = f(t,\omega,x) + \langle g(t,\omega,x), u \rangle, \tag{1.6}$$

and the vector fields f and g are potential fields with respect to x, i.e., there exist functions $F = F(t,\omega,x)$ and $G = G(t,\omega,x)$ such that

$$f = F'_x \quad \text{and} \quad g = G'_x. \tag{1.7}$$

The functions F, G, $F'_x = f$, and $G'_x = g$ are measurable with respect to $t \in I$ for each $\omega \in Q$, Borel-measurable in ω, and satisfy the following conditions:

(growth): For every $r > 0$ there exist a function $\bar{h} \in L^1$ and a constant $C > 0$ such that for a.e. $t \in I$, $\omega \in Q$ and $|x| \leq r$ each of the functions

$$|F(t,\omega,x)|, |F'_x(t,\omega,x)|, |G(t,\omega,x)|, |G'_x(t,\omega,x)|$$

is bounded by

$$C(1 + |\omega|^p) + \bar{h}(t). \tag{1.8}$$

(coercive): There exist functions $\beta \in L^2$, $\gamma \in L^1$ and a constant $\alpha < 1/2$ such that for a.e. $t \in I$, $\omega \in Q$, $u \in M$ and $x \in \mathbb{R}^n$ we have

$$F(t,\omega,x) + \langle G(t,\omega,x), u \rangle \geq -\alpha |x|^2 - \langle \beta(t), x \rangle - \gamma(t). \tag{1.9}$$

The function $l : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is continuous.

(B) The integrand $\Phi : I \times \mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ also is affine in the control of the form

$$\Phi(t,\omega,x,\dot{x},u) = \Phi^1(t,\omega,x,\dot{x}) + \left\langle \Phi^2(t,\omega,x,\dot{x}), u \right\rangle . \tag{1.10}$$

The functions Φ^i, $i = 1, 2$, are measurable with respect to $t \in I$ for each $(\omega, x, \dot{x}) \in Q \times \mathbb{R}^n \times \mathbb{R}^n$ and are continuous in $(\omega, x, \dot{x})$ on $Q \times \mathbb{R}^n \times \mathbb{R}^n$ for a.e. $t \in I$. Furthermore, they satisfy the following growth assumption:

(growth): For every $r > 0$ there exist a constant $C > 0$ and a function $\breve{h} \in L^1$ such that for a.e. $t \in I$, all $\omega \in Q$, all $x \in \mathbb{R}^n$ which satisfy $|x| \le r$, and all $\dot{x} \in \mathbb{R}^n$ we have

$$\left|\Phi^i(t,\omega,x,\dot{x})\right| \le C(1 + |\omega|^p + |\dot{x}|^2) + \breve{h}(t). \tag{1.11}$$

In this paper we give conditions under which the problem $\mathcal{P}^{\omega,a,b}$ has an optimal solution and then investigate whether the sets of optimal processes depend continuously (or semicontinuously) on the parameters (ω, a, b). The existence of a solution is established under suitable convexity assumptions: By (1.7), the system (1.4) can be rewritten as

$$\ddot{x}(t) = F'_x(t,\omega(t),x(t)) + \langle G'_x(t,\omega(t),x(t)), u(t)\rangle , \tag{1.12}$$

with boundary data

$$x(0) = a, \quad x(\pi) = b. \tag{1.13}$$

It is well-known that the system (1.12) is the Euler-Lagrange equation for the functional

$$\mathcal{A}^\omega_u(x) = \int_I \left[\frac{|\dot{x}(t)|^2}{2} + F(t,\omega(t),x(t)) + \langle G(t,\omega(t),x(t)), u(t)\rangle \right] dt \tag{1.14}$$

acting from H^1_0 to $\mathbb{R}$. If the functional $\mathcal{A}^\omega_u$ is convex, then, using properties of the functional of action and the growth conditions above, it can be shown that the sets of possible initial and terminal values of solutions for (1.12), now also including the values of the derivative, are in fact bounded, and then the existence of optimal solutions for the problem $\mathcal{P}$ readily follows from standard results. We therefore also make the following assumption:

(C) The functional $\mathcal{A}^\omega_u$ is convex for all $u \in \mathcal{U}$.

This condition is fulfilled, for example, in the following two cases which often are easy to verify:

(a) The function $F(t,\omega,\cdot)$ is convex and $G(t,\omega,\cdot)$ is linear for a.e. $t \in I$, $\omega \in Q$, or, more generally:

(b) The function

$$x \longmapsto \frac{|x|^2}{2} + F(t,\omega,x) + \langle G(t,\omega,x),u\rangle \tag{1.15}$$

is convex on $\mathbb{R}^n$ for all $\omega \in Q$, $u \in M$ and a.e. $t \in I$.

Once the existence of optimal solutions is established, we are interested in their behavior if the parameters and initial and terminal conditions converge to some limit. Specifically, suppose that the sequence of parameters $\{\omega^k\}_{k=1}^{+\infty}$ converges to some $\bar{\omega}$ in L^p and that the sequence of boundary conditions $\{(a^k,b^k)\}_{k=1}^{+\infty}$ converges to a limit $(\bar{a},\bar{b})$ in $\mathbb{R}^n \times \mathbb{R}^n$ and let $\Xi_*(\omega^k,a^k,b^k)$ denote the set of optimal pairs (x_*,u_*) for the corresponding optimal control problems $\mathcal{P}^k = \mathcal{P}^{\omega_k,a_k,b_k}$, $k \in \mathbb{N}$. We are investigating whether in a suitable topology the sequence $\{\Xi_*(\omega^k,a^k,b^k)\}_{k=1}^{+\infty}$ converges to a limit as $\omega^k \to \bar{\omega}$ and $(a^k,b^k) \to (\bar{a},\bar{b})$ for $k \to \infty$. This is a question about stability of optimal control systems. In short, our main theorem given in Section 3 says that if the functional $\mathcal{A}_u^{\bar{\omega}}$ corresponding to the limit is *strictly* convex for every $u \in \mathcal{U}$, then every accumulation point $(\bar{x},\bar{u})$ of a sequence of optimal pairs $\{(x_*^k,u_*^k)\}_{k=1}^{+\infty}$ taken in the weak topologies is an optimal pair for $\mathcal{P}^{\bar{\omega},\bar{a},\bar{b}}$ and the corresponding optimal values $\mathcal{J}_{\omega^k,a^k,b^k}(x_*^k,u_*^k)$ converge to the optimal value $\mathcal{J}_{\bar{\omega},\bar{a},\bar{b}}(\bar{x},\bar{u})$. Hence the optimal control system is stable with respect to these topologies. The proof of this result is an application of Berge's Theorem [2]. The set-valued map which describes the feasible set will be lower semicontinuous at the limit if equation (1.12) with boundary conditions (1.13) has a unique solution for every $u \in \mathcal{U}$. This is guaranteed if the functional $\mathcal{A}_u^{\bar{\omega}}$ is not only convex, but strictly convex for every $u \in \mathcal{U}$. We will give an example which shows that in general this condition cannot be weakened.

Similar problems for a system described by second order PDE's of the elliptic type with homogeneous Dirichlet boundary data were investigated by us earlier in [13] and [14] using direct variational methods, by Papageorgiou in [18] and [19] using relaxation methods, and also in a broader context in the monograph [8], [9] by Hu and Papageorgiou. The question of existence of optimal solutions to control systems described by semilinear second-order equations was investigated by Idczak. Leaning on methods of spectral analysis, in [10] Idczak demonstrates that trajectories of the system depend continuously on controls and then shows directly that optimal solutions exist. A similar problem for superlinear second order ordinary equations was considered by Nowakowski and Rogowski in [17]. Necessary conditions of optimality for second-order systems of ordinary differential equations with Dirichlet boundary conditions were given by Goebel and Raitums in [7] and by Idczak in [11] (see also the references therein).

As mentioned before, the main results of this paper are sufficient conditions for optimal processes of second-order systems with Dirichlet boundary conditions to depend continuously (or semicontinuously) on parameters and boundary data. To the authors' knowledge this problem was not studied before. Related results concerning first-order differential systems and mathematical programming can be found in the monograph [6] and in the papers [4], [15], and [21] (see also the references therein).

2. Existence of an optimal solution

In this section we consider problem $\mathcal{P}^{\omega,a,b}$ for a fixed parameter $\omega \in L^p$ and fixed boundary values (a,b). Given a control $u \in \mathcal{U}$, denote by X_u the set of all solutions to the boundary value problem (1.12)–(1.13). In general X_u need not be a singleton. Also let $X = \bigcup_{u\in\mathcal{U}} X_u$ denote the set of all solutions to (1.12)–(1.13) for arbitrary controls $u \in \mathcal{U}$. We call the set

$$\Xi(\omega,a,b) = \{(x,u) \in H^1 \times L^2 : u \in \mathcal{U}, x \in X_u\} \tag{2.1}$$

the feasible set of the control system and a pair $(x,u) \in \Xi(\omega,a,b)$ is called admissible. Also let $\Xi_*(\omega,a,b) \subset \Xi(\omega,a,b)$ denote the set of optimal processes for problem $\mathcal{P}^{\omega,a,b}$; i.e., $(x_*,u_*) \in \Xi_*(\omega,a,b)$ iff $u_* \in \mathcal{U}, x_* \in X_{u_*}$ and

$$\mathcal{J}_{\omega,a,b}(x_*,u_*) = \inf\left\{\mathcal{J}_{\omega,a,b}(x,u) : (x,u) \in \Xi(\omega,a,b)\right\}. \tag{2.2}$$

THEOREM 2.1. *If conditions* (A)–(C) *are satisfied for problem* $\mathcal{P}^{\omega,a,b}$, *then:*

(a) *For each admissible control* $u \in \mathcal{U}$ *the set* X_u *is nonempty, i.e., there exists at least one solution* $x \in H^1$ *to* (1.12)–(1.13).
(b) *The set* X *of all solutions is bounded in* H^1, *i.e., there exists a constant* $\rho > 0$ *such that whenever* x_u *is a solution of* (1.12)–(1.13) *corresponding to any* $u \in \mathcal{U}$, *then* $\|x_u\|_{H^1} \le \rho$.
(c) *There exists an optimal process* (x_*,u_*) *for system* $\mathcal{P}^{\omega,a,b}$, *i.e., the set* $\Xi_*(\omega,a,b)$ *is nonempty.*

Proof. Let $u \in \mathcal{U}$ be an arbitrary admissible control. To prove that the corresponding equation (1.12) has a solution it suffices to show that the functional of action in (1.14) has a critical point in the space H^1. For this purpose we shall show that the functional $\mathcal{A}_u^\omega$ attains its lower bound. Since ω, a and b are fixed in this proof, for the moment we drop them in the notation. Due to the Poincaré inequality it is more convenient to consider the problem in the space H_0^1. Let

$$c = \frac{b-a}{\pi}, \quad \mu(t) = \frac{b-a}{\pi}t + a, \quad x(t) = y(t) + \mu(t) \tag{2.3}$$

and

$$\bar{\mathcal{A}}_u(y) = \int_I \Bigg[\frac{|\dot{y}(t)+c|^2}{2} + F(t,\omega(t),y(t)+\mu(t)) + \\ + \langle G(t,\omega(t),y(t)+\mu(t)),u(t)\rangle \Bigg] dt, \tag{2.4}$$

where $y \in H_0^1$. The functional $\bar{\mathcal{A}}_u$ given by (2.4) is continuously differentiable on H^1 [16, Thm. 1.4] and as the sum of a convex, continuous functional and a weakly continuous functional is also weakly lower semicontinuous on H^1 [16, Section 1.5]. It is easy to see that if $y \in H_0^1$ is a minimizer of $\bar{\mathcal{A}}_u$ on the space H_0^1, then $x = y + \mu$ is a minimizer of $\mathcal{A}_u$ on H^1 with boundary conditions $x(0) = a$ and $x(\pi) = b$. It thus suffices to show that $\bar{\mathcal{A}}_u$ has a minimizer in H_0^1 for any $u \in \mathcal{U}$.

By assumption (A, coercive) we have that

$$\bar{\mathcal{A}}_u(y) \geq \int_I \frac{|\dot{y}(t)+c|^2}{2} - \alpha\,|y(t)+\mu(t)|^2 - \langle\beta(t),y(t)+\mu(t)\rangle - \gamma(t)dt. \tag{2.5}$$

It then follows from the Poincaré and Hölder inequalities that there exist constants C^1 and C^2 such that

$$\bar{\mathcal{A}}_u(y) \geq \left(\frac{1}{2} - \alpha\right) \|y\|^2_{H_0^1} + C^1\,\|y\|_{H_0^1} + C^2 = p(y) \tag{2.6}$$

(Recall that $\alpha < 1/2$.) For $r > 0$ sufficiently large, assumption (A, growth) implies that

$$\bar{\mathcal{A}}_u(0) = \int_I \left(\frac{|c|^2}{2} + F(t,\omega(t),\mu(t)) + \langle G(t,\omega(t),\mu(t)),u(t)\rangle \right) dt \tag{2.7}$$

is bounded, say $\bar{\mathcal{A}}_u(0) \leq C^3 < \infty$. Hence for any $u \in \mathcal{U}$ minimizers of $\bar{\mathcal{A}}_u$ necessarily lie in the set

$$\left\{y \in H_0^1 : \bar{\mathcal{A}}_u(y) \leq C^3\right\} \subset \left\{y \in H_0^1 : p(y) \leq C^3\right\}.$$

But the latter set is bounded in H_0^1. Hence there exists a bounded minimizing sequence. This sequence is weakly compact in H_0^1 and thus, since the functional $\bar{\mathcal{A}}_u$ is weakly lower semicontinuous, there exists a minimizer $y \in H_0^1$ of $\bar{\mathcal{A}}_u$ for any $u \in \mathcal{U}$. The function $x = y + \mu$ then is a minimizer of the functional $\mathcal{A}_u$ given by (1.14) and x solves the differential equation (1.12). This proves assertion (a).

Condition (b) also is a direct consequence of the relations (2.6) and (2.7): As was just shown, there exists a $\bar{\rho} > 0$ such that for any $u \in \mathcal{U}$ the set of minimizers of the functional $\bar{\mathcal{A}}_u$ on H_0^1 is contained in the ball $B(0,\bar{\rho}) = \left\{y \in H_0^1 : \|y\|_{H_0^1} \leq \bar{\rho}\right\}$. Using the Poincaré inequality, it follows that

$$\|x\|^2_{H^1} \leq 4\,\|y\|^2_{H_0^1} + 2\,\|\mu\|^2_{H^1},$$

and thus the set X of all minimizers of the functional $\mathcal{A}_u$ on H^1 (for any $u \in \mathcal{U}$) is contained in the ball $\left\{x \in H^1 : \|x\|_{H^1}^2 \leq 4\bar{\rho}^2 + 2\|\mu\|_{H^1}^2\right\}$.

Now we shall prove assertion (c). The system given by equation (1.12)–(1.13) can be represented in the equivalent form

$$\begin{aligned} &\dot{x}(t) = y(t), \qquad x(0) = a,\ x(\pi) = b, \\ &\dot{y}(t) = F_x(t, \omega(t), x(t)) + \langle G_x(t, \omega(t), x(t)), u(t)\rangle, \quad y(0),\ y(\pi)\,\text{free}. \end{aligned} \tag{2.8}$$

For any admissible control $u \in \mathcal{U}$ let x_u denote a solution to (1.12). We have just shown that there exists a constant $\rho > 0$ such that $\|x_u\|_{H^1} \leq \rho$. Hence there exists a constant $\tilde{C}$ such that for all $t \in I$ and all $u \in \mathcal{U}$ we have $|x_u(t)| \leq \tilde{C}$ (e.g., $\tilde{C} = a + \sqrt{\pi}\rho$). Assumption (A, growth) and (1.12) then imply that there exist $C > 0$ and $\bar{h} \in L^1$ such that we have for a.e. $t \in I$ and all $u \in \mathcal{U}$

$$|\ddot{x}_u(t)| \leq C(1 + |\omega(t)|^p) + \bar{h}(t). \tag{2.9}$$

Suppose that the set $\{\dot{x}_u(0), u \in \mathcal{U}\}$ is unbounded. Then there exists a sequence $\{u_s\} \subset \mathcal{U}$ such that $\dot{x}^i_{u_s}(0) \to \pm\infty$ for some $i \in \{1, 2, \ldots, n\}$. Assume that the ith coordinate $\dot{x}^i_{u_s}(0)$ tends to $+\infty$ as $s \to \infty$. Let L be a positive number. Then for s sufficiently large we have the estimate

$$\begin{aligned} \dot{x}^i_{u_s}(t) &= \dot{x}^i_{u_s}(0) + \int_0^t \ddot{x}^i_{u_s}(\tau)d\tau \\ &\geq \dot{x}^i_{u_s}(0) - \int_0^\pi \left[C(1 + |\omega(t)|^p) + \bar{h}(t)\right] dt > L \end{aligned}$$

and thus

$$x^i_{u_s}(t) = a^i + \int_0^t \dot{x}^i_{u_s}(\tau)d\tau \geq a^i + \int_0^t L d\tau = a^i + Lt,$$

where a^i denotes the ith coordinate of the initial vector a. Since a^i and b^i are fixed and L may be arbitrarily large, we obtain

$$x^i_{u_s}(\pi) \geq a^i + L\pi > b^i$$

for s sufficiently large. Thus we have a contradiction with the boundary condition $x_{u_s}(\pi) = b$. This means that the set $\left\{\dot{x}^i_{u_s}(0), u \in \mathcal{U}\right\} \subset \mathbb{R}^n$ is bounded. In a similar way one can show that the set $\left\{\dot{x}^i_{u_s}(\pi), u \in \mathcal{U}\right\} \subset \mathbb{R}^n$ is bounded.

Let B^0 and B^1 be compact sets in $\mathbb{R}^n$ such that $\{\dot{x}_{u_s}(0) : u \in \mathcal{U}\} \subset B^0$ and $\{\dot{x}_{u_s}(\pi) : u \in \mathcal{U}\} \subset B^1$. In this situation the optimal control problem $\mathcal{P}$ can be represented in the following equivalent form: minimize

$$\tilde{\mathcal{J}}(x, y, u) = \int_I \Phi(t, \omega(t), x(t), y(t), u(t))dt + l(a, b) \tag{2.10}$$

subject to

$$\dot{x}(t) = y(t), \quad x(0) = a, \quad x(\pi) = b, \tag{2.11}$$

$$\dot{y}(t) = f(t, \omega(t), x(t)) + \langle g(t, \omega(t), x(t)), u(t) \rangle, \tag{2.12}$$

$$y(0) \in B^0, \quad y(\pi) \in B^1, \tag{2.13}$$

$$u \in \mathcal{U} = \left\{ u \in L^2(I) : u(t) \in M \text{ a.e.} \right\}, \tag{2.14}$$

with B^0 and B^1 compact subsets of $\mathbb{R}^n$. To this optimal control problem well-known classical theorems on existence of solutions apply (for example, see [3, Thm. 5.1 and Cor. 5.1]) and we thus get the existence of an optimal solution (x_*, u_*), completing the proof of the theorem. □

COROLLARY 2.1. *Let $\Omega \subset L^p$ and $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^n$ be bounded sets. There exists a constant $\rho > 0$ such that whenever $\omega \in \Omega$, $a \in A$ and $b \in B$, then for any control $u \in \mathcal{U}$ a corresponding solution x_u of* (1.12)–(1.13) *satisfies $\|x_u\|_{H^1} \leq \rho$, i.e., the set of all solutions is bounded.*

Proof. As above, x_u is a critical point of the functional $\mathcal{A}_u$. If a and b lie in bounded sets, then the constants C^1 and C^2 in the inequality (2.6) can be chosen depending only on the bounds for the sets A and B, but not on the individual points. Also, the H^1 norms of the transforming functions μ in (2.3) will be uniformly bounded with $a \in A$ and $b \in B$. Similarly, for ω in a bounded set the constant C^3 which bounds $\bar{\mathcal{A}}_u^{\omega}(0)$ can be chosen only to depend on the bound for Ω. Thus the proof is as in the proof of assertion (b) of the theorem. □

3. Continuous and semi-continuous dependence of optimal controls and trajectories on variable parameters and boundary data

We now consider the solutions of the optimal control problem $\mathcal{P}^{\omega,a,b}$ for a sequence of time varying parameters $\omega^k \in L^p$ and boundary values $(a^k, b^k) \in \mathbb{R}^n \times \mathbb{R}^n$, $k \in \mathbb{N}$. Let $\Xi^k = \Xi(\omega^k, a^k, b^k)$ denote the corresponding feasible sets and $\Xi_*^k = \Xi_*(\omega^k, a^k, b^k)$ the sets of optimal processes. By Theorem 2.1 each of these sets is nonempty and thus the value

$$V_k = V_{\omega^k, a^k, b^k} = \min \left\{ \mathcal{J}_{\omega^k, a^k, b^k}(x, u) : (x, u) \in \Xi^k \right\}, \; k \in \mathbb{N}, \tag{3.1}$$

of the optimal control problem $\mathcal{P}^k = \mathcal{P}^{\omega^k, a^k, b^k}$ is well-defined and will be attained. We shall prove that if $\omega^k \to \bar{\omega}$ in L^p and $(a^k, b^k) \to (\bar{a}, \bar{b})$ in $\mathbb{R}^n \times \mathbb{R}^n$, then the values V_k converge to the value $\bar{V}$ of the limiting problem and the sets Ξ_*^k of optimal processes converge to the set $\bar{\Xi}_*$ of optimal processes for the limiting problem in the sense of the Painlevé-Kuratowski upper limit of sets (e.g., see [1]), provided the functional $\mathcal{A}_u^{\bar{\omega}}$ corresponding to the limit is *strictly* convex for all $u \in \mathcal{U}$.

DEFINITION 3.1. Let $\{A^k\}_{k=1}^{+\infty}$ be a sequence of subsets of a topological space Z. The set of all cluster points of the sequences $\{z^k\}_{k=1}^{\infty} \subset Z$, such that $z^k \in A^k$, $k = 1, 2, \ldots$, is called the upper limit of the sequence of sets $\{A^k\}$ and denoted by $\limsup A^k$.

THEOREM 3.1. *Suppose conditions* (A)–(C) *are satisfied for the optimal control problem* $\mathcal{P}$. *Let* $\omega^k \to \bar{\omega}$ *in* L^p *and* $(a^k, b^k) \to (\bar{a}, \bar{b})$ *in* $\mathbb{R}^n \times \mathbb{R}^n$ *and suppose that for the limiting problem the functional* $\mathcal{A}_u^{\bar{\omega}}$ *defined in* (1.14) *is strictly convex for all* $u \in \mathcal{U}$. *Then* $\limsup \Xi_*^k$ *taken with respect to the strong topology of* H^1 *in* x *and with respect to the weak topology of* L^2 *in* u *is a nonempty set and* $\limsup \Xi_*^k \subset \bar{\Xi}_*$. *Furthermore, the sequence of optimal values converges,* $\lim_{k\to\infty} V_k = \bar{V}$.

REMARK. If the optimal processes (x_*^k, u_*^k), $k \in \mathbb{N}$, are unique, i.e., the sets Ξ_*^k are singletons, then by (a) and (b) $\{x_*^k\}_{k=1}^{+\infty}$ converges to $\bar{x}_*$ strongly in H^1 (i.e., $x_*^k \to \bar{x}_*$ uniformly on I and $\dot{x}_*^k \to \dot{x}_*^0$ in L^2) and $\{u_*^k\}_{k=1}^{+\infty}$ tends to $\bar{u}_*$ weakly in L^2.

The proof of the theorem follows from an application of Berge's theorem [2] and we first set up the required pieces. Following the notation already introduced, we define a set-valued map or correspondence $\Xi : \mathcal{X} \to \mathcal{Y}$ which associates to the data of the problem the feasible set, i.e., the set of possible solutions of the system. Specifically, let $\mathcal{X} = L^p \times \mathbb{R}^n \times \mathbb{R}^n$, $\mathcal{Y} = L^2 \times H^1$ and let $\Xi(\omega, a, b)$ be the set of all pairs (u, x_u), where $u \in \mathcal{U}$ and x_u is a solution to the boundary value problem (1.12) with parameter given by ω and boundary conditions $x(0) = a$ and $x(\pi) = b$. We fix the strong topology in $\mathcal{X}$, i.e., the strong topology in L^p and the standard (Euclidean) topology in $\mathbb{R}^n$. In the spaces $L^2 \supset \mathcal{U}$ and H^1 we typically consider the corresponding weak topologies, but for the reader's convenience we specify the convergence concepts used since at times we also consider the strong topologies.

LEMMA 3.1. *Under assumptions* (A) *and* (C), *the correspondence* Ξ *is upper semicontinuous (with respect to the weak topologies in* L^2 *and* H^1*).*

Proof. Recall that Ξ is upper semicontinuous in the chosen topology if and only if the following statement holds (cf. [1, Def. 1.4.1]): Whenever $\omega^k \to \bar{\omega}$ in L^p, $a^k \to \bar{a}$ and $b^k \to \bar{b}$ in $\mathbb{R}^n$, and $(u^k, x^k) \in \Xi(\omega^k, a^k, b^k)$ is a pair in the corresponding feasible set such that $u^k \to \bar{u}$ weakly in L^2, and $x^k \to \bar{x}$ weakly in H^1, then $(\bar{u}, \bar{x}) \in \Xi(\bar{\omega}, \bar{a}, \bar{b})$, i.e., $\bar{x}$ is a solution of (1.12) corresponding to $\bar{u}$ with parameter $\bar{\omega}$ and boundary data $(\bar{a}, \bar{b})$.

The set $\mathcal{U}$ is weakly compact in L^2 and thus $\bar{u} \in \mathcal{U}$. We need to show that $\bar{x}$ is an admissible trajectory of the system corresponding to $\bar{u} \in \mathcal{U}$. In fact, since $x^k \in X_{u^k}$ we have for all $h \in H_0^1$ and all $k \in \mathbb{N}$ that

$$\begin{aligned} 0 &= \frac{\partial}{\partial x} \mathcal{A}_{u^k}^{\omega^k}(x^k)h \\ &= \int_I \Big[\left\langle \dot{x}^k(t), \dot{h}(t) \right\rangle + F_x(t, \omega^k(t), x^k(t))h(t) \\ &\qquad + \left\langle G_x(t, \omega^k(t), x^k(t))h(t), u^k(t) \right\rangle \Big] dt. \end{aligned} \tag{3.2}$$

Consider the integral functional

$$\int_I \left\langle G_x(t, \omega^k(t), x^k(t))h(t), u^k(t) \right\rangle dt \tag{3.3}$$

$$= \int_I \left\langle (G_x(t, \omega^k(t), x^k(t)) - G_x(t, \bar{\omega}(t), \bar{x}(t)))h(t), u^k(t) \right\rangle \tag{3.4}$$

$$+ \int_I \left\langle G_x(t, \bar{\omega}(t), \bar{x}(t))h(t), u^k(t) \right\rangle dt. \tag{3.5}$$

We show that for a suitable subsequence $\{k_j\}$ each of the integrals in (3.4) and (3.5) converges to zero. Since x^k tends to $\bar{x}$ weakly in H^1, it follows that x^k tends to $\bar{x}$ uniformly on I [16, Prop. 1.2] and thus there exists a constant r such that $|x^k(t)| \leq r$ for all $t \in I$. Hence by (A, growth) the function $t \mapsto G_x(t, \bar{\omega}(t), \bar{x}(t))h(t)$ lies in L^1. Furthermore, since $u^k \to \bar{u}$ weakly in L^2, there exists a subsequence $\{u^{k_j}\}$ which converges to $\bar{u}$ a.e. on I. Since the controls take values in a compact set, it therefore follows from Lebesgue's Dominated Convergence theorem that

$$\begin{aligned} \lim_{j \to \infty} \int_I \left\langle G_x(t, \bar{\omega}(t), \bar{x}(t))h(t), u^k(t) \right\rangle dt \\ = \int_I \langle G_x(t, \bar{\omega}(t), \bar{x}(t))h(t), \bar{u}(t) \rangle \, dt. \end{aligned} \tag{3.6}$$

Similarly, there exists a constant K such that

$$\begin{aligned} \left| \int_I \left\langle (G_x(t, \omega^k(t), x^k(t)) - G_x(t, \bar{\omega}(t), \bar{x}(t)))h(t), u^k(t) \right\rangle \right| \\ \leq K \int_I \left| G_x(t, \omega^k(t), x^k(t)) - G_x(t, \bar{\omega}(t), \bar{x}(t)) \right| dt, \end{aligned}$$

and for a suitable subsequence, $\{\omega^{k_j}\}$ converges to $\bar{\omega}$ a.e. on I. By condition (A, growth) we have that

$$\begin{aligned} \left| G_x(t, \omega^{k_j}(t), x^k(t)) - G_x(t, \bar{\omega}(t), \bar{x}(t)) \right| \\ \leq C(2 + \left| \omega^{k_j}(t) \right|^p + |\bar{\omega}(t)|^p) + 2\bar{h}(t). \end{aligned}$$

Since ω^{k_j} converges to $\bar{\omega}$ in L^p, if necessary by taking another subsequence, it follows that there exists a $v \in L^p$ such that $\left| \omega^{k_j}(t) \right| \leq v(t)$ a.e. on I (for example, see [5, Thm. IV.9]). Hence for this subsequence the integral in (3.4) converges to zero by Lebesgue's Dominated Convergence Theorem.

Thus for each $h \in H_0^1$ there exists a subsequence $\{k_j\}$ such that

$$\lim_{j\to+\infty}\int_I \left\langle G_x(t,\omega^{k_j}(t),x^{k_j}(t))h(t),u^{k_j}(t)\right\rangle dt = \int_I \langle G_x(t,\bar\omega(t),\bar x(t))h(t),\bar u(t)\rangle\, dt.$$

Similarly it follows that, taking another subsequence if necessary,

$$\lim_{j\to\infty}\int_I \left[\left\langle \dot x^{k_j}(t),\dot h(t)\right\rangle + F_x(t,\omega^{k_j}(t),x^{k_j}(t))h(t)\right] dt = \int_I \left[\left\langle \dot{\bar x}(t),\dot h(t)\right\rangle + F_x(t,\bar\omega(t),\bar x(t))h(t)\right] dt.$$

Thus we have proven that (cf. (3.2))

$$0 = \lim_{j\to\infty}\frac{\partial}{\partial x}\mathcal{A}_{u^{k_j}}^{\omega^{k_j}}(x^{k_j})h = \frac{\partial}{\partial x}\mathcal{A}_{\bar u}^{\bar\omega}(\bar x)h \tag{3.7}$$

for each $h \in H_0^1$. Hence $(\bar x,\bar u)$ is an admissible pair for the control system, i.e., $(\bar x,\bar u)\in \Xi(\bar\omega,\bar a,\bar b)$. This proves the lemma. ☐

Note that this argument also implies the following result:

COROLLARY 3.1. *The feasible sets $\Xi(\omega,a,b)$ are compact in the weak topologies on $\mathcal{Y}=L^2\times H^1$.*

The following is a useful technical statement for addressing lower semicontinuity properties of the relation Ξ and continuity of the functional $\mathcal{J}_{\omega,a,b}(x,u)$, which we state separately.

LEMMA 3.2. *Suppose assumptions* (A) *and* (C) *are satisfied and let $\omega^k \to \bar\omega$ in L^p, $a^k\to\bar a$ and $b^k\to\bar b$ in $\mathbb{R}^n$. Let x^k be a solution of* (1.12) *corresponding to the control $u^k\in\mathcal{U}$ for the parameter ω^k and boundary data (a^k,b^k). If $u^k\to\bar u$ weakly in L^2 and $x^k\to\bar x$ weakly in H^1, then in fact $x^k\to\bar x$ strongly in H^1.*

Proof. Define a function α^k on $[0,\pi]$ by

$$\alpha^k(t)=\frac{1}{\pi}\left[(x^k(\pi)-\bar x(\pi))-(x^k(0)-\bar x(0))\right]t+(x^k(0)-\bar x(0)) \tag{3.8}$$

and set

$$h^k = x^k-\bar x-\alpha^k. \tag{3.9}$$

Note that $h^k\in H_0^1$. It is easy to verify that $\alpha^k\to 0$ strongly in H^1 and hence $h^k\to 0$ weakly in H_0^1. A direct calculation, using (3.2) and (3.7) and the same notation, verifies that

$$
\begin{aligned}
(3.10)\qquad 0 &= \left\langle \frac{\partial}{\partial x}\mathcal{A}_{u^k}^{\omega^k}(x^k) - \frac{\partial}{\partial x}\mathcal{A}_{\bar{u}}^{\bar{\omega}}(\bar{x}), h^k \right\rangle \\
&= \int_I \left|\dot{x}^k(t) - \dot{\bar{x}}(t)\right|^2 dt - \int_I \left\langle \dot{x}^k(t) - \dot{\bar{x}}(t), \alpha^k(t)\right\rangle \\
&\quad + \int_I \left\langle F_x(t,\omega^k(t),x^k(t)) - F_x(t,\bar{\omega}(t),\bar{x}(t)), h^k(t)\right\rangle \\
&\quad + \int_I \left\langle G_x(t,\omega^k(t),x^k(t))h^k(t), u^k(t)\right\rangle dt \\
&\quad - \int_I \left\langle G_x(t,\bar{\omega}(t),\bar{x}(t))h^k(t), \bar{u}(t)\right\rangle dt.
\end{aligned}
$$

Since $h^k \to 0$ weakly in H^1, it follows that $h^k \to 0$ uniformly on I. As in the proof of Lemma 3.1 it can be shown that condition (A, growth) implies that each of the integrands in the last three integrals converges to zero pointwise and is dominated by one L^1-function for all k. Hence each of these terms converges to zero. Moreover, $\alpha^k \to 0$ strongly in H^1. Since $\omega^k \to \bar{\omega}$ in L^p and $a^k \to \bar{a}$ and $b^k \to \bar{b}$ in $\mathbb{R}^n$, the sequences $\{\omega^k\}_{k=1}^\infty$, $\{a^k\}_{k=1}^\infty$ and $\{b^k\}_{k=1}^\infty$ lie in bounded sets Ω, A and B, respectively, and thus by Corollary 2.1 all the solutions $\{x^k\}_{k=1}^\infty$ lie in a ball in H^1. Thus

$$
(3.11)\qquad \int_I \left\langle \dot{x}^k(t) - \dot{\bar{x}}(t), \alpha^k(t)\right\rangle \to 0.
$$

Hence

$$
(3.12)\qquad \lim_{k\to\infty} \int_I \left|\dot{x}^k(t) - \dot{\bar{x}}(t)\right|^2 dt = 0.
$$

Moreover, since $x^k \to \bar{x}$ uniformly on I, it also follows that

$$
(3.13)\qquad \lim_{k\to\infty} \int_I \left|x^k(t) - \bar{x}(t)\right|^2 dt = 0
$$

and thus $x^k \to \bar{x}$ strongly in H^1. $\square$

LEMMA 3.3. *Under assumptions* (A) *and* (C)*, the correspondence* Ξ *is lower semicontinuous with respect to the weak topology for* $u \in L^2$ *and the strong topology for* $x \in H^1$ *at every point* $(\bar{\omega}, \bar{a}, \bar{b}) \in \mathcal{X}$ *where the functional* $\mathcal{A}_{\bar{u}}^{\bar{\omega}}$ *is strictly convex for every* $\bar{u} \in \mathcal{U}$*. More generally, this holds whenever the solution* $x_{\bar{u}}$ *of* (1.12) *corresponding to* $\bar{u}$ *with parameter* $\bar{\omega}$ *and boundary data* $(\bar{a}, \bar{b})$ *is unique for every* $\bar{u} \in \mathcal{U}$*.*

Proof. Recall that Ξ is lower semicontinuous in the chosen topology if and only if the following statement holds [1, Def. 1.4.2]: Whenever $\omega^k \to \bar{\omega}$ in L^p, $a^k \to \bar{a}$ and $b^k \to \bar{b}$ in $\mathbb{R}^n$, and $(\bar{u}, \bar{x}) \in \Xi(\bar{\omega}, \bar{a}, \bar{b})$ is a pair in the feasible set corresponding to the parameter $\bar{\omega}$ and boundary data $(\bar{a}, \bar{b})$, then there exists a sequence $\{(u^k, x^k)\}_{k\in\mathbb{N}} \subset \mathcal{Y}$, $(u^k, x^k) \in \Xi(\omega^k, a^k, b^k)$, the feasible set corresponding to the parameter ω^k and boundary data (a^k, b^k), such that $u^k \to \bar{u}$ weakly in L^2 and $x^k \to \bar{x}$ strongly in H^1.

Pick an arbitrary sequence $\{u^k\}_{k\in\mathbb{N}} \subset \mathcal{U}$ such that $u^k \to \bar{u}$ weakly in L^2. By Theorem 2.1 there exist corresponding solutions x^k such that $(u^k, x^k) \in \Xi(\omega^k, a^k, b^k)$. As above, the sequences $\{\omega^k\}_{k=1}^{\infty}$, $\{a^k\}_{k=1}^{\infty}$ and $\{b^k\}_{k=1}^{\infty}$ lie in bounded sets Ω, A and B, respectively. By Corollary 2.1 all the solutions $\{x^k\}_{k=1}^{\infty}$ lie in a ball in H^1 and thus this sequence is weakly compact. Hence there exist convergent subsequences, say $x^{k_j} \to \hat{x}$. By Lemma 3.1, $\hat{x}$ is a solution of (1.12) corresponding to $\bar{u}$ with parameter $\bar{\omega}$ and boundary data $(\bar{a}, \bar{b})$. But this equation has a unique solution and thus we have $\hat{x} = \bar{x}$. Hence every convergent subsequence converges to $\bar{x}$. Furthermore, by Lemma 3.2, these subsequences all converge strongly to $\bar{x}$. Hence the original sequence itself, $\{x^k\}_{k=1}^{\infty}$, converges strongly to $\bar{x}$ in H^1. This proves the lemma. $\square$

We now show that the objective is continuous on $\mathcal{X} \times \mathcal{Y}$. Recall that

$$\mathcal{J}_{\omega,a,b}(x,u) = l(a,b) + \int_I \Phi^1(t,\omega(t),x(t),\dot{x}(t)) + \left\langle \Phi^2(t,\omega(t),x(t),\dot{x}(t)), u(t)\right\rangle dt. \tag{3.14}$$

Lemma 3.4. *Under assumptions* (A)–(C)*, the objective $\mathcal{J}_{\omega,a,b}(x,u)$ is continuous on $\mathcal{X} \times \mathcal{Y}$ with respect to the weak topologies in $\mathcal{X}$ and $\mathcal{Y}$.*

Proof. Suppose $\omega^k \to \bar{\omega}$ in L^p, $a^k \to \bar{a}$ and $b^k \to \bar{b}$ in $\mathbb{R}^n$, $(u^k, x^k) \in \Xi(\omega^k, a^k, b^k)$, $u^k \to \bar{u}$ weakly in L^2 and $x^k \to \bar{x}$ weakly in H^1. It then follows from Lemma 3.1 that $\bar{x}$ is a solution to (1.12) for $\bar{u}$ with parameter $\bar{\omega}$ and boundary data $(\bar{a}, \bar{b})$, and we need to show that

$$\lim_{k\to\infty} \mathcal{J}_{\omega^k,a^k,b^k}(x^k,u^k) = \mathcal{J}_{\bar{\omega},\bar{a},\bar{b}}(\bar{x},\bar{u}). \tag{3.15}$$

Write

$$\mathcal{J}_{\omega^k,a^k,b^k}(x^k,u^k) - \mathcal{J}_{\bar{\omega},\bar{a},\bar{b}}(\bar{x},\bar{u}) = l(a^k,b^k) - l(\bar{a},\bar{b})$$

$$+ \int_I \Phi^1(t,\omega^k(t),x^k(t),\dot{x}^k(t)) - \Phi^1(t,\bar{\omega}(t),\bar{x}(t),\dot{\bar{x}}(t))dt \tag{3.16}$$

$$+ \int_I \left\langle \Phi^2(t,\omega^k(t),x^k(t),\dot{x}^k(t)) - \Phi^2(t,\bar{\omega}(t),\bar{x}(t),\dot{\bar{x}}(t)), u^k(t)\right\rangle dt \tag{3.17}$$

$$+ \int_I \left\langle \Phi^2(t,\bar{\omega}(t),\bar{x}(t),\dot{\bar{x}}(t)), u^k(t) - \bar{u}(t)\right\rangle dt. \tag{3.18}$$

Since l is continuous, $l(a^k,b^k) \to l(\bar{a},\bar{b})$. We first show that the integral (3.16) converges to zero. Suppose it does not. Then, by considering a subsequence if necessary, we may assume that

$$\left| \int_I \Phi^1(t,\omega^k(t),x^k(t),\dot{x}^k(t)) - \Phi^1(t,\bar{\omega}(t),\bar{x}(t),\dot{\bar{x}}(t))dt \right| > \varepsilon \tag{3.19}$$

for all k. We have $x^k \to \bar{x}$ uniformly on I, and, again by choosing a subsequence if necessary, we may assume that $\omega^k \to \bar{\omega}$ and $\dot{x}^k \to \dot{\bar{x}}$ a.e. on I. But then there exist functions $v_1 \in L^p$ and $v_2 \in L^2$ such that $|\omega^k(t)| \le v_1(t)$ and $|\dot{x}^k(t)| \le v_2(t)$ a.e. on I [5, Thm IV.9]. It then follows from assumption (B) that

$$\left|\Phi^1(t, \omega^k(t), x^k(t), \dot{x}^k(t))\right| \le C(1 + v_1(t)^p + v_2(t)^2) + \breve{h}(t)$$

and

$$\left|\Phi^1(t, \bar{\omega}(t), \bar{x}(t), \dot{\bar{x}}(t))\right| \le C(1 + \bar{\omega}(t)^p + \dot{\bar{x}}(t)^2) + \breve{h}(t).$$

Since the upper bounds lie in L^1, by Lebesgue's Dominated Convergence Theorem

$$\lim_{k\to\infty} \int_I \Phi^1(t, \omega^k(t), x^k(t), \dot{x}^k(t))dt = \int_I \Phi^1(t, \bar{\omega}(t), \bar{x}(t), \dot{\bar{x}}(t))dt,$$

violating (3.19). Since the control set M is bounded, the same reasoning shows that the integral in (3.17) converges to zero. For the integral in (3.18), let φ denote the L^1 function $t \mapsto \Phi^2(t, \bar{\omega}(t), \bar{x}(t), \dot{\bar{x}}(t))$. For any $\epsilon > 0$ there exists an L^2 function ψ so that

$$\int_I |\varphi(t) - \psi(t)|dt < \epsilon.$$

Now

$$\int_I \left\langle \varphi(t), u^k(t) - \bar{u}(t)\right\rangle dt = \int_I \left\langle \varphi(t) - \psi(t), u^k(t) - \bar{u}(t)\right\rangle dt + \int_I \left\langle \psi(t), u^k(t) - \bar{u}(t)\right\rangle dt.$$

Since $u^k \to \bar{u}$ weakly in L^2, the last integral converges to zero. The second integral is bounded by a constant times ϵ since M is bounded. Since ϵ is arbitrary, the limit of the integral in (3.18) is zero as well. This proves the result. $\square$

The value $V : \mathcal{X} \to \mathbb{R}$ of the optimal control problems $\mathcal{P}^{\omega,a,b}$ is defined as

$$V_{\omega,a,b} = \inf\{\mathcal{J}_{\omega,a,b}(x, u) : u \in \mathcal{U}, x \in X_u\}. \tag{3.20}$$

By Theorem 2.1, under assumptions (A)–(C) for each $\omega \in L^p$ and arbitrary boundary conditions $(a, b) \in \mathbb{R}^n \times \mathbb{R}^n$ there exists an optimal solution (x_*, u_*), i.e., $V_{\omega,a,b} = \mathcal{J}_{\omega,a,b}(x_*, u_*)$. Hence we have

$$V_{\omega,a,b} = \min\{\mathcal{J}_{\omega,a,b}(x, u) : (u, x) \in \Xi(\omega, a, b)\}. \tag{3.21}$$

Now define the set-valued map or correspondence $\Xi_* : \mathcal{X} \to \mathcal{Y}$ by assigning to each $(\omega, a, b) \in \mathcal{X}$ the set $\Xi_*(\omega, a, b)$ of optimal pairs (x_*, u_*) for the problem $\mathcal{P}^{\omega,a,b}$. It then more generally follows from Berge's Theorem [2] that the function V is continuous at every point $(\omega, a, b) \in \mathcal{X}$ where the correspondence Ξ is lower semi-continuous and that the correspondence Ξ_* is upper semicontinuous at (ω, a, b). Thus, in particular, we have:

(*) Suppose $\omega^k \to \bar{\omega}$ in L^p, $a^k \to \bar{a}$ and $b^k \to \bar{b}$ in $\mathbb{R}^n$, and let (x_*^k, u_*^k) be optimal solutions for the problems $\mathcal{P}^{\omega^k, a^k, b^k}$. If for every $\bar{u} \in \mathcal{U}$ the solution $x_{\bar{u}}$ of (1.12) corresponding to $\bar{u}$ with parameter $\bar{\omega}$ and boundary data $(\bar{a}, \bar{b})$ is unique, then every accumulation point $(\bar{x}, \bar{u})$ of the sequence $\{(x_*^k, u_*^k)\}_{k\in\mathbb{N}}$ is an optimal solution for the problem $\mathcal{P}^{\bar{\omega}, \bar{a}, \bar{b}}$ and $V_{\omega^k, a^k, b^k} \to V_{\bar{\omega}, \bar{a}, \bar{b}}$ as $k \to \infty$.

Finally, since any such sequence $\{(x_*^k, u_*^k)\}_{k\in\mathbb{N}}$ is weakly compact, accumulation points exist and by Lemma 3.2 the sequence is strongly convergent in H^1. This proves the theorem.

4. Examples

We give two examples to illustrate the results.

EXAMPLE 1. Let

$$v(p) = \begin{cases} 0 & \text{for } |p| \le 1, \\ (p+1)^2 & \text{for } p < -1, \\ (p-1)^2 & \text{for } p > 1, \end{cases} \tag{4.1}$$

so that the function v is continuously differentiable on $\mathbb{R}$. Consider the optimal control problem to minimize the functional

$$\mathcal{J}(x,u) = \int_I x(t)u(t)dt \tag{4.2}$$

subject to

$$\ddot{x}(t) = \omega^k x(t)u(t) + v_x(x(t)), \quad x(0) = x(\pi) = 0, \tag{4.3}$$

where $\omega^k = 1 - \frac{1}{k}$, $k \in \mathbb{N}$, $\bar{\omega} = 1$, $u(t) \in [0,1]$, and the function v is given by formula (4.1). The functional of action related to (4.3) is of the form (cf. (1.14))

$$\mathcal{A}_u^k(x) = \int_I \left[\frac{1}{2} |\dot{x}(t)|^2 - \frac{\omega^k}{2} |x(t)|^2 u(t) + v(x(t)) \right] dt. \tag{4.4}$$

For $k \in \mathbb{N}$ the functional $\mathcal{A}_u^k$ is strictly convex on H_0^1. Thus (4.3) has exactly one solution, namely $x = 0$, for each $u \in \mathcal{U} = \{u \in L^2 : u(t) \in [0,1]\}$. This implies that $\Xi^k = \{(0,u) : u \in \mathcal{U}\} = \Xi_*^k$ and $V_k = V_{\omega^k,0,0} = 0$ for all $k \in \mathbb{N}$. In the limit case, for $\bar{\omega} = 1$ the functional $\mathcal{A}_u^{\bar{\omega}}$ in (4.4) is convex and nonnegative for each $u \in \mathcal{U}$. If $\bar{u}$ is less than 1 on a set $E \subset [0,1]$ of positive measure, then the functional (4.4) is strictly convex and equation (4.3) with $\bar{\omega} = 1$ has only the one solution $\bar{x} = 0$ corresponding to $u = \bar{u}$. However, if $u \equiv 1$, then also the pairs $(c \sin t, 1)$, $|c| \le 1$, are admissible for the control system (4.3). It is easy to see that $\Xi_*(\bar{\omega}, 0, 0) = \{(-\sin t, 1)\}$ and $V_{\bar{\omega},0,0} = -2$. Thus $V_k \nrightarrow V_{\bar{\omega},0,0}$. Clearly, the set-valued map Ξ is not lower semicontinuous at $(\bar{\omega}, 0, 0)$ since the admissible pairs $(c\sin t, 1)$, $|c| \le 1$, cannot be approximated by admissible pairs along the sequence. The optimal control problem (4.2)–(4.3) is ill-posed.

EXAMPLE 2. Consider the optimal control system with perturbations

$$\ddot{x}(t) = -\frac{1}{2}\omega^k(t)x(t)u(t) + 2(\omega^k(t))^2 x(t)e^{x^2(t)} + u(t), \tag{4.5}$$

boundary conditions $x(0) = x(\pi) = 0$ and performance index

$$\mathcal{J}^k(x,u) = \int_I \left[x(t) + \omega^k(t)x^2(t)u(t) + u(t)\right] dt, \tag{4.6}$$

where $\omega^k \in L^p$ takes values in $[0,1]$ and tends to zero in $L^p(\mathbb{R})$ as $k \to \infty$. The control set is $M = [0,1]$. Using Theorem 2.1 one can easily show that the optimal control system $\mathcal{P}^k$ given by (4.5) and (4.6) possesses at least one optimal process (x_*^k, u_*^k) for $k = 1, 2, \ldots$, but it seems difficult to find this process effectively for $k = 1, 2, \ldots$ In the limit case we get the linear system $\bar{\mathcal{P}}$ given by

$$\ddot{x}(t) = u(t), \quad x(0) = x(\pi) = 0, \tag{4.7}$$

$$\bar{\mathcal{J}}(x,u) = \int_I (x(t) + u(t))dt, \tag{4.8}$$

with $u \in [-1,1]$. We can apply the extremum principle (cf. [12]) to the system $\bar{\mathcal{P}}$ and look effectively for an optimal process. In fact, the Lagrange function for the problem $\bar{\mathcal{P}}$ takes the form

$$\mathcal{L}(\lambda_0, \psi, x, u) = \lambda_0 \int_I (x(t) + u(t))dt + \int\limits_I \langle \psi(t), \ddot{x}(t) - u(t)\rangle\, dt \tag{4.9}$$

with $\lambda_0 \ge 0$ and $\psi \in L^2$. Using Theorem 2.1 it is easy to show that the problem $\bar{\mathcal{P}}$ possesses an optimal process (x_*, u_*). By the extremum principle we get

$$\mathcal{L}_x(\lambda_0, \psi, x_*, u_*)h = \int_I \left[\lambda_0 h(t) + \psi(t)\ddot{h}(t)\right] = 0 \tag{4.10}$$

for each $h \in H_0^2$. Moreover,

$$\mathcal{L}(\lambda_0, \psi, x_*, u_*) = \inf_{u \in \mathcal{U}} \mathcal{L}(\lambda_0, \psi, x_*, u), \tag{4.11}$$

where $\mathcal{U} = \left\{u \in L^2 : u(t) \in [-1,1]\right\}$. Integrating (4.10) by parts we obtain

$$\int_I \left[\int_0^t \int_0^{t_1} \lambda_0 d\tau dt_1 + \psi(t)\right] \ddot{h}(t)dt = 0. \tag{4.12}$$

Applying the fundamental lemma of the second order (cf. [22]) to (4.12) we get

$$\frac{1}{2}\lambda_0 t^2 + \psi(t) = c_1 t + c_0,$$

where c_0 and c_1 are constants. Thus $\psi(t) = -(1/2)\lambda_0 t^2 + c_1 t + c_0$ and the infimum condition (4.11) takes the form

$$\int_I \left[\lambda_0 - \left(-\frac{1}{2}\lambda_0 t^2 + c_1 t + c_0\right)\right] u_*(t)dt$$
$$= \inf_{u\in\mathcal{U}} \int_I \left[\lambda_0 - \left(-\frac{1}{2}\lambda_0 t^2 + c_1 t + c_0\right)\right] u(t)dt.$$

The switching function $\varphi(t) = (1/2)\lambda_0 t^2 - c_1 t + \bar{c}_0$, $\bar{c}_0 = -c_0 + \lambda_0$, has no more than two roots. Thus the optimal control $\bar{u}_*$ is piecewise constant, takes only the values -1 or $+1$ and has no more than two jump points. By a direct calculation we can show that $\bar{u}_*(t) = -1$, $\bar{x}_*(t) = -\frac{1}{2}t^2 + \frac{\pi}{2}t$ for $t \in [0,\pi]$ and $\bar{V} = \frac{1}{12}\pi^3 - \pi \approx -0.56$.

Let us return to the perturbed system $\mathcal{P}^k$. By Theorem 2.1 the set of optimal processes $X_*^k \times \mathcal{U}_*^k$ for the system $\mathcal{P}^k$, $k \in \mathbb{N}$, is nonempty. Let $\left\{(x_*^k, u_*^k)\right\}_{k=1}^{\infty}$ be a sequence of optimal processes, i.e., $(x_*^k, u_*^k) \in \Xi_*^k$ and x_*^k is a trajectory of the system (4.5) corresponding to the optimal control u_*^k and to the parameter ω^k. Since the optimal process for the system $\bar{\mathcal{P}}$ is uniquely determined, i.e., the set $\bar{\Xi}_*$ is the singleton $(\bar{x}_*, \bar{u}_*)$, Theorem 3.1 implies that x_*^k tends to $\bar{x}_*(t) = -\frac{1}{2}t^2 + \frac{\pi}{2}t$ uniformly on $I = [0,\pi]$, $\frac{d}{dt}x_*^k(t)$ tends to $\frac{d}{dt}\bar{x}_*(t) = -t + \frac{\pi}{2}$ weakly in L^2, $u_*^k(t)$ tends to $\bar{u}_* \equiv -1$ weakly in L^2 and the sequence of the optimal values $\left\{V^k\right\}_{k=1}^{+\infty}$ converges to $\bar{V} = \frac{\pi^3}{12} - \pi$.

Acknowledgement. We would like to thank an anonymous referee for pointing out Berge's theorem to us. This has greatly streamlined the reasonings of the paper.

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