Illinois Journal of Mathematics
Volume 46, Number 2, Summer 2002, Pages 467–475
S 0019-2082

A VIRTUALIZED SKEIN RELATION FOR JONES POLYNOMIALS

NAOKO KAMADA, SHIGEKAZU NAKABO, AND SHIN SATOH

Abstract. We give a skein relation for Jones polynomials among positive, negative, and virtual crossings with some restrictions. We apply this relation to study some properties of virtual knots obtained by replacing a real crossing by a virtual crossing.

1. Introduction

L. H. Kauffman [4] introduced virtual knot theory as a generalization of classical knot theory. He defined the Jones polynomials of oriented virtual links by state-sum models, which are also called f-polynomials. We denote by $V_K(A) \in \mathbf{Z}[A, A^{-1}]$ the Jones polynomial of an oriented virtual link represented by a diagram K. It is known that there is a skein relation for virtual link diagrams,

$$A^4 V_{K_+}(A) - A^{-4} V_{K_-}(A) = (A^{-2} - A^2) V_{K_0}(A),$$

where (K_+, K_-, K_0) is a skein triple as shown in Figure 1.

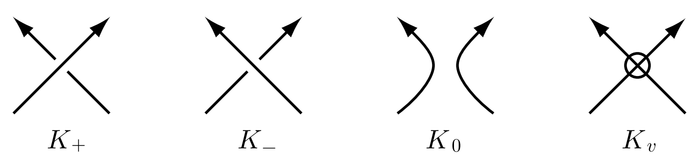


Figure 1

In this paper, we give a skein relation for (K_+, K_-, K_v) (see Figure 1 again), in the case when K_+ and K_- are classical diagrams, that is, when all their crossings are classical ones.

Theorem 1. *If K_+ and K_- are classical diagrams, then we have*

$$A^3 V_{K_+}(A) + A^{-3} V_{K_-}(A) = (A^3 + A^{-3}) V_{K_v}(A).$$

Received July 5, 2001; received in final form February 7, 2002.
2000 *Mathematics Subject Classification.* Primary 57M25. Secondary 57M27.

For a virtual link diagram K we have $V_K(A) \in \mathbf{Z}[A^2, A^{-2}]$ in general, and $V_K(A) \in \mathbf{Z}[A^4, A^{-4}] \cdot A^{2\mu-2}$ if K represents a μ-component classical link. From Theorem 1 we obtain the following result.

COROLLARY 2. *Let K_+ and K_- be classical diagrams of μ-component classical links. Then we have $V_{K_v}(A) \in \mathbf{Z}[A^4, A^{-4}] \cdot A^{2\mu-2}$ if and only if $V_{K_+}(A) = V_{K_-}(A) = V_{K_v}(A)$.*

By Corollary 2, we see that if $V_{K_+}(A) \neq V_{K_-}(A)$ then K_v does not represent a classical link. Hence all virtual knot diagrams illustrated in Figure 2 do not represent classical knots.

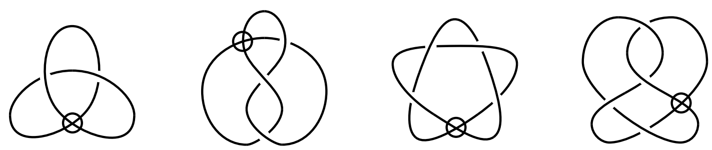


FIGURE 2

This paper is organized as follows. The notion of classical diagrams is extended to that of *normal* diagrams, which are defined in [2]. In Section 2, we review the notion of normality for virtual link diagrams. In Section 3, we prove Theorem 1 and Corollary 2 generalized to the case when K_+ and K_- are normal diagrams (Theorem 9 and Corollary 10). Section 4 contains several applications.

2. Normal diagrams of virtual links

A *state* of a virtual link diagram K is a union of immersed loops in $\mathbf{R}^2$ with only virtual crossings, which is obtained by splicing all classical crossings of K. At each spliced crossing we attach a *chord* labeled A or B to represent the splicing direction as shown on the left of Figure 3.

For a state σ of a virtual link diagram K, we set $\langle K|\sigma\rangle = A^{a(\sigma)}B^{b(\sigma)}d^{|\sigma|-1}$ with $B = A^{-1}$ and $d = -A^2 - A^{-2}$, where $a(\sigma)$ and $b(\sigma)$ denote the number of A-splices and B-splices of σ, respectively, and $|\sigma|$ denotes the number of the immersed loops of σ. For example, we have $\langle K|\sigma\rangle = A^3B^1d^{2-1} = -A^4 - 1$ for the state σ illustrated on the right of Figure 3.

Let $\mathcal{S}$ denote the set of all states of a virtual link diagram K. For a subset $\mathcal{S}'$ of $\mathcal{S}$ we set $\langle K|\mathcal{S}'\rangle = \sum_{\sigma\in\mathcal{S}'}\langle K|\sigma\rangle$. In particular, the *Kauffman bracket* $\langle K\rangle$ is defined as $\langle K|\mathcal{S}\rangle$. For a state σ of K and a classical crossing x of K there are three types with respect to the loop(s) of σ spliced at x as shown in (1)–(3) of Figure 4, where the label X denotes A or B in the figure.

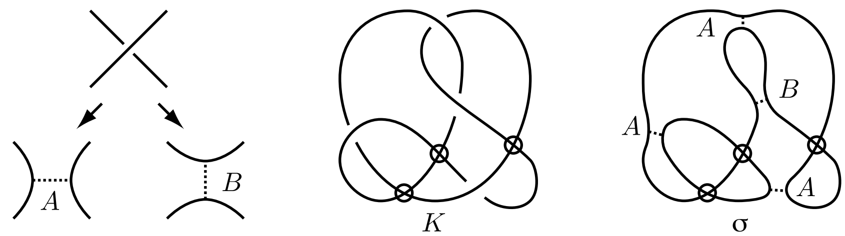
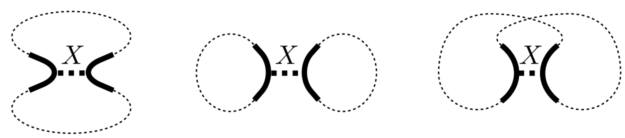


FIGURE 3



(1) (2) (3)

FIGURE 4

DEFINITION 3 (cf. [2]). A state σ of a virtual link diagram K is *normal* if for any classical crossing x of K, the loop(s) of σ spliced at x are of type (1) or (2) in Figure 4. A virtual link diagram K is *normal* if every state of K is normal.

By taking another splicing at a classical crossing, the type (1) and (2) are interchanged and type (3) does not change. Of course, not every virtual link diagram is normal. For example, all diagrams illustrated in Figure 2 are not normal. The following lemma shows that the family of normal diagrams contains that of classical diagrams.

LEMMA 4. *Any classical diagram is normal.*

Proof. Let K be a classical link diagram. Since K has no virtual crossing, any state σ of K is a disjoint union of loops embedded in $\mathbf{R}^2$. Hence σ has no loops of type (3). $\square$

There exist some characterizations of normality of virtual link diagrams (cf. [1], [2]). In the present paper, we give another criterion for normality as follows. For a virtual link diagram K we denote by $\overline{K}$ the union of immersed circles in $\mathbf{R}^2$ obtained by ignoring the over- and under-information at classical crossings of K and leaving the virtual information unchanged.

DEFINITION 5. $\overline{K}$ admits an *alternate orientation* if all edges (when regarding $\overline{K}$ as a 4-valent planar graph) can be oriented as shown in Figure 5.

FIGURE 5

PROPOSITION 6. *A virtual link diagram K is normal if and only if $\overline{K}$ admits an alternate orientation.*

Proof. We prove the proposition by an argument similar to that used in [2]. Assume that a virtual link diagram K is normal. We take a state σ of K such that $|\sigma|$ is minimal among all states of K. Let $\{\ell_1, \dots, \ell_s\}$ be the set of loops of σ. From the minimality it follows that there is no chord (which represents a splicing at a classical crossing) between ℓ_i and ℓ_j for $i \neq j$. We give an arbitrary orientation to $\{\ell_i\}_{i=1,\dots,s}$. Since the state σ is normal, the loop attached by every chord is of type (1). Hence we can give an alternate orientation to $\overline{K}$, which is induced by the above orientation of $\{\ell_i\}_{i=1,\dots,s}$; see Figure 6.

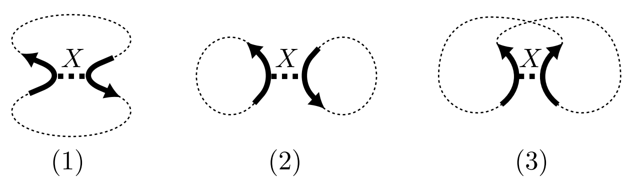


FIGURE 6

Conversely, assume that $\overline{K}$ admits an alternate orientation. Then each loop of any state σ of K has the orientation which is induced by the alternate orientation of $\overline{K}$. Hence we see that there are no loops of type (3) in σ. □

COROLLARY 7. *Let $K = K_1 \cup \cdots \cup K_\mu$ be a virtual link diagram of a μ-component virtual link. If K is normal, then the number of all classical crossings between K_i and $K \setminus K_i$ is even.*

Proof. This is an immediate consequence of Proposition 6. □

For an oriented virtual link diagram K let $w(K)$ denote the sum of the signs of all classical crossings of K. The *Jones polynomial* of K, denoted by $V_K(A)$, is defined to be $(-A^3)^{-w(K)}\langle K\rangle$. We see that $V_K(A) \in \mathbf{Z}[A^2, A^{-2}]$. In particular, we have the following result.

PROPOSITION 8 (cf. [1]). *If K is a normal diagram of a μ-component virtual link, then we have* $V_K(A) \in \mathbf{Z}[A^4, A^{-4}] \cdot A^{2\mu-2}$. $\square$

3. A virtualized skein relation

Let K_v be a virtual link diagram which has a virtual crossing x. We denote by K_+ and K_- the diagrams obtained from K_v by replacing x with a positive crossing and a negative crossing, respectively; see Figure 1. Since any classical diagram is normal (see Lemma 4), it suffices to prove the following result, which is a generalization of Theorem 1.

THEOREM 9. *If K_+ and K_- are normal diagrams, then we have*

$$A^3 V_{K_+}(A) + A^{-3} V_{K_-}(A) = (A^3 + A^{-3}) V_{K_v}(A).$$

Proof. Let $\mathcal{S}_+, \mathcal{S}_-$ and $\mathcal{S}_v$ denote the sets of all states of K_+, K_-, and K_v, respectively. Since K_+ and K_- are normal, $\mathcal{S}_v$ falls into two disjoint subsets $\mathcal{S}'_\varepsilon$ and $\mathcal{S}''_\varepsilon$ as shown in Figure 7. For each state of K_v there are two states of

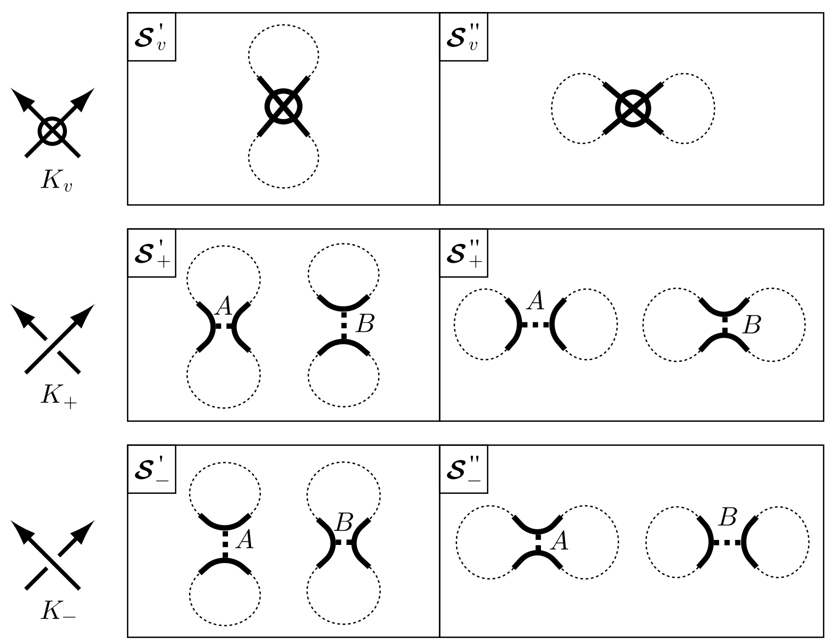


FIGURE 7

K_ε ($\varepsilon = \pm$) obtained by the type of splicing at the virtual crossing. Let $\mathcal{S}'_\varepsilon$ and $\mathcal{S}''_\varepsilon$ denote the subsets of $\mathcal{S}_\varepsilon$, which correspond to $\mathcal{S}'_v$ and $\mathcal{S}''_v$, respectively. Then we have

$$\langle K_+\rangle = \langle K_+|\mathcal{S}'_+\rangle + \langle K_+|\mathcal{S}''_+\rangle = (A+Bd)\langle K_v|\mathcal{S}'_v\rangle + (Ad+B)\langle K_v|\mathcal{S}''_v\rangle,$$
$$\langle K_-\rangle = \langle K_-|\mathcal{S}'_-\rangle + \langle K_-|\mathcal{S}''_-\rangle = (Ad+B)\langle K_v|\mathcal{S}'_v\rangle + (A+Bd)\langle K_v|\mathcal{S}''_v\rangle,$$

and adding these equations we obtain

$$\langle K_+\rangle + \langle K_-\rangle = (A+B)(1+d)(\langle K_v|\mathcal{S}'_v\rangle + \langle K_v|\mathcal{S}''_v\rangle) = -(A^3+A^{-3})\langle K_v\rangle.$$

Since $w(K_v) = w(K_+) - 1 = w(K_-) + 1$, it follows that

$$\begin{aligned}(-A^3)^{-w(K_+)+1}\langle K_+\rangle &+ (-A^3)^{-w(K_-)-1}\langle K_-\rangle \\ &= -(A^3+A^{-3})(-A^3)^{-w(K_v)}\langle K_v\rangle,\end{aligned}$$

which is equivalent to the desired relation. □

We next prove the following result, which is a generalization of Corollary 2.

COROLLARY 10. *Let K_+ and K_- be normal diagrams of μ-component virtual links. Then we have $V_{K_v}(A) \in \mathbf{Z}[A^4, A^{-4}] \cdot A^{2\mu-2}$ if and only if $V_{K_+}(A) = V_{K_-}(A) = V_{K_v}(A)$.*

Proof. By Proposition 8 and the assumption, we may put

$$V_{K_+}(A) = \left(\sum_{i\in\mathbf{Z}} a_i A^{4i}\right)\cdot A^{2\mu-2},$$
$$V_{K_-}(A) = \left(\sum_{i\in\mathbf{Z}} b_i A^{4i}\right)\cdot A^{2\mu-2},$$
$$V_{K_v}(A) = \left(\sum_{i\in\mathbf{Z}} c_i A^{4i}\right)\cdot A^{2\mu-2}.$$

Then it follows from Theorem 9 that

$$\sum_{i\in\mathbf{Z}} a_i A^{4i+3} + \sum_{i\in\mathbf{Z}} b_i A^{4i-3} = \sum_{i\in\mathbf{Z}} c_i A^{4i+3} + \sum_{i\in\mathbf{Z}} c_i A^{4i-3}.$$

Since $(4\mathbf{Z}+3)\cap(4\mathbf{Z}-3) = \emptyset$, we have $a_i = b_i = c_i$ for any i. □

Proposition 8 and Corollary 10 imply that even if K_+ and K_- are normal diagrams, K_v is not necessarily a normal diagram. More precisely, we have the following result.

PROPOSITION 11. *If K_+, K_-, and K_v are normal diagrams, then the virtual links represented by these diagrams have diagrams as shown in Figure* 8. *In particular, if both of K_+ and K_- are classical diagrams, then the three diagrams represent the same classical link.*

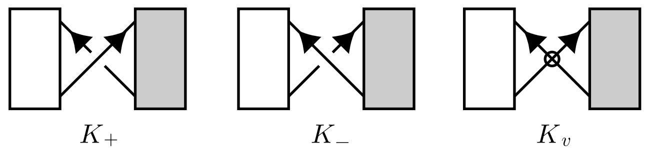


FIGURE 8

Proof. We put $\overline{K} = \overline{K}_+ = \overline{K}_-$. Let x be the crossing of $\overline{K}$ and $\overline{K}_v$ such that x is positive in K_+, negative in K_-, and virtual in K_v. Let C be a component of $\overline{K}$ and $\overline{K}_v$ such that x is on C. We denote by λ the number of classical crossings between C and $\overline{K} \setminus C$, and by λ_v the number of classical crossings between C and $\overline{K}_v \setminus C$. Since $\overline{K}$ and $\overline{K}_v$ admit alternate orientations, we have $\lambda \equiv \lambda_v \equiv 0 \pmod 2$ by Corollary 7. Hence x is a self-intersection of C, for we have $\lambda = \lambda_v + 1$ if x is a crossing between C and another component different from C.

We fix alternate orientations of $\overline{K}$ and $\overline{K}_v$, respectively. By removing the crossing x from C, we obtain a pair of arcs, say $A \cup B$. We may assume that A has the same alternate orientation in $\overline{K}$ and $\overline{K}_v$, and that B has the opposite alternate orientation in $\overline{K}$ and $\overline{K}_v$; see Figure 9. It follows that there are no classical crossings between A and B.

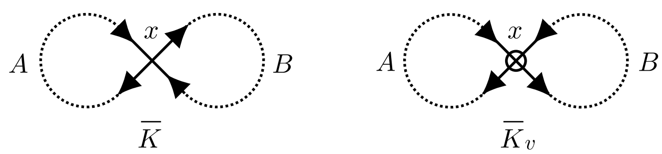


FIGURE 9

More generally, we will show the following result. We define a *component* of $\overline{K} \setminus \{x\} = \overline{K}_v \setminus \{x\}$ to be one of the sets A, B, or a component of $\overline{K}$ and $\overline{K}_v$ different from C. Let C_A be a component of $\overline{K} \setminus \{x\} = \overline{K}_v \setminus \{x\}$ such that there is a finite sequence of components connecting A and C_A in which any adjacent components have a common classical crossing. Similarly, we take a component C_B related to B as above. Then C_A has the same alternate orientation in $\overline{K}$ and $\overline{K}_v$, and C_B has the opposite alternate orientation in $\overline{K}$ and $\overline{K}_v$. It follows that there are no classical crossings between C_A and C_B. Thus we have a division of the components of $\overline{K} \setminus \{x\} = \overline{K}_v \setminus \{x\}$ into two classes such that there are no classical crossings between them. By the argument used in [3], there is a finite sequence of virtual Reidemeister moves [4] that preserve a neighborhood of x such that K_+, K_- and K_v are transformed into the diagrams illustrated in Figure 8. In particular, if K_+ and K_- are classical diagrams, then we may assume that the boxed tangle diagrams in Figure 8 are classical. It is easy to see that those diagrams represent the same classical link. □

4. Applications

For a virtual link diagram K we denote by K^* the virtual link diagram obtained by interchanging the over- and under-information at all classical crossings of K while keeping the orientation of K. If K and K^* represent the same virtual link, then the virtual link is called *amphicheiral*. By definition, we have $V_{K^*}(A) = V_K(A^{-1})$ for any diagram K, and hence $V_K(A) = V_K(A^{-1})$ if K is amphicheiral. The following result gives a necessary condition for K_v to represent an amphicheiral virtual link; indeed, we see that if $V_{K_+}(A) \neq V_{K_-}(A^{-1})$ for normal diagrams K_+ and K_-, then K_v does not represent an amphicheiral virtual link by the corollary.

Corollary 12. *Let K_+ and K_- be normal diagrams. Then we have $V_{K_v}(A) = V_{K_v}(A^{-1})$ if and only if $V_{K_+}(A) = V_{K_-}(A^{-1})$.*

Proof. Since K_+ and K_- are normal diagrams, K_+^* and K_-^* are also normal diagrams. Hence we have the skein relation for the triple (K_-^*, K_+^*, K_v^*),

$$A^3 V_{K_-^*}(A) + A^{-3} V_{K_+^*}(A) = (A^3 + A^{-3}) V_{K_v^*}(A),$$

which is equivalent to

$$A^3 V_{K_-}(A^{-1}) + A^{-3} V_{K_+}(A^{-1}) = (A^3 + A^{-3}) V_{K_v}(A^{-1}).$$

From this and the skein relation for (K_+, K_-, K_v) in Theorem 9 we have $V_{K_v}(A) = V_{K_v}(A^{-1})$ if and only if

$$A^3 V_{K_-}(A^{-1}) + A^{-3} V_{K_+}(A^{-1}) = A^3 V_{K_+}(A) + A^{-3} V_{K_-}(A),$$

which is equivalent to $V_{K_+}(A) = V_{K_-}(A^{-1})$ by an argument similar to the proof of Corollary 10. □

A virtual link diagram K is *alternating* if, when we travel along each component of K, the transverse arcs which we meet are over, under, over, under, … at the consecutive classical crossings. A virtual link is called *alternating* if it is represented by some alternating diagram (cf. [1]). From Corollary 12 and the following result we see that none of the diagrams illustrated in Figure 2 represent an amphicheiral, classical, or alternating virtual link.

Corollary 13. *Let K_+ and K_- are normal diagrams. If $V_{K_+}(A) \neq V_{K_-}(A)$, then K_v represents neither a classical nor an alternating virtual link.*

Proof. If K_v represents a classical or alternating virtual link, then we have $V_{K_v}(A) \in \mathbf{Z}[A^4, A^{-4}] \cdot A^{2\mu - 2}$, for any classical or alternating diagram is normal (see Lemma 4 and [1]). By Corollary 10, it follows that $V_{K_+}(A) = V_{K_-}(A)$. □

We replace a classical crossing of a virtual link diagram K with a virtual crossing so that we obtain another virtual link diagram K_v. We say that K_v is obtained from K by *virtualizing* the classical crossing of K. When we virtualize all classical crossings of K, we obtain a virtual link diagram that represents a trivial link (cf. [4]). Hence virtualizing classical crossings is an unknotting operation for virtual links. The following is also an immediate consequence of Corollary 10.

Corollary 14. *Let K be a normal diagram of a μ-component classical link. If $V_K(A) \neq (-A^2 - A^{-2})^{\mu-1}$, then any virtual link diagram obtained from K by virtualizing a classical crossing of K does not represent a trivial link.* □

Acknowledgments. The first and third authors would like to thank to Department of Mathematics and Statistics at the University of South Alabama for hospitality.

References

[1] N. Kamada, *On the Jones polynomials of checkerboard colorable virtual links*, Osaka J. Math. **39** (2002), 325–333.
[2] N. Kamada and S. Kamada, *On virtual links and links in thickened surfaces admitting checkerboard coloring*, preprint.
[3] S. Kamada, *Braid presentation of virtual knots and welded knots*, preprint, `arXiv:math.GT/0008092`.
[4] L. H. Kauffman, *Virtual knot theory*, Europ. J. Combinatorics **20** (1999) 663–690.

Naoko Kamada, Department of Mathematics, Osaka City University, Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan
E-mail address: naoko@sci.osaka-cu.ac.jp

Shigekazu Nakabo, Kurume National College of Technology, Fukuoka 830-8555, Japan
E-mail address: nakabo@kurume-nct.ac.jp

Shin Satoh, Department of Mathematics, Chiba University, Inage, Chiba 606-8502, Japan
E-mail address: satoh@math.s.chiba-u.ac.jp