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ON THE FIRST EIGENVALUE OF THE LINEARIZED OPERATOR OF THE HIGHER ORDER MEAN CURVATURE FOR CLOSED HYPERSURFACES IN SPACE FORMS

LUIS J. ALÍAS AND J. MIGUEL MALACARNE

Abstract. In this paper we derive sharp upper bounds for the first positive eigenvalue of the linearized operator of the higher order mean curvature of a closed hypersurface immersed into a Riemannian space form. Our bounds are extrinsic in the sense that they are given in terms of the higher order mean curvatures and the center(s) of gravity of the hypersurface, and they extend previous bounds recently given by Veeravalli [24] for the first positive eigenvalue of the Laplacian operator.

1. Introduction

A classical result of Reilly [21] establishes that the first positive eigenvalue λ_1 of the Laplacian operator Δ of a closed (that is, compact and without boundary) hypersurface M^n immersed into the Euclidean space $\mathbb{R}^{n+1}$ satisfies

$$\lambda_1 \leq \frac{n}{\mathrm{vol}(M)} \int_M H^2 \mathrm{d}M,$$

where H denotes the mean curvature of M, with equality if and only if M is a round sphere in $\mathbb{R}^{n+1}$ (see also [7] for an earlier version due to Bleecker and Weiner). More generally, Reilly obtained that

$$\lambda_1 \left(\int_M H_r \mathrm{d}M \right)^2 \leq n \ \mathrm{vol}(M) \int_M H_{r+1}^2 \mathrm{d}M,$$

for every $0 \leq r \leq n-1$, where H_r stands for the r-th mean curvature of the hypersurface, and equality holds precisely when M is a round sphere (recall that $H_0 = 1$ by definition, and $H_1 = H$). More recently, Veeravalli [24] has extended Reilly's inequalities to the case of hypersurfaces immersed into

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hyperbolic and spherical spaces. See also [16], [10], [14] and [11] for extensions of Reilly's inequalities to some other ambient spaces, including hyperbolic space, given by Heintze, by El Soufi and Ilias, by Grosjean, and by Giménez, Miquel and Orengo, respectively.

Let us recall that the Laplacian operator Δ of a hypersurface M immersed into a Riemannian space form naturally arises as the linearized operator of the mean curvature for normal variations of the hypersurface. From this perspective, Δ can be thought of as the first of a sequence of n operators, $L_0 = \Delta, L_1, \dots, L_{n-1}$, where L_r stands for the linearized operator of the $(r+1)$-th mean curvature arising from normal variations of the hypersurface (see, for instance, [20], [22] and [3]). These operators are given by $L_r(f) = \mathrm{div}_M(T_r \nabla f)$ for every smooth function f on M, where T_r denotes the r-th classical Newton transformation associated to the second fundamental form of the hypersurface. In general, the operators L_r are not elliptic, but under appropriate natural geometric hypotheses on the hypersurface, they are elliptic, which makes it possible to consider the first positive eigenvalue $\lambda_1^{L_r}$ of L_r. Inspired by Reilly's inequalities and their subsequent generalizations and extensions, our objective here is to derive sharp upper bounds for $\lambda_1^{L_r}$, not only for hypersurfaces in Euclidean spaces but also in hyperbolic and spherical spaces. Like Veeravalli's bounds, our bounds are extrinsic in the sense that they are given in terms of the total higher order mean curvatures and the center(s) of gravity of the hypersurface. We refer the reader to [2], [12], [13] and [1] for other previous work on the subject.

The main results of this paper are contained in Sections 4, 5, and 6. Specifically, in Section 4, Lemma 6, we establish a general version of a classical result of Reilly (Main Lemma in [21]) for the case of the linearized operator L_r of the $(r+1)$-th mean curvature of a hypersurface in Euclidean space $\mathbb{R}^{n+1}$, in the case when L_r is elliptic (see also equation (1.7) in [2]). As an application of this result, in Section 5, Theorem 9 and Theorem 10, we derive Reilly-type inequalities for the case of the first positive eigenvalue of L_r, which extend Theorem 1.1 and Theorem 1.3 by Alencar, do Carmo and Rosenberg [2]. Our inequalities are sharp, with equality holding if and only if the hypersurface is a round sphere in $\mathbb{R}^{n+1}$. On the other hand, it is not difficult to see that, for every $r = 0, \dots, n-2$, the $(r+2)$-th mean curvature of a round sphere $\mathbb{S}^n(\varrho)$ of radius ϱ in the Euclidean space $\mathbb{R}^{n+1}$ satisfies $c_r H_{r+2} = \lambda_1^{L_r}$, $c_r = (n-r)\binom{n}{r}$ (see Remark 7). Motivated by this fact, in Section 5, Theorem 12, we prove that the first positive eigenvalue of L_r of a positively Ricci curved hypersurface in $\mathbb{R}^{n+1}$ is bounded from above by $c_r \max H_{r+2}$, and equality holds if and only if the hypersurface is a round sphere in $\mathbb{R}^{n+1}$. This is a new upper bound for the first positive eigenvalue of L_r which, for the case of the Laplacian operator $(r=0)$ and the scalar curvature S (recall that $S = n(n-1)H_2$), was first given by Deshmukh [9].

Finally, in Section 6, we consider the cases of hypersurfaces in the sphere and hypersurfaces in the hyperbolic space. Using Veeravalli's approach for a center of gravity of the hypersurface [24], we are able to extend Lemma 6 to the spherical and hyperbolic cases (see Lemma 13 and Lemma 14, respectively). In particular, when $r = 0$ we recover the results of Veeravalli [24] for the first positive eigenvalue of the Laplacian operator. As already observed by Veeravalli in the case $r = 0$, the result is sharp for hypersurfaces in the sphere, but not for hypersurfaces in the hyperbolic space. Observe that the same holds for the upper bound for the first positive eigenvalue given by Alencar, do Carmo and Rosenberg [2, Theorem 3.1] for hypersurfaces in hyperbolic space. As an application of Lemma 13, we derive Reilly-type inequalities for the first positive eigenvalue of the operator L_r of a closed hypersurface in a sphere $\mathbb{S}^{n+1}$. Our inequalities are sharp, with equality holding if and only if the hypersurface is a geodesic sphere in $\mathbb{S}^{n+1}$.

2. Preliminaries

Let $\mathbb{M}^{n+1}_\kappa$ be an $(n+1)$-dimensional Riemannian space form with constant sectional curvature κ, and let $\psi : M^n \to \mathbb{M}^{n+1}_\kappa$ be a connected hypersurface immersed into $\mathbb{M}^{n+1}_\kappa$. Let A be the shape operator (or the second fundamental form) of the hypersurface with respect to a (locally defined) normal unit vector field $\mathbf{N}$. As is well known, A_p is a self-adjoint linear operator on each tangent plane T_pM, for every $p \in M$, and its eigenvalues, $\kappa_1(p), \ldots, \kappa_n(p)$, are the principal curvatures of the hypersurface at the point p. Associated to the shape operator there are n algebraic invariants given by

$$S_r(p) = \sigma_r(\kappa_1(p), \ldots, \kappa_n(p)), \quad 1 \leq r \leq n,$$

where σ_r is the elementary symmetric function defined on $\mathbb{R}^n$ by

$$\sigma_r(x_1, \ldots, x_n) = \sum_{i_1 < \cdots < i_r} x_{i_1} \ldots x_{i_n}.$$

Observe that the characteristic polynomial of A_p can be written in terms of the S_r's as

$$\det(tI - A_p) = \sum_{r=0}^{n} (-1)^r S_r(p) t^{n-r}.$$

The r-th mean curvature $H_r(p)$ of the hypersurface at p is then defined by

$$\binom{n}{r} H_r(p) = S_r(p).$$

In particular, when $r = 1$, then $H_1(p) = (1/n)\mathrm{trace}(A_p) = H(p)$ is the mean curvature, which is the main extrinsic curvature of the hypersurface. On the other hand, when $r = 2$, $H_2(p)$ defines a geometric quantity which is related to the (intrinsic) scalar curvature of the hypersurface. Indeed, it follows from the Gauss equation that the Ricci curvature of M is given by

$$\mathrm{Ric}_p(v,w) = (n-1)\kappa\langle v,w\rangle + nH(p)\langle A_p v,w\rangle - \langle A_p v, A_p w\rangle, \tag{1}$$

for $v,w \in T_pM$. Therefore, the scalar curvature S of the hypersurface M at p is $S(p) = \mathrm{trace}(\mathrm{Ric}_p) = n(n-1)(\kappa + H_2(p))$. In general, the Gauss equation implies that when r is odd, H_r is extrinsic (and its sign depends on the chosen local orientation), while when r is even, H_r is intrinsic. H_n is classically called the Gauss-Kronecker curvature of M.

The classical Newton transformations $T_r(p) : T_pM \to T_pM$ are defined inductively from A_p by

$$T_0(p) = I_p \quad \text{and} \quad T_r(p) = S_r(p)I_p - A_pT_{r-1}(p), \quad 1 \leq r \leq n,$$

where I_p denotes the identity on T_pM, or equivalently by

$$T_r(p) = S_r(p)I_p - S_{r-1}(p)A_p + \cdots + (-1)^{r-1}S_1(p)A_p^{r-1} + (-1)^r A_p^r.$$

Note that by the Cayley-Hamilton theorem, we have $T_n(p) = 0$. As each $T_r(p)$ is polynomial in A_p, these transformations are also self-adjoint linear operators which commute with A_p. Indeed, A_p and $T_r(p)$ can be simultaneously diagonalized: if $\{e_1,\ldots,e_n\} \subset T_pM$ are eigenvectors of A_p associated to the eigenvalues $\kappa_1(p),\ldots,\kappa_n(p)$, respectively, then they are also eigenvectors of $T_r(p)$ associated to the eigenvalues $\mu_{1,r}(p),\ldots,\mu_{n,r}(p)$ of $T_r(p)$, where

$$\mu_{i,r}(p) = \frac{\partial\sigma_{r+1}}{\partial x_i}(\kappa_1(p),\ldots,\kappa_n(p)) = \sum_{i_1<\cdots<i_r, i_j\neq i} \kappa_{i_1}(p)\cdots\kappa_{i_r}(p), \tag{2}$$

for every $1 \leq i \leq n$. From this it can be easily seen that

$$\mathrm{trace}(T_r(p)) = (n-r)S_r(p) = c_rH_r(p)$$

and

$$\mathrm{trace}(A_pT_r(p)) = (r+1)S_{r+1}(p) = c_rH_{r+1}(p),$$

where $c_r = (n-r)\binom{n}{r} = (r+1)\binom{n}{r+1}$. For the details, we refer the reader to the classical paper by Reilly [20] (see also [5] and [22] for a more accessible modern treatment).

In particular, when M is orientable we may choose a globally defined normal unit vector field $\mathbf{N}$ on M, and the shape operator A and its Newton transformations T_r determine globally defined tensor fields $A : \mathcal{X}(M) \to \mathcal{X}(M)$ and $T_r : \mathcal{X}(M) \to \mathcal{X}(M)$.

On the other hand, the divergence of T_r is defined by

$$\mathrm{div}_M\, T_r = \mathrm{trace}(\nabla T_r) = \sum_{i=1}^n (\nabla_{e_i}T_r)(e_i),$$

where $\{e_1,\ldots,e_n\}$ is a local orthonormal frame on M. Another interesting property of the Newton transformations of a hypersurface in a space form is that they are divergence free, that is, $\operatorname{div}_M T_r = 0$ for each r, as shown by Rosenberg [22]. We also refer the reader to [4] for a general computation of $\operatorname{div}_M T_r$, in the case of a hypersurface immersed into an arbitrary Riemannian, not necessarily with constant sectional curvature.

The Newton transformations T_r allow us to consider an interesting kind of second order differential operators acting on the smooth functions on M. For each $r = 0, \ldots, n-1$, consider $L_r : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$, the operator given by

$$L_r(f) = \operatorname{div}_M(T_r \nabla f).$$

As the Newton transformations are divergence free, it was shown in [22] that the operator L_r is the linearized operator of the $(r+1)$-th mean curvature of M. Note that $L_0 = \Delta$ is the Laplacian operator on M with respect to the induced metric. As is well known, the Laplacian operator Δ is always an elliptic operator on M. However, if $r \geq 1$, the operator L_r does not in general have this property. In the following section we will discuss several geometric conditions which guarantee the ellipticity of L_r.

3. Ricci curvature, convexity and ellipticity

A classical theorem of Hadamard [15] gives three equivalent conditions on a closed connected hypersurface M^n immersed into the Euclidean space $\mathbb{R}^{n+1}$ which imply that M is a *convex hypersurface*. Here, by a convex hypersurface in $\mathbb{R}^{n+1}$ we mean that M is embedded in $\mathbb{R}^{n+1}$and is the boundary of a convex body.

THEOREM 1 (Hadamard theorem). *Let $\psi : M^n \to \mathbb{R}^{n+1}$ be a closed connected hypersurface immersed into the Euclidean space. The following assertions are equivalent:*

(i) *The second fundamental form is definite at every point of M.*
(ii) *M is orientable and its Gauss map is a diffeomorphism onto $\mathbb{S}^n$.*
(iii) *The Gauss-Kronecker curvature never vanishes on M.*

Moreover, any of the above conditions implies that M is a convex hypersurface.

For a proof of the Hadamard theorem see [8]. Here we observe that the convexity of a hypersurface in $\mathbb{R}^{n+1}$ is also closely related to the Ricci curvature. In fact, we have the following result.

THEOREM 2. *Let $\psi : M^n \to \mathbb{R}^{n+1}$ be a closed connected hypersurface immersed into the Euclidean space. The following assertion is also equivalent to any of the above assertions in the Hadamard theorem, and therefore it also implies that M is a convex hypersurface:*

(iv) *The Ricci curvature of M is positive everywhere on M.*

Proof. Let us show that condition (i) implies (iv), and that condition (iv) implies (iii). In both proofs we will make use of the Gauss equation (1), which in our case ($\kappa = 0$) can be written as

$$\mathrm{Ric}_p(v, w) = \langle A_p T_1(p) v, w \rangle. \tag{3}$$

First, suppose that condition (i) holds. At every point $p \in M$, choose a unit normal vector ξ_p so that A_{ξ_p} is positive definite. Since the second fundamental form is definite everywhere, we conclude that such a vector field ξ globally exists and is continuous on M. Therefore the principal curvatures $\kappa_1(p), \ldots, \kappa_n(p)$ of M are positive at each point $p \in M$. Hence from (2) the eigenvalues of T_1 are also positive at each $p \in M$. Let $\{e_1, \ldots, e_n\} \subset T_pM$ be an orthonormal basis of eigenvectors of A at a point $p \in M$. From (3) we see that

$$\mathrm{Ric}_p(e_i, e_j) = \langle A_p T_1(p) e_i, e_j \rangle = \kappa_i(p) \mu_{i,1}(p) \delta_{ij}, \qquad i, j = 1, \ldots, n, \tag{4}$$

which implies that the Ricci curvature tensor of M is positive definite at every $p \in M$.

Now, suppose that condition (iv) holds, and let us show that this implies (iii), completing the proof. Let $p \in M$ and take an orthonormal basis of eigenvectors of A in T_pM with ordered eigenvalues $\kappa_1(p) \leq \cdots \leq \kappa_n(p)$, which are continuous functions on M. Using (3) again, we have

$$\mathrm{Ric}_p(e_i, e_i) = \kappa_i(p) \mu_{i,1}(p) > 0, \qquad i = 1, \ldots, n,$$

so that $\kappa_i(p) \neq 0$ for every $i = 1, \ldots, n$. Therefore, the Gauss-Kronecker curvature of M, $H_n(p) = \kappa_1(p) \cdots \kappa_n(p)$, does not vanish on M. □

In particular, if the Ricci curvature of a closed hypersurface in $\mathbb{R}^{n+1}$ is positive, then it is orientable and, after an appropriate choice of its orientation, its second fundamental form is positive definite. By (2), this implies that every Newton transformation T_r is also positive definite, and so we have the following corollary.

Corollary 3. *Let $\psi : M^n \to \mathbb{R}^{n+1}$ be a closed connected hypersurface immersed into the Euclidean space. If M has positive Ricci curvature, then the operators L_r are all elliptic and all the r-th mean curvatures are positive.*

For the case of hypersurfaces immersed into $\mathbb{M}^{n+1}_\kappa$ with $\kappa \neq 0$, equation (3) becomes

$$\mathrm{Ric}_p(v, w) - \kappa \langle v, w \rangle = \langle A T_1 v, w \rangle.$$

Therefore, in that case it is also true that the operators L_r are all elliptic, under the assumption that the Ricci curvature of the hypersurface is greater than the curvature of the ambient space, κ.

On the other hand, as observed by Barbosa and Colares in [5, Proposition 3.2], if $\psi : M^n \to \mathbb{M}^{n+1}_\kappa$ is a closed connected hypersurface immersed into a

Riemannian space form $\mathbb{M}^{n+1}_\kappa$ (if $\kappa > 0$, assume further that $\psi(M)$ is contained in an open hemisphere) such that $H_{r+1} > 0$ is positive for some r, then each operator L_j is elliptic for $j = 0, \ldots, r$.

REMARK 4. In what follows, when L_r is elliptic on M (or, equivalently, T_r is positive definite on M), we will always assume that the chosen orientation on M is the one for which $H_r > 0$.

4. First results. A Reilly-type inequality for the first eigenvalue of L_r

A classical result of Takahashi [23] establishes that the only immersed hypersurfaces in Euclidean space $\mathbb{R}^{n+1}$ whose coordinate functions are eigenfunctions of the Laplacian operator, associated to the same eigenvalue λ, are minimal hypersurfaces in $\mathbb{R}^{n+1}$ (with $\lambda = 0$) and open pieces of round spheres $\mathbb{S}^n(\varrho) \subset \mathbb{R}^{n+1}$ of radius ϱ (with $\lambda = n/\varrho^2 > 0$). In other words, if $\psi : M^n \to \mathbb{R}^{n+1}$ is an immersed hypersurface in Euclidean space, then $\Delta\psi + \lambda\psi = 0$ for a real constant λ if and only if either $\lambda = 0$ and M is a minimal hypersurface in $\mathbb{R}^{n+1}$, or $\lambda > 0$ and M is an open piece of a round sphere of radius $\varrho = \sqrt{n/\lambda}$ centered at the origin of $\mathbb{R}^{n+1}$. Below we establish the corresponding result for the case of the linearized operator L_r of the $(r+1)$-th mean curvature (since $L_0 = \Delta$, taking $r = 0$ we recover classical Takahashi theorem).

LEMMA 5. *Let $\psi : M^n \to \mathbb{R}^{n+1}$ be an orientable connected hypersurface immersed into the Euclidean space, and let L_r be the linearized operator of the $(r+1)$-th mean curvature of M, for some $r = 0, \ldots, n-1$. Then*

$$L_r\psi + \lambda\psi = 0$$

for a real constant λ if and only if either $\lambda = 0$ and M is $(r+1)$-minimal in $\mathbb{R}^{n+1}$ (that is, $H_{r+1} = 0$ on M), or $\lambda \neq 0$ and M is an open piece of a round sphere $\mathbb{S}^n(\varrho) \subset \mathbb{R}^{n+1}$ of radius $\varrho = (c_r/|\lambda|)^{1/(r+2)}$ centered at the origin of $\mathbb{R}^{n+1}$, where $c_r = (n-r)\binom{n}{r}$.

Proof. For a fixed arbitrary vector $\mathbf{a} \in \mathbb{R}^{n+1}$, let us consider the height function $\langle \psi, \mathbf{a} \rangle$ defined on M. Observe that its gradient is given by $\nabla\langle \psi, \mathbf{a} \rangle = \mathbf{a}^\top$, where $\mathbf{a}^\top = \mathbf{a} - \langle \mathbf{N}, \mathbf{a} \rangle \mathbf{N} \in \mathcal{X}(M)$ denotes the tangent component of $\mathbf{a}$ along the immersion. Therefore, for every $X \in \mathcal{X}(M)$ we have

$$\nabla_X(\nabla\langle \psi, \mathbf{a} \rangle) = \langle \mathbf{N}, \mathbf{a} \rangle AX,$$

so that

$$L_r\langle \psi, \mathbf{a} \rangle = \langle \mathbf{N}, \mathbf{a} \rangle \operatorname{tr}(A \circ T_r) = c_r H_{r+1} \langle \mathbf{N}, \mathbf{a} \rangle. \tag{5}$$

In other words,

$$L_r\psi = c_r H_{r+1} \mathbf{N}. \tag{6}$$

In particular, $L_r\psi = 0$ for every $(r+1)$-minimal hypersurface in $\mathbb{R}^{n+1}$. On the other hand, for a round sphere in $\mathbb{R}^{n+1}$ of radius ϱ centered at a point $\mathbf{c} \in \mathbb{R}^{n+1}$, we have $\mathbf{N} = \pm(1/\varrho)(\psi - \mathbf{c})$ and $H_{r+1} = (\mp 1)^{r+1}/\varrho^{r+1}$, so that by (6) we see that

$$L_r\psi + \lambda\psi = \lambda\mathbf{c},$$

where $|\lambda| = c_r/\varrho^{r+2} > 0$. Therefore, a round sphere in $\mathbb{R}^{n+1}$ satisfies $L_r\psi + \lambda\psi = 0$ if and only if it is centered at the origin of $\mathbb{R}^{n+1}$.

Conversely, assume that $L_r\psi + \lambda\psi = 0$ for a real constant λ. Then it follows from (6) that

$$c_r H_{r+1}\mathbf{N} + \lambda\psi = 0.$$

By taking the covariant derivative here, we obtain that

$$c_r X(H_{r+1})\mathbf{N} - c_r H_{r+1} AX + \lambda X = 0$$

for every tangent vector field $X \in \mathcal{X}(M)$, which implies both that H_{r+1} is necessarily constant on M and that

$$c_r H_{r+1} AX = \lambda X \tag{7}$$

for every $X \in \mathcal{X}(M)$. If $H_{r+1} = 0$, then M is $(r+1)$-minimal and there is nothing else to prove. If H_{r+1} is a non-zero constant, then by (7) M is totally umbilical with non-zero umbilicity factor given by $\lambda/(c_r H_{r+1}) \neq 0$. Therefore, M is contained in a round sphere of $\mathbb{R}^{n+1}$, necessarily centered at the origin of $\mathbb{R}^{n+1}$, of radius $\varrho = (c_r/|\lambda|)^{1/(r+2)}$. □

As a first application of Lemma 5, we establish a general version of a classical result of Reilly (Main Lemma in [21]) for the case of the linearized operator L_r of the $(r+1)$-th mean curvature of a hypersurface in $\mathbb{R}^{n+1}$, in the case when L_r is elliptic (see also equation (1.7) in [2]).

LEMMA 6. *Let $\psi : M^n \to \mathbb{R}^{n+1}$ be an orientable closed connected hypersurface immersed into the Euclidean space, and let $\mathbf{c}$ be its center of gravity,*

$$\mathbf{c} = \frac{1}{\mathrm{vol}(M)} \int_M \psi \mathrm{d}M \in \mathbb{R}^{n+1},$$

where $\mathrm{vol}(M)$ denotes the n-dimensional volume of M. Assume that L_r is elliptic on M for some $r = 0, \ldots, n-1$, and let $\lambda_1^{L_r}$ be the first positive eigenvalue of L_r. Then

$$\lambda_1^{L_r} \int_M |\psi - \mathbf{c}|^2 \mathrm{d}M \leq c_r \int_M H_r \mathrm{d}M, \quad c_r = (n-r)\binom{n}{r}, \tag{8}$$

and equality holds if and only if M is a round sphere in $\mathbb{R}^{n+1}$ centered at $\mathbf{c}$.

Proof. Since L_r is assumed to be elliptic, we can use the minimax characterization of $\lambda_1^{L_r}$, as

$$\lambda_1^{L_r} = \inf\left\{\frac{-\int_M fL_r(f)\mathrm{d}M}{\int_M f^2\mathrm{d}M}; \quad \int_M f\mathrm{d}M = 0\right\}. \tag{9}$$

For every $1 \leq i \leq n+1$, let $f_i = \langle \psi - \mathbf{c}, \mathbf{a}_i\rangle$ be i-th coordinate function of $\psi - \mathbf{c}$ on M, where $\{\mathbf{a}_1, \ldots, \mathbf{a}_{n+1}\}$ is the standard orthonormal basis of $\mathbb{R}^{n+1}$, $\mathbf{a}_i = (0, \ldots, 0, \overset{i}{1}, 0 \ldots, 0)$. Then for every $i = 1, \ldots, n+1$ we have $\int_M f_i\mathrm{d}M = 0$, and by (5) we also get

$$L_r f_i = L_r\langle \psi, \mathbf{a}_i\rangle = c_r H_{r+1}\langle \mathbf{N}, \mathbf{a}_i\rangle.$$

Therefore, using (9) we obtain that

$$\lambda_1^{L_r}\int_M f_i^2\mathrm{d}M \leq -\int_M f_i L_r f_i \mathrm{d}M = -c_r\int_M H_{r+1} f_i\langle \mathbf{N}, \mathbf{a}_i\rangle\mathrm{d}M.$$

Now we sum from $i = 1$ to $n+1$. Since

$$\sum_{i=1}^{n+1} f_i^2 = |\psi - \mathbf{c}|^2, \text{ and } \sum_{i=1}^{n+1} f_i\langle \mathbf{N}, \mathbf{a}_i\rangle = \langle \psi - \mathbf{c}, \mathbf{N}\rangle,$$

we get

$$\lambda_1^{L_r}\int_M |\psi - \mathbf{c}|^2\mathrm{d}M \leq -c_r\int_M H_{r+1}\langle \psi - \mathbf{c}, \mathbf{N}\rangle\mathrm{d}M. \tag{10}$$

On the other hand, let us consider now the function $f = \frac{1}{2}|\psi - \mathbf{c}|^2$ defined on M. Observe that its gradient is given by $\nabla f = (\psi - \mathbf{c})^\top$, where

$$(\psi - \mathbf{c})^\top = \psi - \mathbf{c} - \langle \psi - \mathbf{c}, \mathbf{N}\rangle\mathbf{N} \in \mathcal{X}(M)$$

denotes the tangent component of $\psi - \mathbf{c}$ along the immersion ψ. Therefore, for every $X \in \mathcal{X}(M)$ we have

$$\nabla_X(\nabla f) = X + \langle \psi - \mathbf{c}, \mathbf{N}\rangle AX,$$

so that

$$L_r f = \mathrm{tr}(T_r) + \langle \psi - \mathbf{c}, \mathbf{N}\rangle\,\mathrm{tr}(A \circ T_r) = c_r\left(H_r + H_{r+1}\langle \psi - \mathbf{c}, \mathbf{N}\rangle\right). \tag{11}$$

Integrating this equality over M, the divergence theorem implies the well-known Minkowski formulae, which were first obtained by Hsiung [17],

$$\int_M (H_r + H_{r+1}\langle \psi - \mathbf{c}, \mathbf{N}\rangle)\mathrm{d}M = 0, \quad r = 0, \ldots, n-1. \tag{12}$$

We will refer to (12) as the r-th Minkowski formula, and the integrand in (12) will be called the r-th Minkowski-Hsiung integrand.

Finally, using the r-th Minkowski formula in (10), we conclude that

$$\lambda_1^{L_r}\int_M |\psi - \mathbf{c}|^2\mathrm{d}M \leq c_r\int_M H_r\mathrm{d}M,$$

as desired. Moreover, equality holds if and only if

$$L_r\langle\psi-\mathbf{c},\mathbf{a}_i\rangle+\lambda_1^{L_r}\langle\psi-\mathbf{c},\mathbf{a}_i\rangle=0$$

for every $i=1,\dots,n+1$, that is, if and only if $L_r(\psi-\mathbf{c})+\lambda_1^{L_r}(\psi-\mathbf{c})=0$, which by Lemma 5 means that ψ is a round sphere centered at $\mathbf{c}$. □

Remark 7. As a first consequence of Lemma 6 and its proof, observe that, for every $r=0,\dots,n-1$, the first positive eigenvalue of the operator L_r on a round sphere $\mathbb{S}^n(\varrho)\subset\mathbb{R}^{n+1}$ of radius ϱ is given by $\lambda_1^{L_r}=c_r/\varrho^{r+2}=c_rH_{r+2}$, where $c_r=(n-r)\binom{n}{r}$, and the $n+1$ coordinate functions on $\mathbb{S}^n(\varrho)$ are eigenfunctions of L_r associated to $\lambda_1^{L_r}$.

In fact, since $\mathbb{S}^n(\varrho)\subset\mathbb{R}^{n+1}$ is totally umbilical with $A=(1/\varrho)I$, the r-th Newton transformation is simply $T_r=(c_r/n\varrho^r)I$ and L_r is a multiple of the Laplacian operator, $L_r=(c_r/n\varrho^r)\Delta$. Now, using the fact that the first positive eigenvalue of the Laplacian operator of an Euclidean n-sphere of radius ϱ is $\lambda_1^\Delta=n/\varrho^2$, we easily conclude that $\lambda_1^{L_r}=(c_r/n\varrho^r)\lambda_1^\Delta=c_rH_{r+2}$.

In view of Corollary 3, Lemma 6 holds in particular for positively Ricci curved hypersurfaces in $\mathbb{R}^{n+1}$, which are necessarily embedded.

Corollary 8. *Let $\psi:M^n\to\mathbb{R}^{n+1}$ be a closed connected hypersurface in Euclidean space with positive Ricci curvature (hence, necessarily embedded), and let $\mathbf{c}$ be its center of gravity. Then for every $r=0,\dots,n-1$ it follows that*

$$\lambda_1^{L_r}\int_M|\psi-\mathbf{c}|^2\mathrm{d}M\leq c_r\int_M H_r\mathrm{d}M,\quad c_r=(n-r)\binom{n}{r},$$

and equality holds if and only if M is a round sphere in $\mathbb{R}^{n+1}$ centered at $\mathbf{c}$.

5. Upper bounds for the first eigenvalue of L_r

Let $\psi:M^n\to\mathbb{R}^{n+1}$ be an orientable closed connected hypersurface immersed into the Euclidean space, and let λ_1 be the first positive eigenvalue of its Laplacian operator. In [21] Reilly proved that for every $0\leq s\leq n-1$ it follows that

$$\lambda_1\left(\int_M H_s\mathrm{d}M\right)^2\leq n\,\mathrm{vol}(M)\int_M H_{s+1}^2\mathrm{d}M,$$

where $\mathrm{vol}(M)$ denotes the n-dimensional volume of M, and that equality holds if and only if M is a round sphere in $\mathbb{R}^{n+1}$. Lemma 6 allows us to derive the following Reilly-type inequalities for the case of the first positive eigenvalue of L_r, which extend Theorem 1.1 and Theorem 1.3 by Alencar, do Carmo and Rosenberg [2].

Theorem 9. *Let $\psi:M^n\to\mathbb{R}^{n+1}$ be an orientable closed connected hypersurface immersed into the Euclidean space. Assume that L_r is elliptic on*

M, for some $0 \leq r \leq n-1$, and let $\lambda_1^{L_r}$ be the first positive eigenvalue of L_r. Then, for every $0 \leq s \leq n-1$ it follows that

$$\lambda_1^{L_r}\left(\int_M H_s \mathrm{d}M\right)^2 \leq c_r \int_M H_r \mathrm{d}M \int_M H_{s+1}^2 \mathrm{d}M, \quad c_r = (n-r)\binom{n}{r}, \tag{13}$$

and equality holds if and only if M is a round sphere in $\mathbb{R}^{n+1}$.

Theorem 10. *Let $\psi : M^n \to \mathbb{R}^{n+1}$ be an orientable closed connected hypersurface immersed into the Euclidean space, and let $\mathbf{c}$ be its center of gravity,*

$$\mathbf{c} = \frac{1}{\mathrm{vol}(M)} \int_M \psi \mathrm{d}M \in \mathbb{R}^{n+1},$$

where $\mathrm{vol}(M)$ denotes the n-dimensional volume of M. Assume that L_r is elliptic on M, for some $0 \leq r \leq n-1$, and let $\lambda_1^{L_r}$ be the first positive eigenvalue of L_r. Then

$$\lambda_1^{L_r}\left(\int_M \langle \psi - \mathbf{c}, \mathbf{N}\rangle \mathrm{d}M\right)^2 \leq c_r \,\mathrm{vol}(M) \int_M H_r \mathrm{d}M, \tag{14}$$

and equality holds if and only if M is a round sphere in $\mathbb{R}^{n+1}$ centered at $\mathbf{c}$. In particular, if M is embedded in $\mathbb{R}^{n+1}$, then

$$\lambda_1^{L_r} \leq \frac{c_r}{(n+1)^2} \frac{\mathrm{vol}(M)}{\mathrm{vol}(\Omega)^2} \int_M H_r \mathrm{d}M, \tag{15}$$

with equality if and only if M is a round sphere in $\mathbb{R}^{n+1}$. Here Ω is the compact domain in $\mathbb{R}^{n+1}$ bounded by M, and $\mathrm{vol}(\Omega)$ denotes its $(n+1)$-dimensional volume.

Proof of Theorem 9. Let $\mathbf{c}$ be the center of gravity of M. If we multiply both sides of (8) by $\int_M H_{s+1}^2 \mathrm{d}M$ and use the Cauchy-Schwarz inequality, then we obtain

$$\begin{aligned} c_r \int_M H_r \mathrm{d}M \int_M H_{s+1}^2 \mathrm{d}M &\geq \lambda_1^{L_r} \int_M |\psi - \mathbf{c}|^2 \mathrm{d}M \int_M H_{s+1}^2 \mathrm{d}M \\ &\geq \lambda_1^{L_r}\left(\int_M |\psi - \mathbf{c}||H_{s+1}| \mathrm{d}M\right)^2 \\ &\geq \lambda_1^{L_r}\left(\int_M H_{s+1}\langle \psi - \mathbf{c}, \mathbf{N}\rangle \mathrm{d}M\right)^2. \end{aligned}$$

Using now the s-th Minkowski formula we can replace the last integral above by $\int_M H_s \mathrm{d}M$, yielding

$$\lambda_1^{L_r}\left(\int_M H_s \mathrm{d}M\right)^2 \leq c_r \int_M H_r \mathrm{d}M \int_M H_{s+1}^2 \mathrm{d}M.$$

Moreover, if equality occurs in (13), then equality occurs also in (8), which implies that M is a round sphere centered at $\mathbf{c}$. $\square$

Proof of Theorem 10. Multiply both sides of (8) by $\mathrm{vol}(M) = \int_M 1^2 \mathrm{d}M$ and use the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} c_r \,\mathrm{vol}(M) \int_M H_r \mathrm{d}M &\geq \lambda_1^{L_r} \int_M |\psi - \mathbf{c}|^2 \mathrm{d}M \int_M 1^2 \mathrm{d}M \\ &\geq \lambda_1^{L_r} \left(\int_M |\psi - \mathbf{c}| \mathrm{d}M \right)^2 \\ &\geq \lambda_1^{L_r} \left(\int_M \langle \psi - \mathbf{c}, \mathbf{N} \rangle \mathrm{d}M \right)^2, \end{aligned}$$

which yields (14). Furthermore, if equality occurs in (14), then equality occurs also in (8), which implies that M is a round sphere centered at $\mathbf{c}$.

Moreover, in the case when M is embedded in $\mathbb{R}^{n+1}$, let us denote by Ω the compact domain in $\mathbb{R}^{n+1}$ bounded by M, so that $M = \partial\Omega$. Let us consider the vector field $Y(p) = p - \mathbf{c}$ defined on Ω, with Euclidean divergence given by $\mathrm{Div}\, Y = (n+1)$. Therefore

$$(n+1)\,\mathrm{vol}(\Omega) = \int_\Omega \mathrm{Div}\, Y \mathrm{d}\Omega = \int_M \langle \psi - \mathbf{c}, \mathbf{N} \rangle \mathrm{d}M.$$

Using this in (14) yields (15). □

Because of Corollary 3, Theorems 9 and 10 apply in particular to positively Ricci curved hypersurfaces in $\mathbb{R}^{n+1}$.

Corollary 11. *Let $\psi : M^n \to \mathbb{R}^{n+1}$ be a closed connected hypersurface in Euclidean space with positive Ricci curvature (hence, necessarily embedded), let $\mathbf{c}$ be its center of gravity, and let Ω be the convex body in $\mathbb{R}^{n+1}$ bounded by M. Then for every $r = 0, \ldots, n-1$ it follows that*

$$\lambda_1^{L_r} \left(\int_M H_s \mathrm{d}M \right)^2 \leq c_r \int_M H_r \mathrm{d}M \int_M H_{s+1}^2 \mathrm{d}M, \quad 0 \leq s \leq n-1,$$

$$\lambda_1^{L_r} \left(\int_M \langle \psi - \mathbf{c}, \mathbf{N} \rangle \mathrm{d}M \right)^2 \leq c_r \,\mathrm{vol}(M) \int_M H_r \mathrm{d}M,$$

and

$$\lambda_1^{L_r} \leq \frac{c_r}{(n+1)^2} \frac{\mathrm{vol}(M)}{\mathrm{vol}(\Omega)^2} \int_M H_r \mathrm{d}M,$$

where $\mathrm{vol}(M)$ denotes the n-dimensional volume of M and $\mathrm{vol}(\Omega)$ denotes the $(n+1)$-dimensional volume of Ω. Moreover, equality holds in one of these three inequalities if and only if M is a round sphere in $\mathbb{R}^{n+1}$.

On the other hand, as observed in Remark 7, for every $r = 0, \ldots, n-2$, the $(r+2)$-th mean curvature of a round sphere $\mathbb{S}^n(\varrho)$ of radius ϱ in the Euclidean space $\mathbb{R}^{n+1}$ satisfies $c_r H_{r+2} = \lambda_1^{L_r}$. Because of this, it is natural to ask if a closed connected hypersurface in $\mathbb{R}^{n+1}$ with positive Ricci curvature (hence, necessarily embedded) whose $(r+2)$-mean curvature H_{r+2} satisfies

$$c_r H_{r+2} \leq \lambda_1^{L_r}, \quad c_r = (n-r)\binom{n}{r},$$

for some $r = 0, \dots, n-2$, $\lambda_1^{L_r}$ being the first positive eigenvalue of L_r, is necessarily a round sphere in $\mathbb{R}^{n+1}$. This question was positively answered by Deshmukh [9] in the case when $r = 0$, that is, for the case of the scalar curvature S (recall that $S = n(n-1)H_2$) and the Laplacian operator. Below we will see that there is also an affirmative answer to this question in the general case. This allows us to derive a new upper bound for the first positive eigenvalue of L_r.

THEOREM 12. *Let $\psi : M^n \to \mathbb{R}^{n+1}$ be a closed connected hypersurface in Euclidean space with positive Ricci curvature (hence, necessarily embedded). Assume that, for some $r = 0, \dots, n-2$, the $(r+2)$-th mean curvature H_{r+2} of M and the first positive eigenvalue $\lambda_1^{L_r}$ of L_r satisfy*

$$c_r H_{r+2} \leq \lambda_1^{L_r}, \quad c_r = (n-r)\binom{n}{r}.$$

Then M is a round sphere in $\mathbb{R}^{n+1}$ (and equality necessarily holds). Equivalently, for every $0 \leq r \leq n-2$, it follows that

$$\lambda_1^{L_r} \leq c_r \max H_{r+2},$$

and equality holds if and only if M is a round sphere in $\mathbb{R}^{n+1}$.

Proof. As in the proof of Lemma 6, let us consider the function $f = \frac{1}{2}|\psi - \mathbf{c}|^2$ defined on M, and let $g = \langle \psi - \mathbf{c}, \mathbf{N} \rangle$ be the support function on M, with respect to its center of gravity $\mathbf{c}$. We already know (see equation (11)) that

$$L_s f = c_s \left(H_s + g H_{s+1} \right)$$

for every $s = 0, \dots, n-1$. On the other hand, the gradient of g is given by $\nabla g = -A((\psi - \mathbf{c})^\top) = -A(\nabla f)$, so that

$$\begin{aligned} \operatorname{div}(g T_{r+1}(\nabla f)) &= -\langle A \circ T_{r+1}(\nabla f), \nabla f \rangle + g L_{r+1} f \\ &= -\langle A \circ T_{r+1}(\nabla f), \nabla f \rangle + c_{r+1} \left(g H_{r+1} + g^2 H_{r+2} \right). \end{aligned}$$

Then,

$$\int_M \langle A \circ T_{r+1}(\nabla f), \nabla f \rangle \mathrm{d}M = c_{r+1} \int_M \left(g H_{r+1} + g^2 H_{r+2} \right) \mathrm{d}M. \tag{16}$$

Since M has positive Ricci curvature, we have $H_{r+2} > 0$ on M (see Corollary 3 and Remark 4) and from our hypothesis we also have $H_{r+2} \leq \lambda_1^{L_r}/c_r$. Then, taking into account that $|\psi - \mathbf{c}|^2 = |\nabla f|^2 + g^2 \geq g^2$, we obtain that

$$g^2 H_{r+2} \leq \frac{\lambda_1^{L_r}}{c_r} |\psi - \mathbf{c}|^2$$

on M, and by Corollary 8 we have

$$\int_M g^2 H_{r+2} \mathrm{d}M \leq \int_M H_r \mathrm{d}M.$$

Putting this into the integral formula (16), and using the r-th Minkowski formula (12), we conclude that

$$\int_M \langle A \circ T_{r+1}(\nabla f), \nabla f \rangle \mathrm{d}M \leq c_{r+1} \int_M (H_r + gH_{r+1})\, \mathrm{d}M = 0.$$

On the other hand, since the Ricci curvature of M is positive, it follows from Remark 4 that the operator $A \circ T_{r+1}$ is positive definite on M, from which we conclude that $\nabla f \equiv 0$ on M. That is, $|\psi - \mathbf{c}|^2 = \text{constant} = \varrho^2 > 0$ on M, and M is a round sphere in $\mathbb{R}^{n+1}$. ☐

6. Hypersurfaces in the sphere and in the hyperbolic space

In this section we will consider both the case of hypersurfaces immersed into the Euclidean sphere

$$\mathbb{S}^{n+1} = \{x = (x_0, \ldots, x_{n+1}) \in \mathbb{R}^{n+2} : \langle x, x \rangle = 1\},$$

and the case of hypersurfaces immersed into the hyperbolic space $\mathbb{H}^{n+1}$. In this last case, it will be appropriate to use the Minkowski space model of hyperbolic space. Write $\mathbb{R}_1^{n+2}$ for $\mathbb{R}^{n+2}$, with coordinates $(x_0, \ldots, x_{n+1})$, endowed with the Lorentzian metric

$$\langle , \rangle = -dx_0^2 + dx_1^2 + \cdots + dx_{n+1}^2.$$

Then

$$\mathbb{H}^{n+1} = \{x \in \mathbb{R}_1^{n+2} : \langle x, x \rangle = -1, x_0 > 0\}$$

is a complete spacelike hypersurface in $\mathbb{R}_1^{n+2}$ with constant sectional curvature -1. This provides the Minkowski space model for the hyperbolic space.

In order to simplify our notation, let us denote by $\mathbb{M}_\kappa^{n+1}$ either the sphere $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ if $\kappa = 1$, or the hyperbolic space $\mathbb{H}^{n+1} \subset \mathbb{R}_1^{n+2}$ if $\kappa = -1$, and let $\psi : M^n \to \mathbb{M}_\kappa^{n+1} \subset \mathbb{R}^{n+2}$ be an orientable closed connected hypersurface immersed into $\mathbb{M}_\kappa^{n+1}$. We will also denote by $\langle , \rangle$, without distinction, both the Euclidean metric on $\mathbb{R}^{n+2}$ and the Lorentzian metric on $\mathbb{R}_1^{n+2}$, as well as the corresponding (Riemannian) metrics induced on $\mathbb{M}_\kappa^{n+1}$ and on M. Following Veeravalli's approach [24], we define a *center of gravity* of M as a critical point of the smooth function $\mathcal{E} : \mathbb{M}_\kappa^{n+1} \to \mathbb{R}$ given by

$$\mathcal{E}(\mathbf{p}) = \int_M \langle \psi, \mathbf{p} \rangle \mathrm{d}M, \quad \mathbf{p} \in \mathbb{M}_\kappa^{n+1}.$$

Observe that our definition of a center of gravity is equivalent to Veeravalli's, because in our model for $\mathbb{M}_\kappa^{n+1}$ we have (in Veeravalli's notation) $h_\kappa \circ d_{\mathbf{p}} = \kappa - \langle \psi, \mathbf{p} \rangle$ on M.

As for the existence and uniqueness of centers of gravity, observe that when $\kappa = 1$, then every closed hypersurface in $\mathbb{S}^{n+1}$ admits at least two different centers of gravity (the values where $\mathcal{E}$ attains its maximum and minimum values). For instance, it can be easily seen that every geodesic sphere in $\mathbb{S}^{n+1}$ of radius $\varrho < \pi/2$ centered at a point $\mathbf{c} \in \mathbb{S}^{n+1}$ has exactly two centers of gravity, $\mathbf{c}$ and $-\mathbf{c}$. In fact, it is well-known that the position vector field of every oriented hypersurface $\psi : M^n \to \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ immersed into the sphere satisfies

$$\Delta\psi = nH\mathbf{N} - n\psi, \tag{17}$$

which implies that

$$\mathcal{E}(\mathbf{p}) = \int_M \langle \psi, \mathbf{p} \rangle \mathrm{d}M = \int_M H \langle \mathbf{N}, \mathbf{p} \rangle \mathrm{d}M \tag{18}$$

for every $\mathbf{p} \in \mathbb{S}^{n+1}$. In particular, if M is a geodesic sphere in $\mathbb{S}^{n+1}$ of radius $\varrho < \pi/2$ centered at a point $\mathbf{c} \in \mathbb{S}^{n+1}$, then

$$\mathbf{N} = \frac{1}{\sin\varrho}(\mathbf{c} - \cos\varrho\psi)$$

and $H = \cot\varrho$, so that by (18)

$$\mathcal{E}(\mathbf{p}) = \cos\varrho \;\; \mathrm{vol}(M) \langle \mathbf{c}, \mathbf{p} \rangle$$

for every $\mathbf{p} \in \mathbb{S}^{n+1}$. Now, it is clear that the function $\mathcal{E}$ has exactly two critical points on $\mathbb{S}^{n+1}$, which correspond to the points $\mathbf{c}$ and $-\mathbf{c}$, where $\mathcal{E}$ attains its maximum and minimum values, respectively. On the other hand, (18) also implies that for every closed minimal hypersurface M in $\mathbb{S}^{n+1}$ we have $\mathcal{E} \equiv 0$ so that *every* point of $\mathbb{S}^{n+1}$ is a center of gravity of M. In particular, this holds when the radius of the geodesic sphere is $\varrho = \pi/2$ (that is, for an equator of $\mathbb{S}^{n+1}$). More generally, this also holds for every closed (not necessarily minimal) hypersurface of the sphere for which $\int_M \psi \mathrm{d}M = 0 \in \mathbb{R}^{n+2}$, such as, for example, the family of tori $\mathbb{S}^1(r) \times \mathbb{S}^1(\sqrt{1-r^2}) \subset \mathbb{S}^3$, with $0 < r < 1$.

On the other hand, as observed by Veeravalli, when $\kappa = -1$ the function $-\mathcal{E}$ is strictly convex on $\mathbb{H}^{n+1}$, which implies that M admits a unique center of gravity. Even more, let $\psi : M^n \to \mathbb{M}_\kappa^{n+1} \subset \mathbb{R}^{n+2}$ be an oriented closed connected hypersurface immersed into $\mathbb{M}_\kappa^{n+1}$, $\kappa = \pm 1$. Then a point $\mathbf{c} \in \mathbb{M}_\kappa^{n+1}$ is a center of gravity of M if and only if

$$d\mathcal{E}_{\mathbf{c}}(v) = \int_M \langle \psi, v \rangle \mathrm{d}M = \langle \textstyle\int_M \psi \mathrm{d}M, v \rangle = 0$$

for every $v \in T_{\mathbf{c}}\mathbb{M}^{n+1}_\kappa = \mathbf{c}^\perp = \{x \in \mathbb{R}^{n+2} : \langle x, \mathbf{c}\rangle = 0\}$. That is, $\mathbf{c} \in \mathbb{M}^{n+1}_\kappa$ is a center of gravity of M if and only if $\int_M \psi \mathrm{d}M$ is a multiple of $\mathbf{c}$,

$$\int_M \psi \mathrm{d}M = \kappa \langle \textstyle\int_M \psi \mathrm{d}M, \mathbf{c}\rangle \mathbf{c} = \kappa \left(\displaystyle\int_M \langle \psi, \mathbf{c}\rangle \mathrm{d}M \right) \mathbf{c}.$$

Therefore, when $\kappa = 1$, then every closed hypersurface in $\mathbb{S}^{n+1}$ with $\int_M \psi \mathrm{d}M \neq 0 \in \mathbb{R}^{n+2}$ has exactly two centers of gravity, $\mathbf{c}$ and $-\mathbf{c}$, where

$$\mathbf{c} = \frac{1}{|\int_M \psi \mathrm{d}M|} \int_M \psi \mathrm{d}M \in \mathbb{S}^{n+1}.$$

On the other hand, when $\kappa = -1$, then $\langle \psi, \psi\rangle = -1$ implies that

$$\langle \textstyle\int_M \psi \mathrm{d}M, \int_M \psi \mathrm{d}M\rangle < 0,$$

so that the unique center of gravity of M is given by

$$\mathbf{c} = \frac{1}{|\int_M \psi \mathrm{d}M|} \int_M \psi \mathrm{d}M \in \mathbb{H}^{n+1},$$

where

$$\left| \int_M \psi \mathrm{d}M \right| = \sqrt{-\langle \textstyle\int_M \psi \mathrm{d}M, \int_M \psi \mathrm{d}M\rangle} > 0.$$

Using this terminology, we can extend Lemma 6 to the spherical and hyperbolic cases as follows. (When $r = 0$ we recover the results of Veeravalli for the first eigenvalue of the Laplacian operator.)

LEMMA 13. *Let $\psi : M^n \to \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ be an orientable closed connected hypersurface immersed into the sphere, and let $\mathbf{c} \in \mathbb{S}^{n+1}$ be a center of gravity of M. Assume that L_r is elliptic on M, for some $0 \leq r \leq n-1$, and let $\lambda_1^{L_r}$ be the first positive eigenvalue of L_r. Then*

$$\lambda_1^{L_r} \int_M (1 - \langle \psi, \mathbf{c}\rangle^2) \mathrm{d}M \leq c_r \int_M H_r \mathrm{d}M, \quad c_r = (n-r)\binom{n}{r}, \tag{19}$$

and equality holds if and only if M is a geodesic sphere in $\mathbb{S}^{n+1}$ centered at $\mathbf{c}$.

LEMMA 14. *Let $\psi : M^n \to \mathbb{H}^{n+1} \subset \mathbb{R}^{n+2}_1$ be an orientable closed connected hypersurface immersed into the hyperbolic space, and let $\mathbf{c} \in \mathbb{H}^{n+1}$ be its center of gravity. Assume that L_r is elliptic on M, for some $0 \leq r \leq n-1$, and let $\lambda_1^{L_r}$ be the first positive eigenvalue of L_r. Then*

$$\lambda_1^{L_r} \int_M (\langle \psi, \mathbf{c}\rangle^2 - 1) \mathrm{d}M \leq c_r \int_M H_r \mathrm{d}M + \int_M \langle T_r(\mathbf{c}^\top), \mathbf{c}^\top\rangle \mathrm{d}M, \tag{20}$$
$$c_r = (n-r)\binom{n}{r},$$

where $\mathbf{c}^\top = \nabla\langle \psi, \mathbf{c}\rangle$ is the tangent component of $\mathbf{c}$ along the immersion.

As already observed by Veeravalli (in the case $r = 0$), the inequality (20) is not sharp. The same holds for the upper bound for $\lambda_1^{L_r}$ given by Alencar, do Carmo and Rosenberg in [2, Theorem 3.1] for hypersurfaces in $\mathbb{H}^{n+1}$.

Proof of Lemma 13 and Lemma 14. As in the proof of Lemma 6, since L_r is assumed to be elliptic, we can use the minimax characterization of $\lambda_1^{L_r}$ given by (9). In this case, observe that for a fixed arbitrary vector $\mathbf{a} \in \mathbb{R}^{n+2}$, the gradient of the function $\langle \psi, \mathbf{a} \rangle$ defined on M is given by $\nabla \langle \psi, \mathbf{a} \rangle = \mathbf{a}^\top$, where

$$\mathbf{a}^\top = \mathbf{a} - \langle \mathbf{N}, \mathbf{a} \rangle \mathbf{N} - \kappa \langle \psi, \mathbf{a} \rangle \psi \in \mathcal{X}(M)$$

is the tangent component of $\mathbf{a}$ along the immersion. Therefore, for every $X \in \mathcal{X}(M)$ we have

$$\nabla_X(\nabla \langle \psi, \mathbf{a} \rangle) = \langle \mathbf{N}, \mathbf{a} \rangle AX - \kappa \langle \psi, \mathbf{a} \rangle X,$$

and

$$\begin{aligned} L_r \langle \psi, \mathbf{a} \rangle &= \langle \mathbf{N}, \mathbf{a} \rangle \operatorname{tr}(A \circ T_r) - \kappa \langle \psi, \mathbf{a} \rangle \operatorname{tr}(T_r) \\ &= c_r(H_{r+1} \langle \mathbf{N}, \mathbf{a} \rangle - \kappa H_r \langle \psi, \mathbf{a} \rangle). \end{aligned} \tag{21}$$

That is,

$$L_r \psi = c_r H_{r+1} \mathbf{N} - c_r \kappa H_r \psi, \tag{22}$$

for every $0 \leq r \leq n-1$.

Let us consider an orthonormal basis $\{\mathbf{a}_1, \ldots, \mathbf{a}_{n+1}\} \subset \mathbb{R}^{n+2}$ of $T_{\mathbf{c}}\mathbb{M}_\kappa^{n+1} = \mathbf{c}^\perp$, and for every $1 \leq i \leq n+1$, let $f_i = \langle \psi, \mathbf{a}_i \rangle$. Then for every $i = 1, \ldots, n+1$ we have $\int_M f_i \mathrm{d}M = 0$, and by (21) we also get

$$L_r f_i = L_r \langle \psi, \mathbf{a}_i \rangle = c_r(H_{r+1} \langle \mathbf{N}, \mathbf{a}_i \rangle - \kappa H_r \langle \psi, \mathbf{a} \rangle).$$

Therefore, using (9) we obtain that

$$\begin{aligned} \lambda_1^{L_r} \int_M f_i^2 \mathrm{d}M &\leq - \int_M f_i L_r f_i \mathrm{d}M \\ &= c_r \kappa \int_M H_r f_i^2 \mathrm{d}M - c_r \int_M H_{r+1} f_i \langle \mathbf{N}, \mathbf{a}_i \rangle \mathrm{d}M. \end{aligned}$$

Now we sum from $i = 1$ to $n+1$. First, observe that

$$\psi = \sum_{i=1}^{n+1} f_i \mathbf{a}_i + \kappa \langle \psi, \mathbf{c} \rangle \mathbf{c},$$

and

$$\mathbf{N} = \sum_{i=1}^{n+1} \langle \mathbf{N}, \mathbf{a}_i \rangle \mathbf{a}_i + \kappa \langle \mathbf{N}, \mathbf{c} \rangle \mathbf{c},$$

so that $\sum_{i=1}^{n+1} f_i^2 = \kappa(1-\langle\psi,\mathbf{c}\rangle^2)$ and $\sum_{i=1}^{n+1} f_i\langle\mathbf{N},\mathbf{a}_i\rangle = -\kappa\langle\mathbf{N},\mathbf{c}\rangle\langle\psi,\mathbf{c}\rangle$. Therefore, using (21) with $\mathbf{a}=\mathbf{c}$, we obtain that

$$\begin{aligned}\lambda_1^{L_r}\int_M \kappa(1-\langle\psi,\mathbf{c}\rangle^2)\mathrm{d}M &\leq c_r\int_M H_r\mathrm{d}M + \kappa\int_M \langle\psi,\mathbf{c}\rangle L_r(\langle\psi,\mathbf{c}\rangle)\mathrm{d}M \\ &= c_r\int_M H_r\mathrm{d}M - \kappa\int_M \langle T_r(\mathbf{c}^\top),\mathbf{c}^\top\rangle\mathrm{d}M,\end{aligned} \tag{23}$$

where $\mathbf{c}^\top = \nabla\langle\psi,\mathbf{c}\rangle$.

When $\kappa=-1$, then (23) directly gives (20). On the other hand, when $\kappa=1$, then (23) becomes

$$\lambda_1^{L_r}\int_M (1-\langle\psi,\mathbf{c}\rangle^2)\mathrm{d}M \leq c_r\int_M H_r\mathrm{d}M - \int_M \langle T_r(\mathbf{c}^\top),\mathbf{c}^\top\rangle\mathrm{d}M.$$

We now remark that the ellipticity of L_r is equivalent to the positiveness of the quadratic form associated to T_r. Therefore it follows that

$$\lambda_1^{L_r}\int_M (1-\langle\psi,\mathbf{c}\rangle^2)\mathrm{d}M \leq c_r\int_M H_r\mathrm{d}M,$$

with equality if and only if $\mathbf{c}^\top = \nabla\langle\psi,\mathbf{c}\rangle \equiv 0$, that is, if and only if M is a geodesic sphere in $\mathbb{S}^{n+1}$ centered at the point $\mathbf{c}$. □

Remark 15. Integrating (21) over M, the divergence theorem implies the corresponding Minkowski formulae for hypersurfaces immersed in the sphere and in the hyperbolic space, first obtained by Bivens [6] (see also [19] for another approach that is closer to ours, but with a different proof),

$$\int_M H_{r+1}\langle\mathbf{N},\mathbf{a}\rangle\mathrm{d}M = \kappa\int_M H_r\langle\psi,\mathbf{a}\rangle\mathrm{d}M, \tag{24}$$

for each $r=0,\ldots,n-1$ and $\mathbf{a}\in\mathbb{R}^{n+2}$ arbitrary.

As an application of Lemma 13, we derive the following Reilly-type inequalities for the first positive eigenvalue of the operator L_r of a closed hypersurface in sphere, which extend the Theorem in [24].

Theorem 16. *Let $\psi: M^n \to \mathbb{S}^{n+1}\subset\mathbb{R}^{n+2}$ be an orientable closed connected hypersurface immersed into the sphere, and let $\mathbf{c}\in\mathbb{S}^{n+1}$ be a center of gravity of M. Assume that L_r is elliptic on M, for some $0\leq r\leq n-1$, and let $\lambda_1^{L_r}$ be the first positive eigenvalue of L_r. Then we have the following inequalities:*

$$\lambda_1^{L_r}\left(\int_M H_s\langle\psi,\mathbf{c}\rangle\mathrm{d}M\right)^2 \leq c_r\int_M H_r\mathrm{d}M\int_M H_{s+1}^2\mathrm{d}M, \tag{25}$$

for every $0\leq s\leq n-1$, and

$$\lambda_1^{L_r}\left(\int_M \langle\mathbf{N},\mathbf{c}\rangle\mathrm{d}M\right)^2 \leq c_r\,\mathrm{vol}(M)\int_M H_r\mathrm{d}M, \tag{26}$$

where $\mathrm{vol}(M)$ *denotes the* n*-dimensional volume of* M*. In particular, if* M *is embedded in* $\mathbb{S}^{n+1}$*, then* (26) *becomes*

$$\lambda_1^{L_r}\left(\int_\Omega \langle p,\mathbf{c}\rangle \mathrm{d}\Omega(p)\right)^2 \leq \frac{c_r}{(n+1)^2}\,\mathrm{vol}(M)\int_M H_r\mathrm{d}M, \tag{27}$$

where Ω *is any one of the two compact domains in* $\mathbb{S}^{n+1}$ *bounded by* M*. Moreover, equality occurs in one of these three inequalities if and only if* M *is a geodesic sphere in* $\mathbb{S}^{n+1}$ *centered at* $\mathbf{c}$*.*

Proof. If we multiply both sides of (19) by $\int_M H_{s+1}^2\mathrm{d}M$ and use the Cauchy-Schwarz inequality, then we obtain

$$\begin{aligned} c_r\int_M H_r\mathrm{d}M\int_M H_{s+1}^2\mathrm{d}M &\geq \lambda_1^{L_r}\int_M(1-\langle\psi,\mathbf{c}\rangle^2)\mathrm{d}M\int_M H_{s+1}^2\mathrm{d}M \\ &\geq \lambda_1^{L_r}\left(\int_M\sqrt{1-\langle\psi,\mathbf{c}\rangle^2}|H_{s+1}|\mathrm{d}M\right)^2. \end{aligned} \tag{28}$$

Observe now that $\mathbf{c}=\mathbf{c}^\top+\langle\mathbf{N},\mathbf{c}\rangle\mathbf{N}+\langle\psi,\mathbf{c}\rangle\psi$, which implies that $1-\langle\psi,\mathbf{c}\rangle^2=|\mathbf{c}^\top|^2+\langle\mathbf{N},\mathbf{c}\rangle^2$ and

$$\sqrt{1-\langle\psi,\mathbf{c}\rangle^2}\geq|\langle\mathbf{N},\mathbf{c}\rangle|, \tag{29}$$

with equality if and only if $\mathbf{c}^\top=\nabla\langle\psi,\mathbf{c}\rangle\equiv 0$, that is, if and only if M is a geodesic sphere in $\mathbb{S}^{n+1}$ centered at the point $\mathbf{c}$. Putting this into (28), we obtain

$$\begin{aligned} c_r\int_M H_r\mathrm{d}M\int_M H_{s+1}^2\mathrm{d}M &\geq \lambda_1^{L_r}\left(\int_M|\langle\mathbf{N},\mathbf{c}\rangle||H_{s+1}|\mathrm{d}M\right)^2 \\ &\geq \lambda_1^{L_r}\left(\int_M H_{s+1}\langle\mathbf{N},\mathbf{c}\rangle\mathrm{d}M\right)^2 \\ &= \lambda_1^{L_r}\left(\int_M H_s\langle\psi,\mathbf{c}\rangle\mathrm{d}M\right)^2, \end{aligned}$$

where in the last equality we have used the s-th Minkowski formula (24) with $\mathbf{a}=\mathbf{c}$. This finishes the proof of inequality (25).

As for the proof of (26), multiply both sides of (19) by $\mathrm{vol}(M)=\int_M 1^2\mathrm{d}M$ and use the Cauchy-Schwarz inequality and (29) to obtain

$$\begin{aligned} c_r\,\mathrm{vol}(M)\int_M H_r\mathrm{d}M &\geq \lambda_1^{L_r}\int_M(1-\langle\psi,\mathbf{c}\rangle^2)\mathrm{d}M\int_M 1^2\mathrm{d}M \\ &\geq \lambda_1^{L_r}\left(\int_M\sqrt{(1-\langle\psi,\mathbf{c}\rangle^2)}\mathrm{d}M\right)^2 \\ &\geq \lambda_1^{L_r}\left(\int_M\langle\mathbf{N},\mathbf{c}\rangle\mathrm{d}M\right)^2, \end{aligned}$$

which yields (26). Moreover, if equality occurs either in (25) or in (26), then equality occurs also in (19), and M is a geodesic sphere in $\mathbb{S}^{n+1}$ centered at the point $\mathbf{c}$.

Moreover, in the case when M is embedded in $\mathbb{S}^{n+1}$, let us consider the vector field on $\mathbb{S}^{n+1}$ defined by $Y(p) = \mathbf{c} - \langle p, \mathbf{c}\rangle p, p \in \mathbb{S}^{n+1}$. Observe that Y is a conformal vector field on $\mathbb{S}^{n+1}$ with singularities at $\mathbf{c}$ and $-\mathbf{c}$, with spherical divergence given by $\operatorname{Div} Y(p) = -(n+1)\langle p, \mathbf{c}\rangle$. Therefore, if Ω denotes one of the two compact domains in $\mathbb{S}^{n+1}$ bounded by M, then

$$-(n+1)\int_\Omega \langle p, \mathbf{c}\rangle \mathrm{d}\Omega(p) = \int_\Omega \operatorname{Div} Y \mathrm{d}\Omega = \pm \int_M \langle Y, \mathbf{N}\rangle \mathrm{d}M = \pm \int_M \langle \mathbf{c}, \mathbf{N}\rangle \mathrm{d}M,$$

since $\langle Y, \mathbf{N}\rangle|_M = \langle \mathbf{c}, \mathbf{N}\rangle - \langle \psi, \mathbf{c}\rangle\langle \psi, \mathbf{N}\rangle = \langle \mathbf{c}, \mathbf{N}\rangle$. Thus

$$\left(\int_M \langle \mathbf{N}, \mathbf{c}\rangle \mathrm{d}M\right)^2 = (n+1)^2 \left(\int_\Omega \langle p, \mathbf{c}\rangle \mathrm{d}\Omega(p)\right)^2,$$

which together with (26) yields (27). □

Finally, let us remark that equation (21) (or, equivalently, (22)) is the generalization of the well-known formula $\Delta\psi = nH\mathbf{N} - n\kappa\psi$, which holds true for the position vector of every hypersurface $\psi : M^n \to \mathbb{M}^{n+1}_\kappa \subset \mathbb{R}^{n+2}$ immersed into $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ or $\mathbb{H}^{n+1} \subset \mathbb{R}^{n+2}_1$. This allows us to extend the spherical and hyperbolic versions of Takahashi's theorem, given by Markvorsen [18, Corollaries A and B], as follows.

Corollary 17. *Let $\psi : M^n \to \mathbb{M}^{n+1}_\kappa \subset \mathbb{R}^{n+2}$ be an orientable connected hypersurface immersed into $\mathbb{M}^{n+1}_\kappa$, and let L_r be the linearized operator of the $(r+1)$-th mean curvature of M, for some $r = 0, \ldots, n-1$. Then*

$$L_r\psi + f\psi = 0$$

for a smooth function $f \in \mathcal{C}^\infty(M)$ if and only if M is $(r+1)$-minimal in $\mathbb{M}^{n+1}_\kappa$ (and $f = c_r\kappa H_r$ necessarily).

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L.J. Alías, Departamento de Matemáticas, Universidad de Murcia, E-30100 Espinardo, Murcia, Spain
E-mail address: ljalias@um.es

J.M. Malacarne, Departamento de Matemática, Universidade Federal do Espírito Santo, 29060-900 Vitória-ES, Brazil
E-mail address: jmiguel@cce.ufes.br