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BIG DEFORMATIONS NEAR INFINITY

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Abstract. In a related paper we showed that Ruelle's property for a Fuchsian group G fails if the group has a condition we called 'big deformations near infinity'. In this paper we give geometric conditions on $R = \mathbb{D}/G$ which imply this condition. In particular, it holds whenever G is divergence type and R has injectivity radius bounded from below. We will also give examples of groups which do not have big deformations near infinity.

1. Introduction

If G is a Fuchsian group and μ is a bounded measurable function on the unit disk, $\mathbb{D}$, which satisfies $\|\mu\|_\infty < 1$ and $\mu(g(z)) = \mu(z)g'(z)/\overline{g'(z)}$, for every $g \in G$, then we say μ is a G-invariant Beltrami coefficient (or complex dilatation). It is well known that there is a corresponding quasiconformal mapping f_μ which is analytic outside the disk and which conjugates G to a quasi-Fuchsian group G_μ. For convenience, we will write $\delta(\mu) = \delta(G_\mu)$ and $\dim(\mu) = \dim(\Lambda(G_\mu))$ when G is clear from the context (δ denotes the critical exponent and dim the Hausdorff dimension; see Section 2).

We will say that a G-invariant dilatation μ for a Fuchsian group G is a *big deformation* if $\delta(\mu) > 1$. Furthermore, we say a Fuchsian group G has *big deformations near infinity* if there are $\epsilon, \delta > 0$ so that for any compact set $K \subset R = \mathbb{D}/G$, there is a Beltrami coefficient μ supported in $R \setminus K$ with $\|\mu\|_\infty \le 1 - \epsilon$ and $\delta(G_\mu) \ge 1 + \delta$. This condition arose in [7] as a criterion for the non-analytic dependence of $\delta(\mu)$ and $\dim(\mu)$ on μ. In this paper, we will give geometric conditions on R which imply that such deformations exist. For example:

Theorem 1.1. *Suppose G is a torsion free, infinitely generated Fuchsian group and either of the following holds:*

(1) *the injectivity radius is bounded above and below,*

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(2) *G is divergence type and the injectivity radius is bounded below.*

Then G has big deformations near ∞.

This will follow from a more general result. To state it, we need to introduce a few more definitions. If $\{G_n\}$ is a sequence of Kleinian groups acting on 3-dimensional hyperbolic space (consider the ball model), we will say that $\{G_n\} \to G$ geometrically if $G_n(0) \cup S^2$ converges to $G(0) \cup S^2$ in the Euclidean Hausdorff metric. We say that G' is a geometric boundary group of G (and write $G' \in \partial_g G$) if there is a sequence of Möbius transformations $\{g_n\}$ such that $g_n \circ G \circ g_n^{-1} \to G'$. If the injectivity radius at $g_n(0)$ tends to zero then the resulting limit (if it exists) is elementary and non-discrete. Thus in this paper we will only take limits over sequences where the injectivity radius is bounded away from zero. Given any such sequence there is always a subsequence which converges to a discrete group.

In [8] it is proven that if $G_n \to G$ in this sense then $\liminf \delta(G_n) \geq \delta(G)$. Thus if $G' \in \partial_g G$, then $\delta(G) \geq \delta(G')$. Using this one can show (Theorem 3.1) that if $G' \in \partial_g G$ has a big deformation, then G has big deformations near infinity.

We will call a Fuchsian group exceptional if it is the covering group of the sphere minus m disks and n points where $1 \leq m + n \leq 3$ and denote this family by $\mathcal{E} = \bigcup_{m+n \leq 3} \mathcal{E}_{m,n}$. The case $m = 0, n = 3$ (i.e., the thrice punctured sphere) plays a special role because it is the only hyperbolic surface for which the Teichmüller space is trivial. The groups in $\mathcal{E}$ are exceptional from our point of view because by Theorem 3.2 every Fuchsian group not in $\mathcal{E}$ has a big deformation.

Theorem 1.2. *Suppose G is a torsion free, infinitely generated Fuchsian group and either of the following holds:*

(1) $\partial_g G \not\subset \mathcal{E}$,

(2) *G is divergence type and $\mathcal{E}_{0,3} \notin \partial_g G$.*

Then G has big deformations near ∞.

If G has upper or lower bounds on its injectivity radius then the same bounds apply to any geometric limits and by examining each of the groups in $\mathcal{E}$ we see that Theorem 1.1 is an immediate corollary.

We shall say a Fuchsian group G has the Ruelle property if whenever $\{G_t\}$ is an analytic family of quasiconformal deformations of G then $\dim(\Lambda(G_t))$ is a real analytic function of t ($\dim(\Lambda)$ is the Hausdorff dimension of its limit set). Ruelle [15] showed that every cocompact Fuchsian group has this property and Astala and Zinsmeister gave various examples of infinitely generated, convergence type Fuchsian groups for which it fails [2], [3]. In [7] we showed that if G is a torsion free, infinite area Fuchsian group with big deformations near infinity, then the Ruelle property fails. Thus the results of this paper provide new examples where Ruelle's property fails.

We will also give examples of infinitely generated groups which do not have big deformations near infinity. A generalized Y-piece in a Riemann surface R is a region bounded by three simple closed geodesics (or punctures) which is homeomorphic to a 2-sphere minus three disks (or points). If all three boundary components have length $\le L$ we say the Y-piece is L-bounded (punctures count as zero length). We say that R has a L-bounded Y-piece decomposition if it can be written as a union of L-bounded Y-pieces with disjoint interiors.

THEOREM 1.3. *Suppose $R = \mathbb{D}/G$ has a L-bounded Y-piece decomposition and that for every $\epsilon > 0$ all but finitely many of the Y-pieces are ϵ-bounded. Then G does not have big deformations near infinity.*

We say that such surfaces "approximate a thrice puncture sphere" near infinity. It is not hard to see that this is equivalent to saying that $\partial_g G = \mathcal{E}_{0,3}$. It remains open whether such surfaces have Ruelle's property or not.

The remaining sections are organized as follows. In Section 2 we review some basic definitions and results. In Section 3 we show that the non-exception groups have big deformations and prove part (1) of Theorem 1.2. In Section 4 we record a result on random geodesics. In Section 5, we complete the proof of Theorem 1.2 using a theorem of Dennis Sullivan about convex hulls in hyperbolic space. In Section 6, we prove Theorem 1.3.

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2. Background

Now we review some basic definitions. Given a set E, we define

$$\mathcal{H}^\alpha_\delta(E) = \inf \left\{ \sum \operatorname{diam}(U_j)^\alpha : E \subset \bigcup_j U_j, \operatorname{diam}(U_j) \le \delta \right\},$$

where the infimum is over all coverings of E by sets of diameter $\le \delta$. Taking $\delta = \infty$ gives the Hausdorff content $\mathcal{H}^\alpha_\infty(E)$, and the Hausdorff measure of E is given by

$$\mathcal{H}^\alpha(E) = \lim_{\delta \to 0} \mathcal{H}^\alpha_\delta(E).$$

The Hausdorff dimension of E is

$$\dim(E) = \inf\{\alpha : \mathcal{H}^\alpha(E) = 0\} = \inf\{\alpha : \mathcal{H}^\alpha_\infty(E) = 0\}.$$

A Fuchsian group is cocompact if $\mathbb{D}/G$ is compact and is cofinite if the quotient has finite hyperbolic area. A Fuchsian group is called divergence type if

$$\sum_{g\in G} \exp(-\rho(0, g(0))) = \infty,$$

and otherwise it is called convergence type. There are several other conditions which are equivalent to divergence type. For example, a Fuchsian group G is divergence type iff the conical limit set Λ_c has full Lebesgue measure on the circle. Similarly G is divergence type iff the geodesic flow for $R = \mathbb{D}/G$ is ergodic (see [13]).

The group is called first kind if the limit set is the entire circle, and is called second kind otherwise. It is well known that cocompact $\subset$ cofinite $\subset$ divergence type $\subset$ first kind.

Given a Fuchsian group G and a point $z \in \mathbb{D}$, the injectivity radius of G at z is half the distance from z to the nearest distinct G-image of z, i.e.,

$$\operatorname{inj}(z) = \frac{1}{2}\inf\{\rho(z, g(z)) : g(z) \neq z\}.$$

Given $\epsilon > 0$ we define the ϵ-thick and ϵ-thin parts of $R = \mathbb{D}/G$ to be the points where the injectivity radius is $\geq \epsilon$ or $< \epsilon$, respectively. There is a $\epsilon_M > 0$ (called the Margulis constant) independent of G so that for $\epsilon < \epsilon_M$ the ϵ-thin parts are disjoint and of exactly two possible types: the parabolic thin parts which are horoballs in $\mathbb{D}$ tangent to the unit circle at a parabolic fixed point and the hyperbolic thin parts which are fixed distance neighborhoods of a hyperbolic geodesic (see Figure 1). Later we will want to use the following easy estimate.

Lemma 2.1. *Suppose $P \subset \mathbb{D}$ is a parabolic or hyperbolic thin part not containing the origin. Then the set of radial geodesics $\gamma(\theta)$ such that the hyperbolic length of $\gamma \cap P$ is $\geq n$ has angle measure $\leq C\operatorname{diam}(P)e^{-n/2}$.*

Proof. For parabolic thin parts this is immediate from the geometry of a horoball. For hyperbolic thin parts it follows from the previous case and the observation that any hyperbolic thin part is contained in the union of two horoballs of comparable diameter (see Figure 1). □

The critical exponent (or Poincaré exponent) of the group is defined as

$$\delta(G) = \inf\left\{ s : \sum_{g\in G} \exp(-s\rho(0, g(0)))\right\} < \infty.$$

It is shown in Lemma 2.2 of [9] that if Ω is a bounded simply connected invariant component then one can also take

$$\delta(G) = \inf\left\{ s : \sum_{g\in G} \operatorname{dist}(g(z_0), \partial\Omega)^s < \infty\right\}, \tag{1}$$

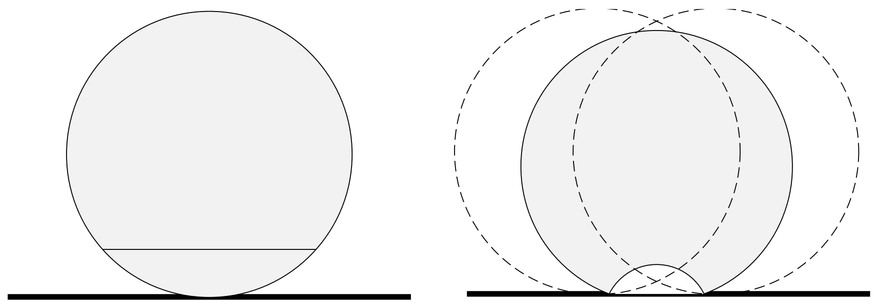

FIGURE 1. Parabolic and hyperbolic thin parts.

for any $z_0 \in \Omega$ and where dist denotes Euclidean distance.

The conical limit set, Λ_c, is defined to be those limit points $x \in \Lambda$ for which some subsequence of the orbit of 0 approaches x within a non-tangential cone with vertex x. Equivalently, $x \in \Lambda_c$ iff the radial segment ending at x projects to a geodesic ray on the quotient orbifold which returns to some compact subset infinitely often (the complement consists of rays which eventually leave every compact set and is called the escaping limit set Λ_e). By Theorem 1.1 in [8], δ is equal to the Hausdorff dimension of the conical limit set. It is also proven there that if $G_n \to G$ (in the sense that each point of $G(0)$ is a limit of points from $\{G_n(0)\}$), then

$$\liminf_n \delta(G_n) \geq \delta(G).$$

A K-quasiconformal map from $\mathbb{R}^2$ to itself is one which satisfies

$$\limsup_{r\to 0} \frac{\max_{|x-y|=r} |f(x) - f(y)|}{\min_{|x-y|=r} |f(x) - f(y)|} \leq K.$$

Such a map is well known to be differentiable almost everywhere and we let $\mu = f_z/f_{\bar{z}}$ denote its complex dilatation. It is also known that $\|\mu\|_\infty \leq (K-1)/(K+1)$ and that the map f is determined (up to composition with Möbius transformations) by its dilatation. If we assume three points are fixed (e.g., 0, 1, ∞) then the map is uniquely determined by μ.

We shall say that a dilatation μ supported on the unit disk is G-invariant if

$$\mu(g(z)) = \mu(z) g'(z)/\overline{g'(z)},$$

for every $g \in G$. If μ is invariant under the action of a Fuchsian group G, then $f_\mu \circ g \circ f_\mu^{-1}$ is Möbius for every $g \in G$ and f_μ is called a quasi-conformal deformation of G. Given such a μ we let $\delta(\mu)$ denote the critical exponent of the group $G_\mu = f_\mu \circ G \circ f_\mu^{-1}$ and let $\dim(\mu) = \dim(\Lambda(G_\mu))$. Since any two groups associated to the same μ are conjugate by a Möbius transformation, these functions are well defined.

A quasiconformal conjugacy of G is a quasiconformal mapping of the disk to itself which conjugates G to another Fuchsian group. Later we will use the theorem of Pfluger [14] that any quasiconformal conjugate of a divergence type group is also divergence type.

Given an interval $I \subset \mathbb{T}$, the corresponding Carleson "square" is

$$Q = Q_I = \{z \in \mathbb{D} : z/|z| \in I, 1 - |z| \leq |I|\}.$$

The "top half" of Q is defined as

$$T(Q) = \left\{ z \in \mathbb{D} : z/|z| \in I, \frac{1}{2}|I| \leq 1 - |z| \leq |I| \right\}.$$

A stopping time region is a domain of the form $W = Q_I \setminus \bigcup_j Q_{I_j}$ where I_j is a collection of pairwise disjoint intervals in I.

We will also need the following result from [10]. It basically says that at points far from the support of μ the corresponding map f_μ behaves like a Hölder function with exponent close to 1. Given $z \in S^2$, let z^* denote it reflection across the unit circle, $\mathbb{T}$.

LEMMA 2.2. *Given $K < \infty$ and $\eta > 0$ there are $C < \infty$ and $r < \infty$ so that the following holds. Suppose f is a conformal map of the disk with a K-quasiconformal extension to the plane with Beltrami coefficient μ. Also suppose that* $\operatorname{diam}(f(\mathbb{D})) \leq K|f'(0)|$. *Suppose $W = Q \setminus \bigcup_j Q_j$ is a stopping time region, $W^* = \{z^* : z \in W\}$, and $\rho(W^*, \operatorname{supp}(\mu)) \geq r$. Then for any collection $\mathcal{C} \subset \{Q_j\}$*

$$\sum_{\mathcal{C}} \operatorname{diam}(f(Q_j))^{1+\eta} \leq C \operatorname{diam}(f(Q))^{1+\eta} \sum_{\mathcal{C}} \frac{\operatorname{diam}(Q_j)}{\operatorname{diam}(Q)}.$$

Two Y-pieces, with a pairing of the boundary components, are close to each other if their boundary lengths are close, i.e., we define

$$d(Y_1, Y_2) = \max\left(\left|\log \frac{a_1}{a_2}\right|, \left|\log \frac{b_1}{b_2}\right|, \left|\log \frac{c_1}{c_2}\right| \right).$$

For generalized Y-pieces we interpret $|\log a_1/a_2|$ as zero if $a_1 = a_2 = 0$ and as $+\infty$ if one is zero and the other is not. Similarly for the b and c terms. The following is Corollary 6.3 of [6].

LEMMA 2.3. *Suppose Y_1 and Y_2 are two L-bounded generalized Y-pieces and let $D = d(Y_1, Y_2)$. Then there is a quasiconformal map $f : Y_1 \to Y_2$ with constant $K = K(L, D)$ which is affine on each of the boundary components. Moreover, the dilatation K_f of f satisfies*

$$|K_f(z)| \leq 1 + C(L, D) \exp(-2\operatorname{dist}(z, \partial Y_1)).$$

In particular, if Y_1 is 1-bounded and D is fixed, then the quasiconformal dilatation is $\leq 1+\nu$ except on a A-neighborhood of the boundary, where $A < \infty$ depends only on D and ν. If Y_1 has very short boundaries, this means the dilatation is concentrated near the boundaries and the map is almost conformal on most of Y_1.

3. Groups with big deformations

LEMMA 3.1. *Suppose $\{G_n\}$ is a sequence of Fuchsian groups which converges geometrically to a group G. If G has a deformation G_μ such that $\delta(G_\mu) > s > 1$, then so does G_n for all sufficiently large n.*

Proof. Let $\{K_j\}$ be a compact exhaustion of a convex, open fundamental domain $\mathcal{F}$ for the group G (also note that $\mathcal{F}$ has boundary of zero area by convexity). By considering the deformations $\mu_j = \mu\chi_{K_j}$ and using the lower semi-continuity of δ to deduce

$$\liminf_j \delta(\mu_j) \geq \delta(\mu) > s,$$

we may assume that μ is compactly supported on a compact set $K \subset \mathcal{F}$. Then for n large enough K is a subset of a fundamental domain $\mathcal{F}_n$ for G_n and thus may be extended to a G_n invariant dilatation. Moreover, the corresponding dilatations on the disk converge to μ and hence the corresponding quasiconformal maps converge uniformly to the map corresponding to μ. Thus the deformations of G_n converge geometrically to the deformation of G and so by the lower semi-continuity of δ we have $\delta > s$ for all sufficiently large n. □

Next we recall that $\mathcal{E}_{m,n}$ denotes the class of Fuchsian groups which covers the sphere minus m disks and n punctures and we let $\mathcal{E} = \bigcup_{m+n\leq 3} \mathcal{E}_{m,n}$. These groups are "exceptional" in the following sense.

THEOREM 3.2. *If G is a torsion free Fuchsian group not in $\mathcal{E}$ then G has a deformation G_μ with $\delta(G_\mu) > 1$.*

Proof. The classification theorem for hyperbolic Riemann surfaces given in [1] says that every such surface is a union of funnels, half-disks and an open set which can be exhausted by geodesic domains, and only the disk and annulus have no Y-pieces. The geodesic domains, in turn, are finite unions of geodesic Y-pieces (where one or more of the boundaries may be a puncture). This easily implies that every hyperbolic surface not in $\mathcal{E}$ is either (1) a genus 1 torus with a puncture, (2) a genus 1 torus with a disk removed, or (3) it contains two Y-pieces joined along a common boundary geodesic. In case (1), there is a big deformation because the punctured torus has finite area and the Teichmüller space is not trivial. In case (2), the surface can be deformed so it converges to the punctured torus, and then we apply the previous case and Lemma 3.1.

In case (3), when there are two Y-pieces joined along a common geodesic, the interior of this union is topologically the sphere minus k simple closed geodesics and j punctures where $k+j=4$. If $k=0$, then R is the 4-punctured sphere and thus has non-trivial Teichmüller space [12] and hence has a deformation with $\delta > 1$. If $k > 0$ then the k closed geodesics can be deformed to punctures by quasiconformal deformations and hence G has a sequence of deformations $\{G_n\}$ which converge geometrically to a 4-punctured sphere group. Thus some G_n has a deformation with $\delta > 1$ by Lemma 3.1. Since G_n is a deformation of G we see that G also has a deformation with $\delta > 1$. $\square$

Corollary 3.3. *Suppose G is a torsion free Fuchsian group and $\partial_g G$ contains a group $G' \notin \mathcal{E}$. Then G has big deformations near ∞, i.e., there are dilatations $\{\mu_n\}$ whose supports leave every compact set in $R = \mathbb{D}/G$, and $\liminf_n \delta(\mu_n) > 1$.*

Proof. Let $G_n = g_n \circ g \circ g_n^{-1}$ denote a sequence of renormalizations of G which converge to G' and let μ be a deformation of G' such that $\delta(\mu) > 1$. Just as in the proof of Lemma 3.1 we may assume that μ is supported on a compact set K in an open fundamental domain for G'. Thus just as before, K will be a subset of a fundamental domain for G_n, and will extend to a deformation μ_n such that $\delta(\mu_n) > 1$ for n sufficiently large. Considering $\{\mu_n\}$ as a sequence of dilatations on G instead, we see that they have supports which leave every compact set, as desired. $\square$

Thus the only infinitely generated Fuchsian groups which might not have big deformations near ∞ are those with $\partial_g G \subset \mathcal{E}$. This is the first part of Theorem 1.2. To prove the second part of Theorem 1.2 we will prove:

Theorem 3.4. *If G is a torsion free divergence type group such that $\partial_g G \subset \mathcal{E} \setminus \mathcal{E}_{0,3}$, then G has big deformations near ∞.*

Recall that $\mathcal{E}_{0,3}$ denotes the covering group of the thrice punctured sphere. Thus for infinitely generated divergence groups, at least, this group is the only obstruction to the failure of Ruelle's property. In particular, if G has a lower bound on its injectivity radius then $\mathcal{E}_{0,3}$ cannot possibly be a boundary group of G and so we deduce that for such groups Ruelle's property always fails.

4. Sets hit by a random geodesic

In this section we record a technical result that we will use in the next section for the proof of Theorem 3.4. It basically says that a set which is uniformly distributed in $R = \mathbb{D}/G$ will frequently be hit by almost every geodesic ray. This will follow from the strong law of large numbers and a few simple facts about the geometry of thin parts. This is probably not a new result, but since I don't know a reference I will give the proof for completeness.

To be more precise, let $\gamma(t, \theta)$ be the point in the disk which is at hyperbolic distance t from the origin in the direction $e^{i\theta}$.

LEMMA 4.1. *Suppose G is an infinite area, torsion free Fuchsian group and that $K \subset R = \mathbb{D}/G$ is compact. Suppose $\mathcal{B}$ is a collection of hyperbolic ϵ-balls in R_{thick} such that every point in $R_{\text{thick}} \setminus K$ is within hyperbolic distance M of some ball in $\mathcal{B}$. Let Ω be the lift of $\bigcup_{\mathcal{B}} B$ to the disk. Then there is a $\eta > 0$ (depending on ϵ and M, but not on K) such that*

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t \chi_\Omega(\gamma(s, \theta)) ds \geq \eta > 0,$$

for (Lebesgue) almost every θ. In other words, a random geodesic ray in R spends at least a fixed fraction of its time inside Ω.

We will prove this in several steps. The first step says the chance of being in a compact subset of an infinite area surface is zero.

LEMMA 4.2. *Suppose G is an infinite area group and suppose $E \subset \mathbb{D}$ is a G-invariant set such that E/G is compact. Then*

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \chi_E(\gamma(s, \theta)) ds = 0.$$

Proof. If G is convergence type there is nothing to do because in this case the conical limit set has Lebesgue measure zero, i.e., almost every geodesic ray leaves every compact set eventually. Hence we may assume G is divergence type. Suppose $E \subset F$ and F is also compact and G invariant. Then since the geodesic flow for G is ergodic (see, e.g., Chapters 7 and 8 of [13]) and $\chi_E, \chi_F \in L^1$, for almost every choice of origin and almost every θ,

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \chi_E(\gamma(s, \theta)) ds \leq \lim_{t\to\infty} \frac{\int_0^t \chi_E(\gamma(s, \theta)) ds}{\int_0^t \chi_F(\gamma(s, \theta)) ds} = \frac{\text{area(E)}}{\text{area(F)}},$$

where area refers to hyperbolic area. Since we can take F to have as large area as we wish (since $R = \mathbb{D}/G$ has infinite area) we deduce that the left hand side is zero, as desired.

Actually, the ergodicity of the geodesic flow states that the argument above is valid for almost every choice of base point. However, if we choose two different base points x and y the geodesic rays from these points which land at the same point of the boundary come together exponentially fast. Thus by expanding E to E' by taking a 1-neighborhood, we see that the ray from x spends at least as much time in E' as the one from y spends in E. Thus if the result is true for base point x, it is also true for y. Thus the result holds for every base point and not just almost every point. $\square$

LEMMA 4.3. *Suppose G is an infinite area Fuchsian group and let $W \subset \mathbb{D}$ denote the points where the injectivity radius is $\geq \epsilon_0$ (i.e., W is the lift of the thick part of R). Assume that ϵ_0 has been chosen so small that the complementary thin parts are at least hyperbolic distance* 10 *apart. Then*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \chi_W(\gamma(s,\theta))ds = \eta_0 > 0,$$

for some $\eta_0 > 0$ which only depends on ϵ_0.

Proof. This follows from the strong law of large numbers and simple geometric properties of thin parts. The version of the law of large numbers we wish to use is the following well-known result.

THEOREM 4.4. *Suppose $\{f_n\} \subset L^2(X,\mu)$ is a sequence of orthogonal functions on a finite measure space and* $\sup_n \|f_n\|_2 < \infty$. *Then $\frac{1}{n}\sum_{k=1}^n f_n(x) \to 0$ for almost every x.*

To apply the result, we will replace W by a subregion V which is the union of tops of Carleson squares contained in W. It will be enough to prove the results for angles θ corresponding to an interval I_0 such that the corresponding Carleson square Q_0 satisfies $T(Q_0) \subset W$, for almost every point of the circle can be covered by such intervals.

Fix such a Carleson square Q_0 and form collections of dyadic subsquares inductively by saying $\mathcal{C}(Q)$ is the maximal collection of dyadic subsquares of $Q' \subset Q$ such that the top half of Q' consists entirely of points with injectivity radius $\geq \epsilon_0$. We write $\mathcal{C} = \bigcup_n \mathcal{C}_n$ by setting $\mathcal{C}_0 = Q_0$ and putting Q into $\mathcal{C}_n$ if $Q \in \mathcal{C}$ and it is a maximal proper subsquare of a square in $\mathcal{C}_{n-1}$.

By examining parabolic and hyperbolic thin parts, it is easy to check that the number of squares in $\mathcal{C}(Q)$ of size $2^{-n}\ell(Q)$ is at most $2^{n/2}$ and so their bases have total length at most $2^{-n/2}\ell(Q)$. See Lemma 2.1. This estimate is the main point; the rest of the proof is a simple calculation.

Define f_I on I' as $\log \ell(Q')/\ell(Q)$ and let $g_I(x) = f_I(x) - m_I(f_I)$ where m_I denotes the average value over I. Because of our estimates,

$$0 < m \leq m_I(f_I) \leq M < \infty,$$

with m and M independent of I. Let $g_n = \sum_{I\in\mathcal{C}_n} g_I$. Then g_I has mean value 0 over the interval I and so g_n has mean value zero over intervals where g_k, $k < n$, is constant. Thus $\{g_n\}$ is an orthogonal sequence and it is easy to see that these functions are in $L^2(\mathbb{T})$ with a uniform bound. Thus by the strong law of large numbers

$$\frac{1}{n}\sum_{k=1}^n g_k(\theta) \to 0,$$

for almost every θ. Unwinding the definitions, this means that for almost every θ,

$$mn \leq \sum_{k=1}^{n} f_k(\theta) \leq Mn.$$

Setting $t = \sum_{k=1}^{n} f_k(\theta)$, we see that for t large enough (depending on θ) the geodesic corresponding to angle θ enters at least t/M distinct $T(Q)$'s in V and hence spends at least time t/M in the thick part of R, as desired. $\square$

LEMMA 4.5. *Let G, K, M and $\mathcal{B}$ be as in Lemma 4.1. Then there are $C_1, C_2 < \infty$ so that the following holds. Suppose Q is a Carleson square such that $T(Q)$ is in W and the distance from $T(Q)$ to K is $\geq C_1$. Then there is a lift of a ball in $\mathcal{B}$ which is contained in Q and is at most distance C_2 from $T(Q)$.*

Proof. All we have to do is find a point z in $W \cap Q$ which is more than distance M from ∂Q and is exactly distance C_3 from $T(Q)$ (this describes an arc L in the interior of Q). Then taking any ball in $\mathcal{B}$ which is within distance M of this point will prove the lemma with $C_1 = C_2 + M$, $C_2 = C_3 + M$.

If there were no thick points on the line segment L then the whole of L must be in a single thin component. However, for a fixed M and taking C_3 large enough, it is easy to see that this implies $T(Q)$ is contained in the same thin component (since the component is hyperbolically convex and so contains the hull of L which contains a point of $T(Q)$ if its Euclidean diameter is close to that of the base of Q). This proves the lemma. $\square$

Proof of Lemma 4.1. Again we will use the strong law of large numbers. Let $W' \subset W$ be the union of top halves of Carleson squares which are more than distance C_1 from K. We can write $W = \bigcup_{Q\in\mathcal{C}} T(Q)$, and $W' = \bigcup_{Q\in\mathcal{C}'} T(Q)$, where $\mathcal{C}' \subset \mathcal{C}$ is the set of squares Q such that $T(Q)$ is more than hyperbolic distance C_1 from K, i.e., the squares which are disjoint from the compact set $K' = \{z : \rho(z, K) \leq C_1\}$. Note that

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{t}\int_0^t \chi_{W'}(\gamma(s,\theta))ds \\
&\qquad \geq \liminf_{t\to\infty} \frac{1}{t}\int_0^t \chi_{W}(\gamma(s,\theta))ds - \limsup_{t\to\infty} \frac{1}{t}\int_0^t \chi_{K'}(\gamma(s,\theta))ds \\
&\qquad \geq \eta_1,
\end{aligned}$$

by Lemmas 4.2 and 4.3.

Suppose $T(Q_j) \subset W'$ and let $B_j \subset Q_j$ be the corresponding ball given by the previous lemma. Let $J_j \subset I_j$ be the radial projection of $\frac{1}{2}B_j$ onto the circle. Let

$$f_I(\theta) = \chi_{J_j}(\theta) - \frac{|J_j|}{|I_j|}\chi_{I_j}(\theta),$$

and let

$$f_n = \sum_{I \in \mathcal{C}_n} f_I(\theta).$$

Then the strong law of large numbers applies to $\{f_n\}$ and we deduce that for almost every θ,

$$\int_0^t \chi_\Omega(\gamma(s,\theta))ds \geq C_4 \int_0^t \chi_W(\gamma(s,\theta))ds,$$

where C_4 is a uniform lower bound for $|J_j|/|I_j|$. Thus the number of times a ray of length t hits the center half of a ball $B \in \mathcal{B}$ is bounded below by a constant times the number of times it hits a $Q \in \mathcal{C}$. As noted above, the latter is bounded below by $\eta_1 t$ and this gives the desired result. $\square$

5. Proof of Theorem 3.4

Suppose G is a Fuchsian group of the first kind and that $G_\mu = f_\mu \circ G \circ f_\mu^{-1}$ is a quasi-Fuchsian deformation. Suppose $\Omega = f_\mu(\mathbb{D})$ and let $C(\Lambda_\mu) \subset \mathbb{H}^3$ denote the hyperbolic convex hull of $\partial\Omega = \Lambda(G_\mu) = f_\mu(\mathbb{T})$ and let S be the boundary component of $C(\Lambda_\mu)$ which faces Ω. By a result of Thurston (see [17] or [11]) the path metric ρ_S on S makes it isomorphic to the usual hyperbolic disk. Let $\iota : \mathbb{D} \to S$ be such an isomorphism.

Thus G_μ acting on S induces a Fuchsian group $H_\mu = \iota^{-1} \circ g_\mu \circ \iota$ acting on the disk. Moreover, a theorem of Sullivan ([4], [5], [11], [16]) states that S with its path metric is quasiconformal to Ω with its hyperbolic metric. Thus H_μ is quasiconformally conjugate to the group G. By a well known result of Pfluger [14], this implies that H_μ is divergence type iff G is.

Consider the case of $\Omega_0 = B(0,1) \cup ([0,10] \times [-1,1]) \cup B(10,1)$. Let S_0 denote the boundary component of the convex hull of $\partial\Omega_0$ which faces Ω_0. Then S_0 is easy to describe; it is the union of a Euclidean half cylinder (over the central rectangle) and two quarter spheres (over each half disk). In the central portion the only curves where the path metric agrees with the metric in $\mathbb{H}^3$ are the circles which lie in the (y,z) plane. Thus if we choose two points z_1, z_2 which are unit distance apart in the path metric but which have different x coordinates, their distance in $\mathbb{H}^3$ is < 1. In particular, it is easy to see by compactness that if we take two unit balls B_1 and B_2 on S_0 (say centered at the points above $(0,3)$ and $(0,6)$) then there is an $\epsilon > 0$ with the following property: for any $z \in S_0$ there is an $i \in \{1,2\}$ so that if $\gamma_0 = [z_1, z_2]$ is a unit length segment of a geodesic ray from z and $\gamma_0 \cap B_i \neq \emptyset$ then $\rho_{\mathbb{H}^3}(z_1, z_2) \leq 1 - \epsilon$. If Ω_1 is another domain which approximates Ω_0, the corresponding surface S_1 will approximate S_0 and if the approximation is close enough then we can find balls $B_1, B_2 \subset S_1$ with the same property (but perhaps with a slightly smaller ϵ). This is because if $S_n \to S$ and there are points $x_n, y_n \in S_n$ which converge to points $x_0, y_0 \in S_0$, then paths between x_n and y_n have a convergent subsequence to a path between x_0 and y_0. Thus the path metric between x_0 and y_0 is $\le \liminf \rho_{S_n}(x_n, y_n)$. Moreover, the three dimensional hyperbolic metric clearly converges.

For $x, y \in \mathbb{D}$, set $\Upsilon(x, y) = \rho_{\mathbb{H}^3}(\iota(x), \iota(y))$, and note that $\Upsilon \le \rho$. Clearly

$$\delta(G_\mu) = \inf \left\{ s : \sum_{h \in H_\mu} \exp(-s\Upsilon(0, h(0))) \right\} < \infty.$$

On the other hand, since H is divergence type,

$$1 = \delta(H_\mu) = \inf \left\{ s : \sum_{h \in H_\mu} \exp(-s\rho(0, h(0))) \right\} < \infty.$$

Given a compact set $K \subset R = \mathbb{D}/G$, we want to define a dilatation μ supported off K such that $\|\mu\|_\infty \le k$ for some $k < 1$ independent of K and with $\delta(\mu) > 1 + \delta_0$ for some $\delta_0 > 0$ independent of K. To do this we will use our assumption that $\partial_g G \subset \mathcal{E} \setminus \mathcal{E}_{0,3}$. This implies that given any $r > 0$, there is a compact $K_1 \subset R$ so that at every point z outside K_1, R approximates a surface from $\mathcal{E} \setminus \mathcal{E}_{0,3}$ on a ball of radius r around z. In particular, given any $r_1 > 0$ there is a $r_2 > 0$ and a compact $K_1 \subset R$ such that every thick point z in $R \setminus K_1$ is within distance r_2 of a point w where the injectivity radius is $\ge r_1$.

This means that we can choose a collection of radius r_1 balls in $R \setminus K_1$ which are topological disks and so that any thick point of $R \setminus K_1$ is within distance r_3 of one of the balls. The dilatation μ will be supported on the union of these balls. To define the dilatation μ on a particular ball B, consider a lift to the unit disk and assume B is centered at the origin. Take a quasiconformal mapping f of the disk to the region Ω_0 described above and which extends to be conformal outside the unit disk and let μ on B be the restriction of the dilatation of this map to B. If r_1 is large enough then when we extend μ to be G-invariant, it still maps the disk to a uniform approximation Ω_1 of Ω_0. Thus the surface S_1 which bounds the convex hull of Ω_1 is close to the convex hull of Ω_0. As explained above, this implies that we can choose two unit balls B_1, B_2 on S_1 about unit distance apart and a $\epsilon_0 > 0$ with the property that given any $z \in S$ there is an $i = 1, 2$ so that for any geodesic ray γ in S based at z and any unit length segment γ_0 of γ which hits B_i, the hyperbolic distance between its endpoints is $\le 1 - \epsilon_0$. Thus although S_1 may contain geodesic segments for the path metric which also look like geodesics for the hyperbolic metric, we can always move a short distance and find segments for which this is not true.

In terms of the group H_μ what we have shown is the following. There is a H_μ-invariant $E \subset \mathbb{D}$ which is compact modulo H_μ such that every point outside E is within a uniformly bounded hyperbolic distance of a unit hyperbolic ball $B \in \mathcal{B}$ with the property that every radial line segment $[x, y]$ of unit hyperbolic length which hits B has the property that $\Upsilon(x,y) \leq 1-\epsilon_0$. Now consider the radial geodesic from 0 to $\gamma(n,\theta)$ and let $x_k = \gamma(k,\theta)$ for $k = 0,\dots,n$. Then since Υ is a metric, the triangle inequality implies

$$\Upsilon(0,\gamma(n,\theta)) \leq \sum_{k=0}^{n-1} \Upsilon(x_k, x_{k+1}) \leq n - N(n,\theta)\epsilon_0,$$

where $N(n,\theta)$ is the number of balls $B \in \mathcal{B}$ which the segment hits. By Lemma 4.1, $N(n,\theta) > cn$ for some $c > 0$ for almost every θ and all n sufficiently large. Thus we get that

$$\limsup_{t\to\infty} \frac{1}{t}\Upsilon(0,\gamma(t,\theta)) \leq 1-\epsilon_1,$$

for almost every θ and some $\epsilon_1 > 0$. We claim this implies $\delta(G_\mu) > (1-\epsilon_1)^{-1} > 1$.

Given a point $z \in \mathbb{D} \setminus \{0\}$ let I_z be the interval on $\mathbb{T}$ centered at $z/|z|$ and of length $1-|z|$. Since H_μ is quasiconformally conjugate to the divergence type group G, it is also divergence type by a well known result of Pfluger [14]. Thus orbits of H_μ are non-tangentially dense almost everywhere on the unit circle and hence the intervals $\{I_{h(0)}\}_{h\in H_\mu}$ cover almost every point of $\mathbb{T}$ infinitely often. Since

$$\limsup_{t\to\infty} \frac{1}{t}\Upsilon(0,\gamma(t,\theta)) \leq 1-\epsilon_1,$$

for almost every θ, for any $\epsilon > 0$, the set of intervals $\{I_h : h \in H_\mu, \Upsilon(0,h(0)) \leq (1-\epsilon_1+\epsilon)\rho(0,h(0))\}$ also covers almost every point infinitely often, and hence $\sum(1-|h(0|) = \infty$. Let $\{h_j\}$ be an enumeration of these elements of H_μ. Thus

$$\sum_j \exp\left(\frac{-1}{1-\epsilon_1+\epsilon}\Upsilon(0,h_j(0))\right) \geq \sum_j \exp(-\rho(0,h_j(0))) = \infty.$$

Thus $\delta(\mu) \geq (1-\epsilon_1+\epsilon)^{-1}$. Taking $\epsilon \to 0$ gives $\delta(G_\mu) \geq (1-\epsilon_1)^{-1}$. This proves there are big deformations near infinity.

6. Proof of Theorem 1.3

It is convenient to prove Theorem 1.3 in a slightly more general form. Recall that we say that R has a L-bounded Y-piece decomposition if it can be written as a union of L-bounded Y-pieces with disjoint interiors. Let Γ be the union of all simple closed geodesics which occur as boundary arcs in the Y-piece decomposition and let $\Gamma^\epsilon \subset \Gamma$ denote all those with lengths $\geq \epsilon$. By the collar lemma there is a $C > 0$ (depending only on L) so that the hyperbolic C-neighborhoods of elements of Γ are pairwise disjoint.

Theorem 6.1. *Given $L, K < \infty$ and $\eta > 0$ there are $\epsilon > 0$ and $r < \infty$ so that the following holds. Suppose $R = \mathbb{D}/G$ is a Riemann surface which has a decomposition into L-bounded Y-pieces. Suppose $F : R \to S$ is* *a K-quasiconformal map with Beltrami coefficient μ and* $\mathrm{dist}(\mathrm{supp}(\mu), \Gamma^\epsilon) > r$*. Then the corresponding quasi-Fuchsian deformation of G has limit set of dimension $\leq 1 + \eta$.*

Theorem 1.3 follows because Γ^ϵ is compact and hence eventually the supports of $\{\mu_n\}$ will be far from it and because $\delta(G) \leq \dim(\Lambda)$. Note that the following is also a special case of Theorem 6.1.

Corollary 6.2. *Given $K < \infty$ and $\eta > 0$ there is an $\epsilon > 0$, so that if $R = \mathbb{D}/G$ is a Riemann surface which has an ϵ-bounded Y-piece decomposition, then* $\dim(f(\mathbb{T})) \leq 1 + \eta$ *for every K-quasiconformal deformation of G.*

We can think of this as a quantified version of the well known fact that the thrice punctured sphere has trivial Teichmüller space, i.e., there are no quasi-Fuchsian deformations of $\mathcal{E}_{0,3}$. We now proceed with the proof of Theorem 6.1.

Suppose $F : R \to S$ is K-quasiconformal and let $f : \mathbb{D} \to \mathbb{D}$ be a lift to the unit disk. Suppose $Y_1 \subset R$ is a Y-piece. Let Ω_1 be a lift of (the interior of) Y_1 to the unit disk. See Figure 2. Suppose γ is a boundary geodesic of Y_1 and that γ lifts to $(-1, 1) \subset \partial\Omega_1$. Normalize f so that it fixes -1 and 1.

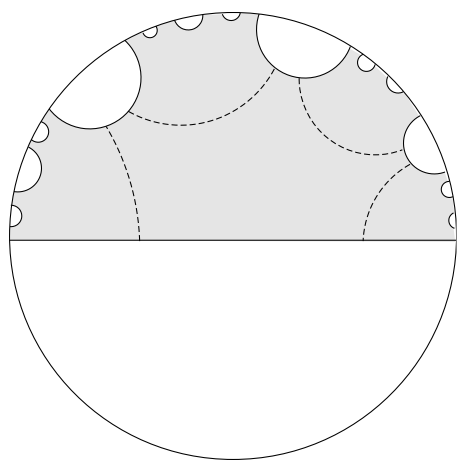

Figure 2. Ω, a lift of a Y-piece

Let $E_1 = \partial\Omega_1 \cap \mathbb{T}$ and let $E_2 = f(E_1)$. Let Ω_2 be the hyperbolic convex hull of E_2. Then Ω_2 projects to a Y-piece Y_2 in S and by Lemma 2.3 there is a quasiconformal map $g : \Omega_1 \to \Omega_2$ which agrees with f on E_1 and projects to a map from Y_1 to Y_2. The map g is affine on the components of $\partial\Omega_1 \cap \mathbb{D}$ and has quasiconformal constant depending only on $\mathrm{dist}(Y_1, Y_2)$. Let Γ be as in the introduction, and given $\gamma \in \Gamma$, let $\lambda_\gamma = \ell(g(\gamma))/\ell(\gamma)$.

By piecing together maps corresponding to all lifts of all Y-pieces in the decomposition of R, we obtain a map of the disk to the disk which agrees with f on the boundary, but which may be discontinuous across the lift of any boundary geodesic $\gamma \in \Gamma$. The maps on either side of γ are both affine, so the discontinuity consists of a translation along γ, say of length t_γ (we think of γ as oriented and t_γ as a signed distance). Thus the map can be made continuous across γ by composing with a skew map of the following form. Assume γ is the positive imaginary axis in the upper half plane and Ω lies in the right quadrant. Define

$$\tau(z) = \begin{cases} e^{a(z)} z, & \pi/2 - \theta_0 \arg(z) \leq \pi/2, \\ z, & \arg(z) < \pi/2 - \theta_0, \end{cases}$$

where θ_0 is chosen so small that $\{z : \pi/2 - \theta_0 \arg(z) \leq \pi/2\}$ lies inside the C-neighborhood of γ which is disjoint from the C-neighborhoods of all other boundary geodesics and $a(z)$ is the linear function of $\arg(z)$ which is $-t_\gamma$ when $\arg(z) = \pi/2$ and is 0 when $\arg(z) = \pi/2 - \theta_0$. The map τ is clearly quasiconformal with constant t_γ/θ_0 except for a discontinuity along of the imaginary axis (which is a hyperbolic translation of size $-t_\gamma$) and is the identity outside a C-neighborhood of γ.

LEMMA 6.3. *Suppose $R = \mathbb{D}/G$ is a Riemann surface with a L-bounded Y-piece decomposition and suppose γ is a boundary arc for one of the Y-pieces. Suppose $f : \mathbb{D} \to \mathbb{D}$ is a K-quasiconformal conjugation of G to another Fuchsian group and the Beltrami coefficient of f is μ. Then $\frac{1}{1+C} \leq \lambda_\gamma \leq 1+C$ and $-C \leq t_\gamma \leq C$ for some C which depends only on K and L. Given any $\epsilon > 0$ there is an $r < \infty$ (depending only on ϵ, K and L) so that $C \leq \epsilon$ if $\rho(\gamma, \mathrm{supp}(\mu)) \geq r$.*

Proof. First we would like to identify λ_γ and t_γ in terms of f's action on the circle. Assume γ lifts to $(-1,1) \subset \mathbb{D}$. Suppose γ is the boundary of two Y-pieces Y_1 and Y_2 (not necessarily distinct) and choose a second boundary component $\gamma_1 \in \partial Y_1$ and $\gamma_2 \in \partial Y_2$ (these may be punctures) and let σ_1 and σ_2 be the shortest curves connecting γ to γ_1 and γ_2, respectively. Let $\tilde\gamma_j$, $j = 1, 2$, be a lift of γ_j which is connected to $(-1, 1)$ by a lift $\tilde\sigma_j$. If γ_j is a puncture then $\tilde\gamma_j$ will be a point on the circle. Since γ has length $\leq L$, we may also choose these lifts so that $\tilde\sigma_1$ and $\tilde\sigma_2$ both hit $(-1, 1)$ within hyperbolic distance $L/2$ of the origin. Let $g \in G$ be an element with translation axis $(-1, 1)$. By replacing g by a power of itself we may assume that $1 \leq \rho(0, g(0)) \leq 1 + L$.

Given an ordered set of six points on the unit circle $E = (x_1, x_2, y_1, y_2, z_1, z_2)$, let γ_x, γ_y and γ_z denote the hyperbolic geodesics in $\mathbb{D}$ with endpoints (x_1, x_2), (y_1, y_2) and (z_1, z_2), respectively. We allow $y_1 = y_2$ or $z_1 = z_2$; in either case, the geodesics γ_y and γ_z are just interpreted as the points on the circle. However, all other pairs must be distinct. Let p_y and p_z be the points on γ_x which are closest to γ_y and γ_z. Let $d(E) = \rho(p_y, p_z)$. See Figure 3.

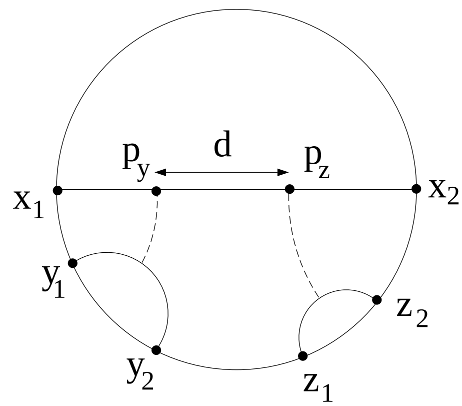

FIGURE 3. Definitions of $d(E)$.

Let $x_1 = -1, x_2 = 1$, let y_1 and y_2 be the endpoints of $\tilde{\gamma}_1$ and let z_1 and z_2 be the endpoints of $g(\tilde{\gamma}_1)$. Then clearly, $\lambda_\gamma = d(f(E))/d(E)$. Since f is quasisymmetric on the circle it is easy to see that this will be bounded and bounded away from zero depending only on the quasiconformal constant of f. If $\rho(\gamma, \mathrm{supp}(\mu)) \geq r$, then with our normalization, μ is supported in the annulus $\{z : 1-Ce^{-r} \leq |z| \leq 1+Ce^{-r}\}$ for some absolute $C < \infty$. As $r \to \infty$ and K remains fixed, the compactness of the family of K-quasiconformal mappings implies f must tend to the identity (we have normalized f so it fixes -1, 1 and ∞). Thus for r large enough, every point on the unit circle is moved less than ϵ by f and hence it is easy to see that $d(f(E))$ differs from $d(E)$ by at most $C\epsilon$. Since $\rho(p_y, p_z) \geq 1$ and both points are at most hyperbolic distance $L/2$ from the orgin, it is easy to deduce that $1-C\epsilon \leq d(f(E))/d(E) \leq 1+C\epsilon$.

We can compute t_γ in almost exactly the same way, except that now we let z_1 and z_2 be the endpoints of $\tilde{\gamma}_2$ and then $t_\gamma = d(f(E)) - d(E)$. As above, this is bounded in terms of K and if γ is far from the support of μ, the same proof as above shows $|t_\gamma| \leq C\epsilon$. This proves the lemma. □

Applying these estimates and Lemma 2.3 to the map g constructed above we obtain a quasiconformal map h of the disk to itself which agrees with f on the circle and which satisfies the following properties:

(1) h maps Y-pieces to Y-pieces.
(2) The quasiconformal constant $C(K)$ of h is bounded depending only on K, the quasiconformal constant of f.

(3) Given K there is an $\epsilon > 0$ such that on any ϵ-bounded Y-piece the quasiconformal dilatation $K_h(z)$ of h is bounded by $1 + C_1(K) \exp(-2\mathrm{dist}(z, \partial Y))$.
(4) Given any $K < \infty$ and $\nu > 0$ there is an $r < \infty$ such that on any Y-piece whose finite boundary is at least distance r from the support of μ, $K_h(z)$ is bounded by $1 + \nu$.

By the assumption in Theorem 6.1, every Y-piece in the decomposition of R is either ϵ-bounded or has boundary geodesics at least distance r from the support of μ. Thus the quasiconformal map h can be factored as $h = h_2 \circ h_1$ where h_1 is also $C(K)$-quasiconformal and has dilatation supported in a C neighborhood of the ϵ-geodesics and h_2 is $(1+\nu)$-quasiconformal. Hence h_2 is Hölder continuous with exponent $(1+\eta)^{-1}$ (if ν is small enough) and hence can only raise the Hausdorff dimension on any set by at most a factor of $1+\eta$. Thus it suffices to show $h_1(\mathbb{T})$ has dimension at most $1+\eta$ (and hence $h(\mathbb{T})$ has dimension at most $(1+\eta)^2$). All we need to prove is the following (which does not involve any group).

Lemma 6.4. *Suppose we are given $K < \infty$, $\eta > 0$ and $s < \infty$. Then there is a $r = r(K, \eta, s) < \infty$ so that the following holds. Suppose $\{\gamma_n\}$ is a collection of geodesics in the hyperbolic disk such that any two of them are at least hyperbolic distance r apart. Suppose f is a quasiconformal map of the plane which is conformal inside the disk and whose dilatation outside the disk satisfies $K_f(z) \le K$ if $\rho(z^*, \bigcup_n \gamma_n) \le s$ and is $= 1$ otherwise. Then the Hausdorff dimension of $f(\mathbb{T})$ is less than $1 + \eta$.*

Proof. Divide the disk into two types of regions; those which are near a geodesic and those which are not. More precisely, let $t \ge s$ (to be chosen below) and let $\{A_n\}$ be the t-neighborhoods of these geodesics. By taking r large enough (e.g., $\ge 2t + 2s + 10$) we may assume that if $n \ne m$ then $\rho(A_n, A_m) \ge 2t + 10$. We will say a dyadic Carleson square Q is type 1 if $T(Q)$ hits $A = \bigcup_n A_n$ and Q is type 2 if $T(Q)$ is disjoint from A. For each type 1 square form a stopping time region $W \subset Q$ by removing all maximal type 2 squares. If Q is type 2, then define $W \subset Q$ by removing all maximal type 1 subsquares. This process divides the disk into type 1 and type 2 regions, as illustrated in Figure 4.

Note that if $W = Q \setminus \bigcup_j Q_j$ is a type 2 region, then $T(Q)$ is within distance $t + 1$ of one geodesic and $T(Q_j)$ is within distance $t+1$ of a different one. Thus the top halves of these squares are more than $r - 2t - 4$ apart. Thus

$$\mathrm{diam}(Q_j) \le C \exp(-r + 2t)\mathrm{diam}(Q) \le C \exp(-r/2)\mathrm{diam}(Q), \tag{2}$$

if we assume $r \ge 4t$. Now suppose $\eta > 0$. The map f is Hölder of some positive order (since it is K-quasiconformal). So if $W = Q \setminus \bigcup_j Q_j$ is a type

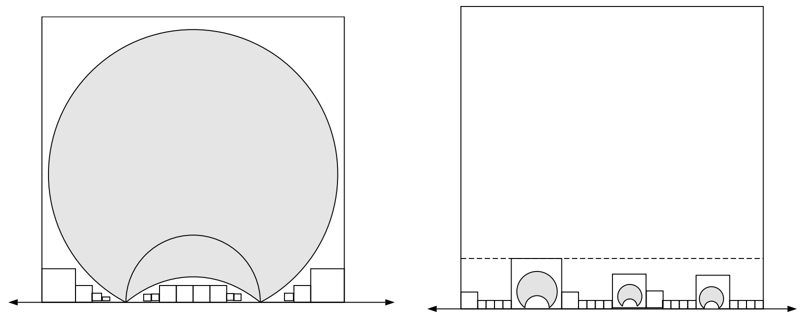

FIGURE 4. Type 1 and type 2 regions

1 region then

$$\sum_j \mathrm{diam}(f(Q_j))^{1+\eta} \le M \mathrm{diam}(f(Q))^{1+\eta},$$

where M depends only on K, η and s, since the diameters of the Q_j's decay geometrically. If W is a type 2 region, we claim that for any $m > 0$,

$$\sum_j \mathrm{diam}(f(Q_j))^{1+\eta} \le m \mathrm{diam}(f(Q))^{1+\eta},$$

holds if r is large enough (depending on K, η, s). To prove the claim, choose r_0 so that Lemma 2.2 holds for K and $\eta/2$. Then for $r \ge 4r_0$, Lemma 2.2 and (2) imply

$$\begin{aligned}\sum_j \left(\frac{\mathrm{diam}(f(Q_j))}{\mathrm{diam}(f(Q))}\right)^{1+\eta} &\le C(K,\eta) \sum_j \left(\frac{\mathrm{diam}(Q_j)}{\mathrm{diam} Q}\right)^{1+\eta/2} \\ &\le C(K,\eta) \sup_j \left(\frac{\mathrm{diam}(Q_j)}{\mathrm{diam}(Q)}\right)^{\eta/2} \sum_j \frac{\mathrm{diam}(Q_j)}{\mathrm{diam} Q} \\ &\le C(K,\eta) C e^{-r\eta/4}.\end{aligned}$$

This proves the claim if r is large enough (depending only on η and K).

If we take m so small that $mM < 1$ then dropping down n generations we get a covering of $f(\mathbb{T})$ by balls B_j which satisfies $\sum_j \mathrm{diam}(B_j)^{1+\eta} < (mM)^n \to 0$. Thus $\delta(\mu) \le \dim(f_\mu(\mathbb{T})) \le 1+\eta$. This completes the proof of Theorem 6.1. □

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