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PSEUDO-ISOTOPY CLASSES OF DIFFEOMORPHISMS OF THE UNKNOTTED PAIRS $(S^{n+2},\ S^n)$ AND $(S^{2p+2},\ S^p \times S^p)$

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Abstract. We consider two pairs, the standard unknotted n-sphere in S^{n+2}, and the product of two p-spheres trivially embedded in S^{2p+2}, and study orientation preserving diffeomorphisms of these pairs. Pseudo-isotopy classes of such diffeomorphisms form subgroups of the mapping class groups of S^n and $S^p \times S^p$, respectively, and we determine the algebraic structure of such subgroups when $n > 4$ and $p > 1$.

1. Introduction

Let M^n be a closed, oriented, smooth manifold of dimension n. We denote the group of all orientation preserving diffeomorphisms of M by $\mathrm{Diff}(M)$. Recall that two elements of this group, say f and h, are called pseudo-isotopic if there exists an orientation preserving diffeomorphism $F \in \mathrm{Diff}(M \times [0,1])$ such that $F(x,0) = f(x)$ and $F(x,1) = h(x)$ for all $x \in M$. The set of pseudo-isotopy classes forms a group, which we call the *mapping class group of M* and denote by $\pi_0 \,\mathrm{Diff}(M)$. For some special types of manifolds this group has been computed by several authors; see, for example, [5], [15], [16], [19], [25], [26], [31]. The classical mapping class group $\mathcal{M}_g$, that is, the group of the isotopy classes of orientation preserving diffeomorphisms of an orientable surface F^2 of genus g, has been studied extensively, and there are also results on the so-called *spin mapping class group* ([7], [9], [22]), which is a subgroup of $\mathcal{M}_g$ that consists of diffeomorphisms preserving the Rokhlin quadratic form $\rho : H_1(F^2;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$ (see the next section for the definition). The spin mapping class group has a geometrical meaning: it consists of the diffeomorphisms that extend to diffeomorphisms of the 4-sphere, if one assumes that the closed 2-surface is trivially embedded in S^4 (see [9] and [22]). From this point of view, there are interesting results of S. Hirose (Theorem A in [8]) and Z. Iwase (Proposition 3.1 in [12]) who have shown that for a non-trivially embedded 2-torus $T^2 \hookrightarrow S^4$ the isotopy classes of diffeomorphisms of T^2 which extend over S^4 form a proper subgroup of the spin mapping class group of the torus.

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One way to obtain such a non-trivial embedding is to take a classical torus knot $(S^3,\ K)$, choose a 3-disk $D^3 \hookrightarrow S^3 \setminus (K \times D^2)$ and construct the *spun* T^2*-knot* in S^4 using an analog of the spinning construction:

$$\big(S^4,\ S(K)\big) := \big((S^3,\ K) \setminus \operatorname{int} D^3\big) \times S^1 \bigcup_{\mathrm{id}} S^2 \times D^2.$$

One can consider 2-surfaces not only in S^4 but in any other closed orientable 4-manifold and ask similar questions. An attractive family of 2-surfaces in a 4-manifold is the family of non-singular algebraic curves, say in $\mathbb{C}P^2$. Theorem 4.3 of [10] claims that for the smooth algebraic curves of odd degree greater than four, the isotopy classes of diffeomorphisms extendable over $\mathbb{C}P^2$ form a proper subgroup of the corresponding mapping class group.

Here we are interested in a higher dimensional analog. Let $M^n,\ n \geq 4$, be a closed oriented locally unknotted submanifold of the standard sphere S^{n+2}; in other words, consider an n-dimensional locally flat differentiable M^n-knot $(S^{n+2},\ M^n)$. The following questions naturally arise:

- Given such a knot, which elements of the mapping class group $\pi_0 \operatorname{Diff}(M^n)$ have representatives that extend to orientation-preserving diffeomorphisms of the ambient sphere S^{n+2}?
- How does the set of such elements depend on the knot type?

It should be mentioned that similar questions appear in the study of the homotopy type of the complement to an algebraic hypersurface in the complex space. For the details, the reader is referred to a recent work of A. Libgober (see [20], §3.2), and to §5 of [4].

In this paper we answer the first question for two "trivial" knots: $(S^{n+2},\ S^n)$ with $n \geq 5$, and $(S^{2p+2},\ S^p \times S^p)$ with $p \geq 2$.

Let us define first the subgroup of the mapping class group of M, which we will be dealing with. Suppose $M^n,\ n \geq 5$, is a closed simply-connected manifold embedded in the standard sphere $S^{n+m},\ m \geq 1$. Consider two orientation preserving diffeomorphisms of M, say ϕ_1 and ϕ_2, which are isotopic. If we assume that ϕ_1 extends to an orientation preserving diffeomorphism of the ambient sphere, then the isotopy extension theorem (see, for example, [11], Ch. 8, Theorem 1.5) guarantees that ϕ_2 can also be extended to a diffeomorphism of S^{n+m}. Moreover, it follows from the disk theorem ([11], Ch. 8, Theorem 3.1) that we can always assume that the extension is isotopic to the identity map of the ambient sphere (by changing the extension on a small disk embedded far away from M). This implies that for an M^n-knot $(S^{n+m},\ M)$ the following subgroup of $\pi_0 \operatorname{Diff}(M)$, denoted by $\mathcal{E}(S^{n+m},\ M)$, is well defined (cf. the definition of $\mathcal{E}(S^4,\ K)$ in [9]).

$$\mathcal{E}(S^{n+m},\ M) := \Big\{[\phi] \in \pi_0 \operatorname{Diff}(M) \mid \text{there exists } \Phi \in \operatorname{Diff}(S^{n+m}) \text{ s. t. } \Phi|_M = \phi\Big\}.$$

When $n \geq 5$, we can replace isotopy by pseudo-isotopy by a result of J. Cerf [3].

In the next section we will use a higher dimensional analog of the Rokhlin quadratic form introduced by J. Levine in [17] to generalize a result of J. Montesinos (see [22]) who showed that for the trivially embedded torus $T^2 \hookrightarrow S^4$ the group $\mathcal{E}(S^4,\ T^2)$ consists of the matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$, where both products $a \cdot b$ and $c \cdot d$ are even integers. In the last section we study diffeomorphisms of $S^p \times S^p$ which act trivially on the homology and extend over the ambient sphere S^{2p+2}. The isotopy classes of such diffeomorphisms form a subgroup of $\mathcal{E}(S^{2p+2},\ S^p \times S^p)$, which will be denoted by $\mathcal{S}E(S^{2p+2},\ S^p \times S^p)$. We will write $\phi \sim \varphi$, and $G \cong H$, to indicate that the diffeomorphisms ϕ and φ are (pseudo) isotopic, and the groups G and H are isomorphic, respectively. All homology and cohomology groups will have integer coefficients unless otherwise stated.

2. Levine-Rokhlin quadratic forms

In this section we discuss a higher dimensional analog of the Rokhlin quadratic form and show that if a diffeomorphism $f \in \mathrm{Diff}(M^n)$ extends to a diffeomorphism of the ambient sphere S^{n+2}, then f_* (the induced homology map) commutes with this quadratic form q. The definition of such a form q is based on the following construction of Levine ([17], §3).

Let us denote the connected sum of k copies of $S^p \times S^p$ by M^n ($p \geq 3$ is assumed to be odd in this section) and assume that M^n is embedded in S^{n+2}. Notice that since M is at least 2-connected, the normal bundle $\nu(S^{n+2}, M^n) \simeq \varepsilon^2$ is trivial for any embedding $M^n \hookrightarrow S^{n+2}$. Hence we can choose a framing $\mathcal{F} = (f_1, f_2)$, two orthonormal vector fields on M, to get a framed submanifold $(M, \mathcal{F})$ of S^{n+2}. Moreover, since M is simply connected, the set of homotopy classes $[M,\ SO(2)]$ consists of the trivial element only and therefore the choice of our framing $\mathcal{F}$ is unique up to homotopy. For such a framed manifold, Levine (see [17], §4 and also [6], §3.2) defined a function

$$\phi(M, \mathcal{F}):\ H_p(M;\mathbb{Z}) \longrightarrow \mathbb{Z}_2,$$

which depends only on the isotopy class of the pair $(M, \mathcal{F})$ and satisfies the formula

$$\phi(\alpha + \beta) = \phi(\alpha) + \phi(\beta) + (\alpha \cdot \beta)_2,$$

where $\alpha, \beta \in H_p(M;\mathbb{Z})$ and $(\alpha \cdot \beta)_2$ is the mod 2 intersection number of α and β. We now recall the definition of Levine in a slightly different form which will suit the needs of this paper. Fix an embedding $M^n \hookrightarrow S^{n+2}$ and a framing $\mathcal{F} = (f_1, f_2)$ of the normal bundle $\nu(S^{n+2}, M^n)$. Take a point pt outside a tubular neighborhood of M in S^{n+2} and choose a (positive) framing $\mathcal{G}$ of $\tau(S^{n+2})|_{S^{n+2}-pt}$ (the tangent bundle restricted to the complement of pt). Let $\mathtt{S} \hookrightarrow M$ be an embedded p-sphere which represents a class $[z] \in H_p(M;\mathbb{Z})$ and consider the subbundle E of $\tau(S^{n+2})|_{\mathtt{S}}$ which is the Whitney sum of the tangent bundle $\tau(\mathtt{S})$ and the line bundle $f_1|_{\mathtt{S}}$. This subbundle has a canonical framing $\Upsilon_1 = (e_1, \ldots, e_p, e_{p+1})$ which comes from the standard framing of $\mathbb{R}^{p+1}$ (cf. [6], §3.2, and recall that we have fixed an embedding $M^n \hookrightarrow S^{n+2}$). This Υ_1 together with the field $f_2|_{\mathtt{S}}$ gives us a framing Υ_2 of the sum $E \oplus f_2|_{\mathtt{S}}$ which is a $(p+2)$-subbundle of $\tau(S^{n+2})|_{\mathtt{S}}$. Comparing this framing $\Upsilon_2(x)$ with the framing $\mathcal{G}(x)$ at each point $x \in \mathtt{S}$ produces an element of the Stiefel manifold $V_{p+2}(\mathbb{R}^{2p+2})$ of orthonormal $(p+2)$-frames in $\mathbb{R}^{2p+2}$. Thus we obtain an element of the homotopy group $\pi_p(V_{p+2}(\mathbb{R}^{2p+2})) \cong \mathbb{Z}_2$ (for the last isomorphism see [2], Theorem IV.1.12). In our case, this construction gives a well defined function

$$\phi(M): \ H_p(M;\mathbb{Z}) \longrightarrow \mathbb{Z}_2,$$

which depends only on the isotopy class of the embedding $M^n \hookrightarrow S^{n+2}$.

Proposition 1 (cf. [22], Proposition 4.1; [9], Theorem 1.2). *Let $p = n/2 \geq 3$ be odd and let M^n be a closed $(p-1)$-connected manifold embedded into S^{n+2} and $[f] \in \mathcal{E}(S^{n+2}, M^n)$. Then $f_* : H_p(M) \to H_p(M)$ preserves the function ϕ.*

Proof. Let $\mathtt{S}^p \hookrightarrow M$ be an embedded sphere representing a cycle $[\mathtt{S}^p] \in H_p(M, \mathbb{Z})$. Since the diffeomorphism f extends to a diffeomorphism of the ambient sphere, we can compare two Υ_2-framings at each point of $f(\mathtt{S}^p)$. This comparison gives us an element of $\pi_p(SO(p+2))$. Since the (f_1, f_2)-restrictions of these two framings are homotopic, it follows from the exact homotopy sequence of the fibration $SO(p) \hookrightarrow SO(2p+2) \twoheadrightarrow V_{p+2}(\mathbb{R}^{2p+2})$ that $\phi([f(\mathtt{S}^p)]) = \phi([\mathtt{S}^p])$. □

Remark. If we denote the boundary operator from this exact homotopy sequence by ∂, then $\partial(\phi([\mathtt{S}^p])) \in \pi_{p-1}(SO(p))$ is the obstruction to trivializing the normal bundle of $\mathtt{S}^p$ in M (see [6], Lemma 3.4).

Let us now assume that our $2p$-manifold $M^n \cong (S^p \times S^p)\# \ldots \#(S^p \times S^p)$ is trivially embedded in S^{n+1}, which is embedded in S^{n+2} as an equator: $M^n \hookrightarrow S^{n+1} \hookrightarrow S^{n+2}$. In particular, we assume that S^{n+1} can be presented as the union of two copies of the handlebody $\mathcal{H} \simeq (D^{p+1} \times S^p)\# \ldots \#(D^{p+1} \times S^p)$ (here # stands for the boundary connected sum) along the boundary $M^n = \partial\mathcal{H}$:

$$S^{n+1} = \mathcal{H}_+ \bigcup_M -\mathcal{H}_-.$$

Taking the tensor product of $H_p(M;\mathbb{Z})$ with $\mathbb{Z}_2$ and using the function ϕ, we obtain a quadratic form $q : H_p(M;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$ which has Arf invariant zero. For a closed orientable 4-manifold X with $H_1(X;\mathbb{Z}_2) \cong 0$ and a closed orientable embedded 2-surface $F \hookrightarrow X^4$ which realizes the element of $H_2(X;\mathbb{Z}_2)$ dual to $w_2(X)$, V. Rokhlin (see [23], §3) defined a function $\psi : H_1(F;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$ which also satisfies the formula $\psi(\alpha + \beta) =$

$\psi(\alpha) + \psi(\beta) + (\alpha \cdot \beta)_2$. The form ψ has Arf invariant zero for a trivially embedded surface too.

We now turn to the case where $M \cong S^p \times S^p$ is standardly embedded in S^{2p+2}. By this we mean that M in S^{2p+2} is the boundary of a tubular neighborhood of a standardly embedded $S^p \hookrightarrow S^{2p+1}$, where S^{2p+1} is an equator of S^{2p+2} (cf. Definition 1.6 of [21]). It then follows from our Proposition 1 and Propositions 4.2 and 4.3 of [22] that if a diffeomorphism $f \in \mathrm{Diff}(M)$ extends to a diffeomorphism of the pair (S^{n+2}, M), then the induced automorphism f_* is an element of the group, which consists of the matrices $\left(\begin{smallmatrix} d_1 & d_2 \\ d_3 & d_4 \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$, where both products d_1d_2 and d_3d_4 are even integers. Clearly, any matrix of this type is congruent modulo 2 either to $\mathrm{Id} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or $V := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and we denote such a subgroup of $\mathrm{SL}(2,\mathbb{Z})$ by $\Gamma_V(2)$. One can use the fact that the corresponding projective group $\Gamma_V(2)/\mathbb{Z}_2 \cong \mathbb{Z}_2 * \mathbb{Z}$ is generated by V and $T := \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ (cf. [27], §3) to show that $\Gamma_V(2)$ admits the presentation $\Gamma_V(2) \cong \langle V, T \mid V^4 = \mathrm{Id},\ V^2T = TV^2\rangle$.

Let $h : \pi_0 \mathrm{Diff}(M) \longrightarrow \mathrm{Aut}(H_p(M;\mathbb{Z}))$ be the obvious homomorphism induced by the action on the homology of M. We denote the restriction of this homomorphism onto the subgroup $\mathcal{E}(S^{n+2},\ M)$ by $h_{\mathcal{E}}$. Evidently, $\mathrm{Im}(h_{\mathcal{E}}) \subset \Gamma_V(2)$. Montesinos showed (see [22], Theorem 5.4) that if M is a standardly embedded torus $S^1 \times S^1 \hookrightarrow S^3 \hookrightarrow S^4$, then $\mathcal{E}(S^4,\ M) \cong \Gamma_V(2)$. Here we prove the following generalization of this result.

LEMMA 1. *Let $p \geq 3$ be odd and let $S^p \times S^p$ be standardly embedded in S^{2p+2}. Then $\mathrm{Im}(h_{\mathcal{E}}) = \Gamma_V(2)$.*

Proof. We will show that there exist diffeomorphisms $\varphi_V \in \mathrm{Diff}(S^{n+2},\ M^n)$ and $\varphi_T \in \mathrm{Diff}(S^{n+2},\ M^n)$ such that $h_{\mathcal{E}}[\varphi_V] = V$ and $h_{\mathcal{E}}[\varphi_T] = T$. Let us start with the matrix T and consider the following commutative diagram that consists of two exact homotopy sequences of the fibration $SO(n) \hookrightarrow SO(n+1) \twoheadrightarrow S^n$

$$\begin{tikzcd}
 & \pi_{p+1}(S^{p+1}) \arrow[d, "\partial"'] \arrow[dr, "\mu=\times 2"] & & \\
\pi_p(SO(p)) \arrow[r, "i_p"'] & \pi_p(SO(p+1)) \arrow[r, "j_p"'] \arrow[d, "i_{p+1}"'] & \pi_p(S^p) \arrow[r] & \pi_{p-1}(SO(p)) \\
 & \pi_p(SO(p+2)) & &
\end{tikzcd}$$

Since p is odd, the map $\mu := j_p \circ \partial$ is multiplication by 2 (see [2], Lemma IV.1.9). It follows from this diagram that there exists a smooth map $m_T : S^p \longrightarrow SO(p+1)$ such that $j_p([m_T]) = \pm 2$ and $i_{p+1}[m_T] = 0$. Hence there exists a smooth map (see the Smooth Approximation Theorem in §II.11 of [1]) $\gamma : D^{p+1} \longrightarrow SO(p+2)$ that extends the composition of m_T with the inclusion $\iota : SO(p+1) \hookrightarrow SO(p+2)$, i.e., $\gamma|_{S^p} = \iota \cdot m_T$. Now we use γ together with m_T to define the following self-diffeomorphism of $S^{2p+2} = (D^{p+1} \times S^{p+1}) \cup (S^p \times D^{p+2})$:

$$\Phi(x,y) := \begin{cases} (x, \iota \cdot m_T(x) \circ y) & \text{if } (x,y) \in S^p \times D^{p+2}, \\ (x, \gamma(x) \circ y) & \text{if } (x,y) \in D^{p+1} \times S^{p+1}. \end{cases}$$

Since the restriction of Φ onto the product $S^p \times S^p \hookrightarrow S^p \times S^{p+1}$ (here $x \times S^p$ is the equator of $x \times S^{p+1}$) is the map $(x,y) \longmapsto (x, m_T(x) \circ y)$ and $j_p([m_T]) = \pm 2$, we can define $\varphi_T := \Phi$ to obtain the equality $h_{\mathcal{E}}[\varphi_T] = \pm T$. The product $S^p \times S^p$ bounds $S^p \times D^{p+1}$ smoothly embedded in S^{2p+2}. Then it follows from Theorem 1.7 of [21] that the pair $(S^{2p+2},\ S^p \times S^p)$ is equivalent to the standard one, where $S^p \times S^p$ is standardly embedded in the equator S^{2p+1}.

For the other case we consider $S^p \times S^p$ standardly embedded into the unit sphere $S^{2p+2} \subset \mathbb{R}^{2p+3}$. Using $\{x_0, x_1, \dots, x_{2p+2}\}$ as the coordinates in $\mathbb{R}^{2p+3}$, we present the equator sphere $S^{2p+1} = \{(x_0, x_1, \dots, x_{2p+2}) \mid x_0 = 0 \;\&\; x_1^2 + \cdots + x_{2p+2}^2 = 1\}$ as the union $S^{2p+1} = \mathcal{H}_1 \cup -\mathcal{H}_2$, where

$$\mathcal{H}_1 = \left\{(x_1, x_2, \dots, x_{2p+2}) \in S^{2p+1} \mid x_1^2 + x_2^2 + \cdots + x_{p+1}^2 \le \frac{1}{2}\right\}$$

and

$$\mathcal{H}_2 = \left\{(x_1, x_2, \dots, x_{2p+2}) \in S^{2p+1} \mid x_1^2 + x_2^2 + \cdots + x_{p+1}^2 \ge \frac{1}{2}\right\}$$

and consider a linear map $\Omega : \mathbb{R}^{2p+3} \longrightarrow \mathbb{R}^{2p+3}$ defined by the square matrix

$$\Omega := \left(\begin{array}{c|cccc|cccc} (-1)^p & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \\ \hline 0 & -1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \end{array}\right)$$

of size $(2p+3) \times (2p+3)$ (where the three blocks in the lowest row have sizes $(p+1) \times 1$, $(p+1) \times (p+1)$, and $(p+1) \times (p+1)$, respectively). Ω acts on vectors of $\mathbb{R}^{2p+3}$ by multiplying a row vector by Ω on the right.

Since Ω is an orthogonal matrix which maps the unit disk D^{2p+3} onto itself preserving orientation ($\det(\Omega) = (-1)^{2p+2} = 1$), restriction of Ω onto the unit sphere S^{2p+2} is an orientation preserving diffeomorphism as well (see

§4.4 of [11]). If we denote a point of $S^p \times S^p$ by (a, b) with $a = (x_1, \dots, x_{p+1})$ and $b = (x_{p+2}, \dots, x_{2p+2})$, then the restriction of Ω onto $S^p \times S^p = \partial\mathcal{H}_i$ maps (a, b) to the point $(\mathrm{R}(b), a)$, where $\mathrm{R}(b)$ stands for the "*reflected*" point $(-x_{p+2}, x_{p+3}, \dots, x_{2p+2})$. Such a diffeomorphism of $S^p \times S^p$ induces in homology the automorphism of the form $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and therefore we can define a diffeomorphism φ_V as the restriction $\varphi_V := \Omega|_{S^{2p+2}}$. □

REMARK. Notice that Ω^4, as well as $\Omega^4|_{S^p \times S^p}$, is the identity map. We will use this fact later to show that certain short exact sequences split.

COROLLARY 1. *Let $2 \leq p < q$, and let $S^p \times S^q$ be standardly embedded in S^{p+q+2}. Then* $\mathrm{Im}(h_{\mathcal{E}}) \cong \mathbb{Z}_2$.

Proof. Indeed, it is again enough to consider the union $S^{p+q+1} = \mathcal{H}_1 \cup -\mathcal{H}_2$, where

$$\mathcal{H}_1 := \left\{ (x_1, x_2, \dots, x_{p+q+2}) \in S^{p+q+1} \mid x_1^2 + x_2^2 + \dots + x_{p+1}^2 \leq \frac{1}{2} \right\}$$

and $\mathcal{H}_2$ is defined analogously, and a linear map $\Omega' : \mathbb{R}^{p+q+3} \longrightarrow \mathbb{R}^{p+q+3}$ defined by the obvious formula

$$\Omega'(x_0, x_1, \dots, x_{p+q+1}, x_{p+q+2}) := \\ (x_0, -x_1, x_2 \dots, x_{p+1}, -x_{p+2}, x_{p+3} \dots, x_{p+q+2}).$$

The restriction of this map onto S^{p+q+2} is an orientation preserving diffeomorphism that restricts on $S^p \times S^q = \partial\mathcal{H}_i$ to a diffeomorphism defined by the formula $(a, b) \longmapsto (\mathrm{R}(a), \mathrm{R}(b))$, where $a \in S^p$ and $b \in S^q$. Such a diffeomorphism generates the group $\mathrm{Im}(h) \cong \mathbb{Z}_2$ (see Proposition 2.1 of [25]). □

COROLLARY 2. *Let $p > 2$ be even, and let $S^p \times S^p$ be standardly embedded in S^{2p+2}. Then* $\mathrm{Im}(h_{\mathcal{E}}) = \mathrm{Im}(h) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Proof. This time we consider the orthogonal $(2p+3) \times (2p+3)$-matrix

$$\widehat{\Omega} := \left(\begin{array}{c|cccc|cccc}
-1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \hline
0 & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 1 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 \\ \hline
0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0
\end{array} \right)$$

which has order two and $\det(\widehat{\Omega}) = 1$. The restriction of this linear map onto S^{2p+2} is an orientation preserving diffeomorphism, which extends the interchange of factors in $S^p \times S^p$ $\big((a,b) \longmapsto (b,a)$ for $(a,b) \in S^p \times S^p$, and we assume again that the equator is $S^{2p+1} = \mathcal{H}_1 \cup -\mathcal{H}_2$, and $S^p \times S^p = \partial\mathcal{H}_i\big)$. This interchange generates one of the $\mathbb{Z}_2$-copies of $\mathrm{Im}(h)$, and the other copy is generated, as above, by the simultaneous orientation reversal of each factor: $(a,b) \longmapsto (\mathrm{R}(a), \mathrm{R}(b))$ (cf. [29], §1; and [19], §1.2). □

3. Trivial action on the homology

Consider a "trivial" M^n-knot $(S^{n+2},\ M^n)$ with $n \geq 5$ ("triviality" here means that for an equator we have $S^{n+1} = W_+ \cup W_-$ with $M^n = \partial W_\pm$). Identification of the group Θ_{n+1} of all homotopy $(n+1)$-spheres with the relative mapping class group $\pi_0 \operatorname{Diff}(D^n, \mathrm{rel}\ \partial)$ (see [28]) gives a homomorphism $\iota : \Theta_{n+1} \longrightarrow \pi_0 \operatorname{Diff}(M^n)$ (if one presents a homotopy sphere Σ_φ as the union of two $(n+1)$-disks glued together via a diffeomorphism $\varphi \in \operatorname{Diff}(D^n, \mathrm{rel}\ \partial)$, then $\iota(\Sigma_\varphi)$ is defined to be the identity outside an embedded n-disk $D^n \hookrightarrow M^n$ and φ on that disk).

Theorem 1. *If a diffeomorphism $\phi \in \mathrm{Im}(\iota)$ extends to a diffeomorphism Φ of the trivial M^n-knot $(S^{n+2},\ M^n)$, then ϕ is pseudo-isotopic to the identity.*

Proof. Take two canonical pairs of the disks $(D_+^{n+3},\ D_+^{n+2})$ and $(D_-^{n+3},\ D_-^{n+2})$ with the boundary $(S^{n+2},\ S^{n+1})$. Push W_+ along the normal vector field inside D_+^{n+3} in such a way that we get the copy $\widetilde{W}_+$ of W_+ embedded in D_+^{n+3} so that $\widetilde{W}_+ \cap S^{n+2} = M^n$. We can assume w.l.o.g. that we have $W_+ \times I \subset D_+^{n+3}$ with $W_+ \times I \cap S^{n+2} = W_+ \times \{0\}$ and $\widetilde{W}_+ = M \times I \cup W_+ \times \{1\}$. Repeat this step with W_- to obtain $\widetilde{W}_- \subset D_-^{n+3}$. Then we glue together two pairs $(D_+^{n+3},\ \widetilde{W}_+)$ and $(D_-^{n+3},\ \widetilde{W}_-)$ via the diffeomorphism Φ to obtain a spherical knot

$$(S^{n+3},\ \Sigma_\phi^{n+1}) := (D_+^{n+3},\ \widetilde{W}_+) \bigcup_\Phi (D_-^{n+3},\ \widetilde{W}_-). \tag{1}$$

We want to show that the homotopy sphere Σ_ϕ bounds a topological disk inside S^{n+3}. Indeed, since $W_+ \cup W_- = S^{n+1}$ bounds a smooth disk D^{n+2} in S^{n+2}, it is clear that the union

$$\mathcal{D}_1^{n+2} := W_+ \times I \bigcup_{W_+ \times \{0\}} D^{n+2} \subset D_+^{n+3}$$

is homeomorphic to the $(n+2)$-disk with the boundary $\partial(\mathcal{D}_1^{n+2}) = \widetilde{W}_+ \bigcup_M W_-$. As a homeomorphism, the map ϕ is isotopic to the identity, which implies that the union $\widetilde{W}_- \cup \Phi(W_-)$ (obtained also as a result of gluing two pairs (1) together) is homeomorphic to the double $\mathcal{D}W_-$ (for a manifold X with boundary the double is defined as the boundary $\partial(X \times I) =: \mathcal{D}X$). Hence, we can assume that this union bounds in the "lower" hemisphere of S^{n+3} a manifold homeomorphic to the product $W_- \times I \subset D_-^{n+3}$, and that the intersection of this product with the equator sphere S^{n+2} is $\Phi(W_-)$. This implies that the union

$$\mathcal{D}_1^{n+2} \bigcup_{\Phi(W_-)} W_- \times I =: \mathcal{D}^{n+2}$$

is a topological disk in the sphere S^{n+3} with the boundary

$$\partial\mathcal{D}^{n+2} = \widetilde{W}_+ \bigcup_{\phi} \widetilde{W}_- = \Sigma_\phi.$$

Now it follows from Corollary 4.16 of [24] that the pair (S^{n+3}, Σ_ϕ^{n+1}) is combinatorially unknotted and therefore the complement $S^{n+3} \setminus \Sigma_\phi^{n+1}$ has the homotopy type of a circle. Using Theorem III from [14] (cf. also [18]) we conclude that the exotic sphere Σ_ϕ^{n+1} is diffeomorphic to the standard one. This implies that ϕ must be isotopic to the identity (cf. [25], §4). □

Corollary 3. *If $n \geq 5$ and (S^{n+2}, S^n) is the unknot, then $\mathcal{E}(S^{n+2},\ S^n) \cong \mathrm{Id}$.*

Remark. If the trivial M^n-knot is actually a spherical knot, i.e., $M^n = S^n$, then it is not hard to see geometrically that the complement $S^{n+3} \setminus \Sigma_\phi^{n+1}$ to the homotopy sphere Σ_ϕ^{n+1} constructed above is a disk bundle over a circle.

Let us continue now with the following example. Take the product $S^{p-2} \times S^{p-1}$ standardly embedded in $S^{p-1} \times S^{p-1}$ (the $(p-2)$-sphere is the equator of the first $(p-1)$-sphere), where $p \geq 9$ and $p \equiv 6 \pmod 8$, and again present the sphere $S^{2p-1} = (D^p \times S^{p-1}) \cup (S^{p-1} \times D^p)$ as the union of two handlebodies. Then the group of the isotopy classes of diffeomorphisms of $S^{p-2} \times S^{p-1}$ which act trivially on the homology is isomorphic to $\pi_{p-1}(SO(p-1)) \oplus \Theta_{2p-2}$ (see, for example, [26], Theorem 3.10). Since $p - 1 \equiv 5 \pmod 8$, we see from [13] that $\pi_{p-1}(SO(p-1)) \cong \mathbb{Z}_2$ and it follows also from [13] that the inclusion $SO(p-1) \overset{\tau}{\hookrightarrow} SO(p)$ induces the trivial homomorphism $\tau_* : \pi_{p-1}(SO(p-1)) \longrightarrow \pi_{p-1}(SO(p))$. Hence there exists a smooth map $\gamma : D^p \longrightarrow SO(p)$ which extends the composition of the generator $g : S^{p-1} \longrightarrow SO(p-1)$ of $\pi_{p-1}(SO(p-1))$ with τ, i.e., $\gamma|_{S^{p-1}} = \tau \circ g$. Now we repeat the construction we used to prove Lemma 1 and define a diffeomorphism Φ of the ambient sphere S^{2p-1} by the formula:

$$\Phi(x, y) := \begin{cases} (\tau(g(y)) \circ x,\ y) & \text{if } (x, y) \in D^p \times S^{p-1}, \\ (\gamma(y) \circ x,\ y) & \text{if } (x, y) \in S^{p-1} \times D^p. \end{cases}$$

The following lemma immediately follows from Corollary 1 and Theorem 1.

Lemma 2. *Let $S^{p-2} \times S^{p-1}$ be standardly embedded in S^{2p-1}, with $p \geq 9$ and $p \equiv 6 \pmod 8$. Then the following short exact sequence splits:*

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathcal{E}(S^{2p-1},\ S^{p-2} \times S^{p-1}) \xrightarrow{h_{\mathcal{E}}} \mathbb{Z}_2 \longrightarrow 0.$$

Next we generalize this example and study diffeomorphisms of $S^p \times S^p$ which extend over S^{2p+2} and act trivially on $H_p(S^p \times S^p; \mathbb{Z})$.

The isotopy classes of the diffeomorphisms which act trivially on $H_p(S^p \times S^p; \mathbb{Z})$ form a normal subgroup of the mapping class group and will be denoted by $\pi_0 \operatorname{SDiff}(S^p \times S^p)$. Theorem 2 of [15] gives the following description of such a subgroup for $M = S^p \times S^p,\ p \geq 3$:

$$(2) \quad 0 \longrightarrow \Theta_{2p+1} \xrightarrow{\iota} \pi_0 \operatorname{SDiff}(M) \xrightarrow{\chi} \operatorname{Hom}\big(H_p(M),\ S\pi_p(SO(p))\big) \longrightarrow 0.$$

Here $S\pi_p(SO(p))$ denotes the image of the map $S : \pi_p(SO(p)) \to \pi_p(SO(p+1))$ induced by the inclusion, and the homomorphism χ is defined (for $p \geq 4$) as follows (cf. Lemma 1 of [15]): Take any $f \in \operatorname{SDiff}(M)$, represent $x \in H_p(M)$ by a sphere $S^p \hookrightarrow M$ and use an isotopy to make $f|_{S^p} = \mathrm{Id}$. The stable normal bundle $\nu(S^p) \oplus \varepsilon^1$ of this sphere in M is trivial and therefore the differential of f gives an element of $\pi_p(SO(p+1))$. It is easy to see that this element lies in the image of S. If $p = 6$ we have $S\pi_p(SO(p)) = 0$, and for all other $p \geq 3$ the groups $S\pi_p(SO(p))$ are given in the following table ([15], p. 644):

$p \pmod 8$	0	1	2	3	4	5	6	7
$S\pi_p(SO(p))$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\mathbb{Z}$

When p is odd, the map $S\pi_p(SO(p)) \hookrightarrow \pi_p(SO(p+1)) \longrightarrow \pi_p(SO(p+2))$ is a monomorphism (see [30]) and therefore the induced map

$$\operatorname{Hom}(H_p(M),\ S\pi_p(SO(p))) \longrightarrow \operatorname{Hom}(H_p(M),\ \pi_p(SO))$$

is a monomorphism too. Hence we can describe $\chi([f])[S^p]$ as the class of the stable normal bundle of $S^p \times S^1$ in the mapping torus M_f of the diffeomorphism f (cf. Lemma 2 of [15] and recall that $f|_{S^p} = \mathrm{Id}$). Assuming that f extends to a diffeomorphism of S^{n+2}, which is isotopic to the identity, we see that $S^p \times S^1 \hookrightarrow M_f \hookrightarrow S^{2p+2} \times S^1$. Since the normal bundle of M_f in $S^{2p+2} \times S^1$ is trivial, and the stable normal bundle of $S^p \times S^1$ in $S^{2p+2} \times S^1$ is also trivial, this implies that the stable normal bundle of $S^p \times S^1$ in M_f is trivial as well. Thus the homomorphism χ, restricted onto the isotopy classes of diffeomorphisms of $S^p \times S^p$ which extend over S^{2p+2} and act trivially on the homology, is the trivial map for odd $p \geq 5$. This result together with Theorem 1 and the exact sequence (2) implies that in this case $\ker(h_{\mathcal{E}}) = \{0\}$. If $p = 3$, we have $S\pi_p(SO(p)) \simeq \mathbb{Z}$ and we can identify $\operatorname{Hom}(H_p(M),\ S\pi_p(SO(p)))$ with $H^p(M)$. Then one can use the Pontrjagin class of the mapping torus M_f

instead of χ (see [15]), to obtain the same conclusion that the kernel of the map $h_{\mathcal{E}}$ is also trivial when $p = 3$.

Let us now turn to the case when p is even and at least 4, and denote the kernel of $h_{\mathcal{E}}$ by

$$\ker(h_{\mathcal{E}}) := \mathcal{S}E(S^{2p+2},\ S^p \times S^p).$$

The following computations are well known (cf. [30]):

p (mod 8)	0	2	4	6
$S\pi_p(SO(p))$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\pi_p(SO(p+1))$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\pi_p(SO(p+2))$	$\mathbb{Z}_2$	0	0	0

It follows from these computations and the paragraph above that in this case the image of the restriction of χ onto the subgroup $\mathcal{S}E(S^{2p+2},\ S^p \times S^p)$ is either zero or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Lemma 3. *If p is even and $p \geq 4$, then $\mathcal{S}E(S^{2p+2},\ S^p \times S^p) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\mathcal{E}(S^{2p+2},\ S^p \times S^p) \cong \mathrm{D}_8 \oplus \mathbb{Z}_2$, where D_8 stands for the dihedral group of order 8.*

Proof. Indeed, as we just mentioned above, the restriction of χ takes values in $\mathrm{Hom}(H_p(M),\ \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and to prove that this restriction is an epimorphism, it is enough to construct a diffeomorphism of the pair $(S^{2p+2},\ S^p \times S^p)$ for each of the generators of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. We will use results from [25] to do that. Let $g : S^p \longrightarrow SO(p+1)$ be a smooth map which represents a nontrivial element of the group $\pi_p(SO(p+1))$ and such that $\tau \circ g$ is homotopic to the identity (τ here is the canonical inclusion $SO(p+1) \hookrightarrow SO(p+2)$). The homotopy class of this map g has order two for each even p, as we just saw in the table above. It then follows from Proposition 3.2 of [25] that the diffeomorphisms of $S^p \times S^p$ defined by the formulas

$$(x,\ y) \overset{\delta_1}{\longmapsto} (x,\ g(x) \circ y) \quad \text{and} \quad (x,\ y) \overset{\delta_2}{\longmapsto} (g(y) \circ x,\ y)$$

with $(x, y) \in S^p \times S^p$ are representatives of the generators of the group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since $\tau \circ g \sim \mathrm{Id}$, there exists a smooth map $\gamma : D^{p+1} \longrightarrow SO(p+2)$ such that $\gamma|_{\partial D^{p+1}} = \tau \circ g$ and we define an orientation preserving diffeomorphism of the sphere S^{2p+2} by the following formula (in the second case the formula should be modified in the obvious way):

$$\Phi(x, y) := \begin{cases} (x,\ \tau(g(x)) \circ y) & \text{if } (x, y) \in S^p \times D^{p+2}, \\ (x,\ \gamma(x) \circ y) & \text{if } (x, y) \in D^{p+1} \times S^{p+1}. \end{cases}$$

Thus we obtain the two required diffeomorphisms of the pair $(S^{2p+2},\ S^p \times S^p)$, and also the exactness of the following sequence:

$$0 \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow \mathcal{E}(S^{2p+2},\ S^p \times S^p) \xrightarrow{h_{\mathcal{E}}} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow 0.$$

Recall that the generators of $\mathrm{Im}(h_{\mathcal{E}})$ are the interchange of factors $u: (x,y) \longmapsto (y,x)$, and the orientation reversal $(x,y) \longmapsto (\mathrm{R}(x), \mathrm{R}(y))$. It is not hard to see that the orientation reversal acts trivially by conjugation on $\ker(h_{\mathcal{E}})$. Hence one of the $\mathbb{Z}_2$-factors of $\mathrm{Im}(h_{\mathcal{E}})$ splits off. As for the conjugation by u, we have

$$(x,\ y) \xmapsto{u} (y,\ x) \xmapsto{\delta_1} (y,\ g(y)\circ x) \xmapsto{u} (g(y)\circ x,\ y),$$

that is, $\delta_2 = u\circ\delta_1\circ u$. This implies that the factor group $\mathcal{E}(S^{2p+2},\ S^p\times S^p)/_{\mathbb{Z}_2}$ admits the following presentation: $\langle a,\ b,\ u \mid a^2 = b^2 = u^2 = e,\ ab = ba,\ au = ub\rangle$. We leave it as an exercise to show that this presentation gives the dihedral group of order eight. $\square$

Let us now summarize what has been proved and state the following theorem:

Theorem 2. *Let $S^p \times S^p$ be standardly embedded in S^{2p+2}, and $p \geq 3$. Then*

$$\mathcal{E}(S^{2p+2},\ S^p\times S^p) \cong \begin{cases} \mathrm{D}_8 \oplus \mathbb{Z}_2 & \text{if } p \text{ is even,} \\ \Gamma_V(2) & \text{if } p \text{ is odd.} \end{cases}$$

Remark 1. The group of pseudo-isotopy classes of diffeomorphisms of $S^2 \times S^2$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ (cf. [29], §2). Our proof of Corollary 2 implies that each of the generators has a representative extendable to a diffeomorphism of S^6.

Remark 2. It is interesting to note that the group $\mathcal{E}(S^{2p+2},\ S^p\times S^p)$ is isomorphic to $\Gamma_V(2)$ for all odd $p \geq 1$ and the "Hopf invariant one" cases (i.e., when $p = 1,\ 3$ or 7) play no special role here (cf. [19], §1.2, for example).

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