Illinois Journal of Mathematics
Volume 51, Number 3, Fall 2007, Pages 831–842
S 0019-2082

TWO-WEIGHT POINCARÉ INEQUALITIES FOR THE PROJECTION OPERATOR AND A-HARMONIC TENSORS ON RIEMANNIAN MANIFOLDS

GEJUN BAO, YUMING XING, AND CONGXIN WU

ABSTRACT. We prove both local and global two-weight Poincaré inequalities for A-harmonic tensors on Riemannian manifolds. We also prove an analogue for the projection operator.

1. Introduction and notation

In recent years the classical Poincaré inequality has been generalized to different versions in $\mathbf{R}^n$; see [1], [2], [5], [11]. In this paper, we shall prove two-weight Poincaré inequalities for A-harmonic tensors on Riemannian manifolds in $\mathbf{R}^n$, $n \geq 2$. Our results can be considered as generalizations of the classical inequality and be used to study the integrability of A-harmonic tensors and the properties of related operators.

In this paper we always assume that M is a Riemannian, compact, oriented and C^∞ smooth manifold without boundary on $\mathbf{R}^n$ and Ω is an open subset of $\mathbf{R}^n$. Balls are denoted by B and σB is the ball with the same center as B and with $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$. For $l = 0, 1, \ldots, n$, we use $\wedge^l = \wedge^l(\mathbf{R}^n)$ to denote the linear space of l-vectors, generated by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}$, corresponding to all ordered l-tuples $I = (i_1, i_2, \ldots, i_l)$, $1 \leq i_1 < i_2 < \cdots < i_l \leq n$. Let $0 < p < \infty$, $0 < \alpha < \infty$. We denote the weighted L^p-norm of a measurable function f over E by

$$\|f\|_{p,E,w^\alpha} = \left(\int_E |f(x)|^p w^\alpha dx \right)^{1/p} \tag{1.1}$$

if the integral exists. Let $\wedge^l M$ be the l-th exterior power of the cotangent bundle and $C^\infty(M, \wedge^l)$ be the space of smooth l-forms on M. We also use $D'(M, \wedge^l)$ to denote the space of all differential l-forms and $L^p(\wedge^l M)$ to denote

Received October 3, 2005; received in final form September 15, 2006.
2000 *Mathematics Subject Classification.* Primary 31C45. Secondary 35J60, 58A10.
Research supported by the NSF of China (#10671046, #10771044).

the l-forms

$$\omega(x)=\sum_I \omega_I dx_I=\sum \omega_{i_1 i_2 \dots i_l}(x)dx_{i_1}\wedge dx_{i_2}\wedge\cdots\wedge dx_{i_l}$$

on M satisfying $\int_M |\omega_I|^p<\infty$ for all ordered l-tuples. Thus, $L^p(\wedge^l M)$ is a Banach space with norm

$$\|\omega\|_{p,M}=\left(\int_M |\omega(x)|^p dx\right)^{1/p}=\left(\int_M\left(\sum_I|\omega_I(x)|^2\right)^{p/2}dx\right)^{1/p}. \tag{1.2}$$

The Hodge star operator $\star:\wedge\to\wedge$ is defined by the rule $\star 1=e_1\wedge e_2\wedge\cdots\wedge e_n$ and $\alpha\wedge\star\beta=\beta\wedge\star\alpha=\langle\alpha,\beta\rangle(\star 1)$ for any $\alpha,\beta\in\wedge$. The codifferential operator $d^\star: D'(M,\wedge^{l+1})\to D'(M,\wedge^l)$ is given by $d^\star=(-1)^{nl+1}\star d\star$ on $D'(M,\wedge^{l+1})$, $l=0,1,\dots,n$, and the Laplace-Beltrami operator $\triangle$ is defined by $\triangle=dd^\star+d^\star d$.

During the last decade, many interesting results have been established in the study of the A-harmonic equation

$$d^\star A(x,d\omega)=0, \tag{1.3}$$

where $A: M\times\wedge^l(\mathbf{R}^n)\to\wedge^l(\mathbf{R}^n)$ satisfies the conditions

$$|A(x,\xi)|\le a|\xi|^{p-1} \text{ and } \langle A(x,\xi),\xi\rangle\ge|\xi|^p \tag{1.4}$$

for almost every $x\in M$ and all $\xi\in\wedge^l(\mathbf{R}^n)$. Here $a>0$ is a constant and $1<p<\infty$ is a fixed exponent associated with (1.3). We call u an A-harmonic tensor on a manifold M if u satisfies the A-harmonic equation (1.3) on M.

The following result appears in [7]: Let $D\subset\mathbf{R}^n$ be a bounded, convex domain. To each $y\in D$ there corresponds a linear operator $K_y: C^\infty(D,\wedge^l)\to C^\infty(D,\wedge^{l-1})$ defined by

$$(K_y\omega)(x;\xi_1,\xi_2,\dots,\xi_l)=\int_0^1 t^{l-1}\omega(tx+y-ty;x-y,\xi_1,\xi_2,\dots,\xi_{l-1})dt$$

and the decomposition $\omega=d(K_y\omega)+K_y(d\omega)$ holds at any y $\in D$. A homotopy operator $T: C^\infty(D,\wedge^l)\to C^\infty(D,\wedge^{l-1})$ is defined by averaging K_y over all points y in D,

$$T\omega=\int_D\varphi(y)K_y\omega dy, \tag{1.5}$$

where $\varphi\in C_0^\infty(D)$ is normalized by $\int_D\varphi(y)dy=1$. Then we have the decomposition

$$\omega=d(T\omega)+T(d\omega), \tag{1.6}$$

and the norm of the homotopy operator can be estimated by

$$\|T\omega\|_{S,D}\le C\operatorname{diam}(D)\|\omega\|_{S,D}. \tag{1.7}$$

For all $\omega \in L^p(D, \wedge^l)$, $1 \le p < \infty$, we define the l-form $\omega_D \in D'(D, \wedge^l)$ by

$$\omega_D = |D|^{-1} \int_D \omega(y)dy, \ l = 0, \ \text{and } \omega_D = d(T\omega), \ l = 1, 2, \dots, n. \tag{1.8}$$

2. The local inequalities

In this section we prove local two-weighted Poincaré inequalities for A-harmonic tensors on Riemannian manifolds. We need the following definition and lemmas.

Definition 2.1. We say a pair of weights $(w_1(x), w_2(x))$ satisfies the $A_{r,\lambda}(\Omega)$-condition in a set $\Omega \subset \mathbf{R}^n$, and we write $(w_1(x), w_2(x)) \in A_{r,\lambda}(\Omega)$, for some $\lambda \ge 1$ and $1 < r < \infty$ with $1/r + 1/r' = 1$, if

$$\sup_{B \subset \Omega} \left(\frac{1}{|B|} \int_B w_1^\lambda dx \right)^{1/\lambda r} \left(\frac{1}{|B|} \int_B \left(\frac{1}{w_2} \right)^{\lambda r'/r} dx \right)^{1/\lambda r'} < \infty.$$

The class of $A_{r,\lambda}(\Omega)$-weights (or the two-weights) appears in [8]. See [6] for more applications of the two-weights. We need the following generalized Hölder inequality.

Lemma 2.1. *Let $0 < \alpha < \infty$, $0 < \beta < \infty$ and $1/s = 1/\alpha + 1/\beta$. If f and g are measurable functions in $\mathbf{R}^n$, then $\|f \cdot g\|_{s,\Omega} \le \|f\|_{\alpha,\Omega} \cdot \|g\|_{\beta,\Omega}$ for any $\Omega \subset \mathbf{R}^n$.*

The following weak reverse Hölder inequality appears in [9].

Lemma 2.2. *Let u be an A-harmonic tensor on a manifold M, $\rho > 1$, and $0 < s, t < \infty$. Then there exists a constant C, independent of u, such that $\|u\|_{s,B} \le C|B|^{(t-s)/ts} \|u\|_{t,\rho B}$ for all balls or cubes B with $\rho B \subset M$.*

We also need the following lemma from [6].

Lemma 2.3. *Let $w \in A_r$. Then exist constants $\beta > 1$ and C independent of w, such that $\|w\|_{\beta,B} \le C|B|^{(1-\beta)/\beta} \|w\|_{1,B}$ for all balls $B \subset \mathbf{R}^n$.*

The following lemma appears in [8].

Lemma 2.4. $(w_1(x), w_2(x)) \in A_r(\Omega)$ *iff*

$$\left(\frac{1}{|B|} \int_B w_1 dx \right) \left(\frac{1}{|B|} \int_B \left(\frac{1}{w_2} \right)^{r'/r} dx \right)^{r/r'} \le C$$

for any ball $B \subset \Omega$.

LEMMA 2.5. *Let $u \in D'\left(B, \wedge^l\right)$ and $du \in L^p\left(B, \wedge^{l+1}\right)$. Then $u - u_B$ is in $W_p^1\left(B, \wedge^l\right)$ with $1 < p < \infty$ and $\|u - u_B\|_{p,B} \le C\left(n,p\right)|B|^{1/n}\|du\|_{p,B}$ for B a ball or a cube in $\mathbf{R}^n$, $l = 0, 1, 2, \ldots, n$.*

Now we prove a local two-weight Poincaré inequality for A-harmonic tensors on Riemannian manifolds. This will be used to prove Theorem 3.2 in the next section.

THEOREM 2.6. *Let $u \in D'\left(M, \wedge^l\right)$ be an A-harmonic tensor on a manifold M and $du \in L^s\left(M, \wedge^{l+1}\right)$, $l = 0, 1, 2, \ldots, n$, $1 + \left(\alpha\left(r-1\right)\right)/\lambda < s < \infty$. Assume that $\rho > 1$ and $\left(w_1\left(x\right), w_2\left(x\right)\right) \in A_{r,\lambda}\left(\Omega\right)$ for some $\lambda \ge 1$ and $1 < r < \infty$. Then there exists a constant C, independent of u, such that*

$$\|u - u_B\|_{s,B,w_1^\alpha} \le C|B|^{1/n}\|du\|_{s,\rho B,w_2^\alpha} \tag{2.1}$$

for all balls B with $\rho B \subset \Omega$ and any real number α with $0 < \alpha \le \lambda$.

Proof. First, we prove that (2.1) is true for $0 < \alpha < \lambda$. Choose $t = \lambda s/(\lambda - \alpha)$. Then $1 < s < t$. Using Hölder's inequality, we find that

$$\begin{aligned}\|u - u_B\|_{s,B,w_1^\alpha} &\le \left(\int_B |u - u_B|^t dx\right)^{1/t}\left(\int_B w_1^{\alpha/s\cdot st/(t-s)}dx\right)^{(t-s)/st} \\ &= \|u - u_B\|_{t,B}\left(\int_B w_1^\lambda dx\right)^{\alpha/s\lambda}.\end{aligned} \tag{2.2}$$

Note that u_B is a closed form and u is a solution of (1.3). Therefore $u - u_B$ is still a solution of (1.3). Taking $m = \lambda s/\left(\lambda + \alpha\left(r-1\right)\right)$ we have $m < s < t$. Using Lemma 2.2 and Lemma 2.5, we obtain

$$\begin{aligned}\|u - u_B\|_{t,B} &\le C_1|B|^{(m-t)/mt}\|u - u_B\|_{m,\rho B} \\ &\le C_2|B|^{(m-t)/mt}|B|^{1/n}\|du\|_{m,\rho B}\end{aligned} \tag{2.3}$$

for all balls B with $\rho B \subset \Omega$. Since $1/m = 1/s + \left(s-m\right)/sm$, by Hölder's inequality again, we obtain

$$\begin{aligned}\|du\|_{m,\rho B} &= \left(\int_{\rho B}\left(|du|w_2^{\alpha/s}w_2^{-\alpha/s}\right)^m dx\right)^{1/m} \\ &\le \left(\int_{\rho B}|du|w_2^\alpha dx\right)^{1/s}\left(\int_{\rho B}\left(\frac{1}{w_2}\right)^{\lambda/(r-1)}dx\right)^{\alpha(r-1)/s\lambda}\end{aligned} \tag{2.4}$$

for all balls B with $\rho B \subset \Omega$. From (2.2), (2.3), and (2.4) we get

(2.5)

$$\|u - u_B\|_{s,B,w_1^\alpha} \le C_2 |B|^{(m-t)/mt} |B|^{1/n} \left(\int_B w_1^\lambda dx \right)^{\alpha/s\lambda} \times \left(\int_{\rho B} |du| w_2^\alpha du \right)^{1/s} \left(\int_{\rho B} \left(\frac{1}{w_2} \right)^{\lambda/(r-1)} dx \right)^{\alpha(r-1)/s\lambda}.$$

Since $(w_1(x), w_2(x)) \in A_{r,\lambda}(\Omega)$, we have

$$\left(\int_B w_1^\lambda dx \right)^{\alpha/s\lambda} \left(\int_{\rho B} \left(\frac{1}{w_2} \right)^{\lambda/(r-1)} dx \right)^{\alpha(r-1)/s\lambda} \le C_3 |B|^{\alpha r/\lambda s}. \tag{2.6}$$

Substituting (2.6) into (2.5) and using $(m-t)/mt = -\alpha r/\lambda s$, we obtain

$$\begin{aligned} \|u - u_B\|_{s,B,w_1^\alpha} &\le C_4 |B|^{(m-t)/mt} |B|^{1/n} |B|^{\alpha r/\lambda s} \left(\int_{\rho B} |du|^s w_2^\alpha dx \right)^{1/s} \\ &\le C_4 |B|^{1/n} \left(\int_{\rho B} |du|^s w_2^\alpha dx \right)^{1/s}, \end{aligned}$$

which is equivalent to (2.1).

By Lemma 2.4, we have $\left(w_1^\lambda(x), w_2^\lambda(x)\right) \in A_r(\Omega)$.

We now consider the case $\lambda = \alpha$. By Lemma 2.3, there exist constants $\beta > 1$ and $C_5 > 0$ such that

$$\|w_1^\lambda\|_{\beta,B} \le C_5 |B|^{(1-\beta)/\beta} \|w_1^\lambda\|_{1,B} \tag{2.7}$$

for any cube or any ball $B \subset \mathbf{R}^n$. Choose $t = s\beta/(\beta-1)$. Then $1 < s < t$ and $\beta = t/(t-s)$. Since $1/s = 1/t + (t-s)/ts$, by Lemma 2.1 and (2.7), we have

$$\begin{aligned} \left(\int_B |u - u_B|^s w_1^\lambda dx \right)^{1/s} &\le \left(\int_B |u - u_B|^t dt \right)^{1/t} \left(\int_B \left(w_1^{\lambda/s} \right)^{ts/(t-s)} dx \right)^{(t-s)/ts} \\ &\le C_5 \|u - u_B\|_{t,B} \cdot |B|^{(1-\beta)/\beta s} \|w_1^\lambda\|_{1,B}^{1/s}. \end{aligned} \tag{2.8}$$

Now, choosing $m = s/r$, we have $m < s$. By Lemma 2.2 and Lemma 2.5, we have

$$\begin{aligned} \|u - u_B\|_{t,B} &\le C_6 |B|^{(m-t)/mt} \|u - u_B\|_{m,\rho B} \\ &\le C_7 |B|^{(m-t)/mt} |B|^{1/n} \|du\|_{m,\rho B}. \end{aligned} \tag{2.9}$$

Applying Hölder's inequality again, we obtain

$$\|du\|_{m,\rho B} = \left(\int_{\rho B} \left(|du| w_2^{\lambda/s} w_2^{-\lambda/s}\right)^s dx\right)^{1/m} \tag{2.10}$$
$$\leq \left(\int_{\rho B} |du|^s w_2^{\lambda} dx\right)^{1/s} \left(\int_{\rho B} \left(\frac{1}{w_2}\right)^{\lambda r'/r} dx\right)^{r/r's}.$$

Combining (2.9) and (2.10) yields

$$\|u - u_B\|_{t,B} \leq C_8 |B|^{\frac{m-t}{mt}} |B|^{1/n} \left(\int_{\rho B} |du|^s w_2^{\lambda} dx\right)^{1/s} \tag{2.11}$$
$$\times \left(\int_{\rho B} \left(\frac{1}{w_2}\right)^{\lambda r'/r} dx\right)^{r/r's}.$$

Substituting (2.11) into (2.8), we find that

$$\left(\int_B |u - u_B|^s w_1^{\lambda} dx\right)^{1/s} \tag{2.12}$$
$$\leq C_9 |B|^{(1-\beta)/\beta s} |B|^{(m-t)/mt} |B|^{1/n} \left(\int_B w_1^{\lambda} dx\right)^{1/s}$$
$$\times \left(\int_{\rho B} |du|^s w_2^{\lambda} dx\right)^{1/s} \left(\int_{\rho B} \left(\frac{1}{w_2}\right)^{\lambda r'/r} dx\right)^{r/r's}.$$

Using the condition $(w_1(x), w_2(x)) \in A_{r,\lambda}(\Omega)$, we obtain

$$\left(\int_B w_1^{\lambda} dx\right)^{1/s} \left(\int_{\rho B} \left(\frac{1}{w_2}\right)^{\lambda r'/r} dx\right)^{r/r's} \leq C_{10} |B|^{r/s}. \tag{2.13}$$

Combining (2.12) and (2.13), we conclude that

$$\left(\int_B |u - u_B|^s w_1^{\lambda} dx\right)^{1/s} \leq C_{11} |B|^{(1-\beta)/\beta s} |B|^{(m-t)/mt} |B|^{1/n} |B|^{r/s}$$
$$\times \left(\int_{\rho B} |du|^s w_2^{\lambda} dx\right)^{1/s}$$
$$= C_{11} |B|^{1/n} \left(\int_{\rho B} |du|^s w_2^{\lambda} dx\right)^{1/s}.$$

This ends the proof of Theorem 2.6. □

The following lemma appears in [4].

LEMMA 2.7. *Assume that u is an A-harmonic tensor on a manifold M, $\sigma > 1$ and $0 < s,t < \infty$, $l = 1, 2, \ldots, n$, $1 < s < \infty$. Then there exists a constant C, independent of u, such that*

$$\|du\|_{s,B} \le C|B|^{(t-s)/ts}\|du\|_{t,\rho B} \tag{2.14}$$

for all balls with $\rho B \subset M$.

Similar to the proof of Theorem 2.6, but using (1.7) and Lemma 2.7 instead of Lemma 2.2, we obtain the following norm inequality for the homotopy operator. We omit the dails.

THEOREM 2.8. *Let $u \in D'(M, \wedge^l)$ be an A-harmonic tensor on M, $du \in L^s_{loc}(M, \wedge^{l+1})$, $l = 1, 2, \ldots, n$, $1 < s < \infty$, and T be the homotopy operator defined in (1.5). Assume that $\rho > 1$ and $(w_1(x), w_2(x)) \in A_{r,\lambda}(M)$ for some $\lambda \ge 1$ and $1 < r < \infty$. Then there exists a constant C, independent of u, such that*

$$\|T(du)\|_{s,B,w_1^\alpha} \le C \operatorname{diam}(B) \|du\|_{s,\rho B,w_2^\alpha} \tag{2.15}$$

for any real number α with $0 < \alpha \le \lambda$.

Note that Theorems 2.6 and 2.8 contain two weights, $w_1(x)$ and $w_2(x)$, and two parameters, λ and α. These features make the Poincaré inequalities more flexible and more useful.

3. Inequalities for the projection operator

We say that $u \in L^1_{\mathrm{loc}}(\wedge^l M)$ has a generalized gradient if, for each coordinate system, the pullbacks of the coordinate function of u have a generalized gradient in the familiar sense. We write

$$\mathcal{W}(\wedge^l M) = \{u \in L^1_{\mathrm{loc}}(\wedge^l M) : u \text{ has a generalized gradient}\}$$

and define the harmonic l-field by

$$\mathcal{H} = \mathcal{H}(\wedge^l M) = \{u \in \mathcal{W}(\wedge^l M) : du = d^\star u = 0, u \in L^p \text{ for some } 1 < p < \infty\}.$$

We always use G to denote Green's operator and H to denote the harmonic projection operator acting on differential forms on manifolds. From [10, Chapter 6] we know that the projection operator, Green's operator, and the Laplace-Beltrami operator satisfy Poisson's equation

$$H(u) = u - \triangle G(u). \tag{3.1}$$

THEOREM 3.1. *Let $u \in D'(M, \wedge^l)$, $l = 1, 2, \ldots, n$, be an A-harmonic tensor on a manifold M. Assume that $\rho > 1$, $1 < s < \infty$, and $(w_1(x), w_2(x)) \in A_{r,\lambda}(M)$ for some $\lambda \ge 1$ and $1 < r < \infty$. Then there exists a constant C, independent of u, such that*

$$\|\triangle(G(du))\|_{s,B,w_1^\alpha} \le C\|du\|_{s,\rho B,w_2^\alpha} \tag{3.2}$$

for any ball $B \subset M$ and any real number α with $0 < \alpha \leq \lambda$.

Proof. From [3], we know that for any smooth l-form ω,

$$\|\triangle G(du)\|_{s,B} \leq C_1 \|du\|_{s,B}. \tag{3.3}$$

We first show that (3.2) holds for $0 < \alpha < \lambda$. Let $t = \lambda s/(\lambda - \alpha)$. Using Lemma 2.1 and (3.3), we have

$$\begin{aligned} \|\triangle G(du)\|_{s,B,w_1^\alpha} &\leq \left(\int_B |\triangle G(du)|^t dx\right)^{1/t} \left(\int_B w_1^{\alpha/s \cdot ts/(t-s)} dx\right)^{(t-s)/ts} \\ &\leq C_1 \|du\|_{t,B} \left(\int_B w_1^\lambda dx\right)^{\alpha/\lambda s}. \end{aligned} \tag{3.4}$$

Let $m = \lambda s/(\lambda + \alpha(r-1))$. Then $m < s$. Applying Lemma 2.6 yields

$$\|du\|_{t,B} \leq C_2 |B|^{m-t)/mt} \|du\|_{m,\rho B}. \tag{3.5}$$

Substituting (3.5) into (3.4), we have

$$\|\triangle G(du)\|_{s,B,w_1^\alpha} \leq C_3 |B|^{(m-t)/mt} \left(\int_B w_1^\lambda dx\right)^{\alpha/\lambda s}. \tag{3.6}$$

Using Lemma 2.1 again with $1/m = 1/s + (s-m)/sm$, we obtain

$$\begin{aligned} \|du\|_{m,\rho B} &\leq \left(\int_{\rho B} |du|^s w_2^\alpha dx\right)^{1/s} \left(\int_{\rho B} \left(\frac{1}{w_2}\right)^{\alpha m/(s-m)} dx\right)^{(s-m)/sm} \\ &= \|du\|_{s,\rho B,w_2^\alpha} \left(\int_{\rho B} \left(\frac{1}{w_2}\right)^{\lambda/(r-1)} dx\right)^{\alpha(r-1)/s\lambda} \end{aligned} \tag{3.7}$$

for all balls B with $\rho B \subset M$. Substituting (3.7) into (3.6) gives

$$\begin{aligned} \|\triangle G(du)\|_{s,B,w_1^\alpha} \leq C_3 |B|^{(m-t)/mt} &\left(\int_B w_1^\lambda dx\right)^{\alpha/\lambda s} \\ &\times \left(\int_{\rho B} \left(\frac{1}{w_2}\right)^{\lambda/(r-1)} dx\right)^{\alpha(r-1)/s\lambda} \|du\|_{s,\rho B,w_2^\alpha}. \end{aligned} \tag{3.8}$$

Since $(w_1(x), w_2(x)) \in A_{r,\lambda}(M)$,

$$\left(\int_B w_1^\lambda dx\right)^{\alpha/\lambda s} \left(\int_{\rho B} \left(\frac{1}{w_2}\right)^{\lambda/(r-1)} dx\right)^{\alpha(r-1)/s\lambda} \leq C_4 |B|^{\alpha r/\lambda s}. \tag{3.9}$$

Combining (3.9) and (3.8), we conclude that

$$\|\triangle G(du)\|_{s,B,w_1^\alpha} \leq C_5 \|du\|_{s,\rho B,w_2^\alpha} \tag{3.10}$$

for all balls B with $\rho B \subset M$. We have proved that (3.2) is true if $0 < \alpha < \lambda$.

Similar to the proof of Theorem 2.6, we can prove that (3.2) is also true for $\alpha = \lambda$. That is, we have

$$\|\triangle(G(du))\|_{s,B,w_1^\lambda} \le C_{13}\|du\|_{s,\rho B,w_2^\lambda}$$

for all balls B with $\rho B \subset M$. The proof of Theorem 3.1 is complete. □

From [3], we know that d commutes with G or $\triangle$. Thus we have

$$d(\triangle(G(u)) = (\triangle d(G(u)) = \triangle(G(du)). \tag{3.11}$$

Then from (1.6), (1.7), (1.8), (3.11) and (3.3) we obtain

$$\begin{aligned} \|\triangle(G(u)) - (\triangle(G(u))_B\|_{s,B} &= \|\triangle(G(u)) - d(T(\triangle(G(u)))\|_{s,B} \\ &= \|T(d(\triangle(G(u)))\|_{s,B} \\ &= \|T(\triangle(G(du)))\|_{s,B} \\ &\le C_{14}\,\mathrm{diam}(B)\|\triangle(G(du))\|_{s,B} \\ &\le C_{15}\,\mathrm{diam}(B)\|du\|_{s,B}. \end{aligned} \tag{3.12}$$

Using the same method as in the proof of Theorem 3.1, and Lemma 2.1 and (3.12), we obtain the Poincaré inequality for the composition of $\triangle$ and G.

LEMMA 3.2. *Let $u \in D'(M, \wedge^l)$, $l = 1, 2, \dots, n$, be an A-harmonic tensor on a manifold M. Assume that $\rho > 1$, $1 < s < \infty$, and $(w_1(x), w_2(x)) \in A_{r,\lambda}(M)$ for some $\lambda > 1$ and $1 < r < \infty$. Then there exists C, independent of u, such that*

$$\|\triangle(G(u)) - (\triangle(G(u)))_B\|_{s,B,w_1^\alpha} \le C\,\mathrm{diam}(B)\|du\|_{s,\rho B,w_2^\alpha} \tag{3.13}$$

for all balls $B \subset M$ with $\rho B \in M$ and any real number α with $0 < \alpha \le \lambda$.

Now we are ready to prove the following local Poincaré inequality for the projection operator applied to A-harmonic tensors on manifolds.

THEOREM 3.3. *Let $u \in D'(M, \wedge^l)$ be an A-harmonic tensors on a manifolds M, $du \in L^s(M, \wedge^{l+1})$, $l = 0, 1, \dots, n$, and let H be the projection operator. Assume that $(w_1(x), w_2(x)) \in A_{r,\lambda}(M)$ for some $\lambda \ge 1$, $1 < r < \infty$, and $1 + (\alpha(r-1))/\lambda < s < \infty$. Then there exists C, independent of u, such that*

$$\|H(u) - (H(u))_B\|_{s,B,w_1^\alpha} \le C|B|^{1/n}\|du\|_{s,\rho B,w_2^\alpha} \tag{3.14}$$

for all balls $B \subset M$ with $\rho B \subset M$, $0 < \alpha \le \lambda$.

Proof. Applying Theorem 2.6 and Lemma 3.2 and using (3.2), we have

$$\begin{aligned}\|H(u)-(H(u))_B\|_{s,B,w_1^\alpha} &= \|u-\triangle(G(u))-(u-\triangle(G(u)))_B\|_{s,B,w_1^\alpha}\\ &= \|u-\triangle(G(u))-(u_B-\triangle(G(u)))_B\|_{s,B,w_1^\alpha}\\ &\le \|u-u_B\|_{s,B,w_1^\alpha}+\|\triangle(G(u))-(\triangle(G(u)))_B\|_{s,B,w_1^\alpha}\\ &\le C_1|B|^{1/n}\|du\|_{s,\rho B,w_2^\alpha}.\end{aligned}$$

The proof of Theorem 3.3 has been completed. □

4. The global inequalities

In this section, we extend our main local results to global ones. For this purpose, we need the following covering lemma appearing in [9].

LEMMA 4.1. *Each Ω has a modified Whitney cover of cubes $V=\{Q_i\}$ such that $\bigcup_i Q_i=\Omega$ and $\sum_{Q\in V}\chi_{\sqrt{5/4}Q}(x)\le N\chi_\Omega(x)$ for all $x\in\mathbf{R}^n$ and some $N>1$, where χ_E is the characteristic function for a set E.*

THEOREM 4.2. *Let $u\in L^s(\wedge^l M)$, $l=1,2,\dots,n$, $1<s<\infty$, be an A-harmonic tensors on a manifolds M and T be a homotopy operator defined by (1.5). Assume that $(w_1(x),w_2(x))\in A_{r,\lambda}(M)$ for some $1\le\lambda$ and $1<r<\infty$. Then there exists a constants C, independent of u, such that*

$$\|T(du)\|_{s,M,w_1^\alpha}\le C\,\mathrm{diam}(M)\|du\|_{s,M,w_2^\alpha} \tag{4.1}$$

for any real number α with $0<\alpha\le\lambda$.

Proof. Since M is compact, there exists a finite coordinate chart cover $\{U_1,U_2,\dots,U_m\}$ of M such that $\bigcup_{k=1}^m U_k=M$. We can equip M with a topology in unique way such that each U_k is open, $k=1,\dots,m$. Applying Lemma 4.1 to U_k (note that $\bigcup_{B\in V}B=U_k$ now) and Theorem 2.8, we find that

$$\begin{aligned}\|T(du)\|_{s,U_k,w_1^\alpha} &\le \sum_{B\in V}\|T(du)\|_{s,B,w_1^\alpha}\\ &\le \sum_{B\in V}\left(C_1\,\mathrm{diam}(B)\|du\|_{s,\rho B,w_2^\alpha}\right)\chi_{\sqrt{5/4}\rho B}(x)\\ &\le C_2\,\mathrm{diam}(U_k)\|du\|_{s,U_k,w_2^\alpha}\sum_{B\in V}\chi_{\sqrt{5/4}\rho B}(x)\\ &\le C_k\,\mathrm{diam}(M)\|du\|_{s,M,w_2^\alpha}.\end{aligned} \tag{4.2}$$

Now, from (4.2), we obtain

$$\begin{aligned} \|T(du)\|_{s,M,w_1^\alpha} &\leq \sum_{k=1}^{m} \|T(du)\|_{s,U_k,w_1^\alpha} \\ &\leq \sum_{k=1}^{m} C_k \operatorname{diam}(M)\|du\|_{s,M,w_2^\alpha} \\ &\leq C \operatorname{diam}(M)\|du\|_{s,M,w_2^\alpha}. \end{aligned} \tag{4.3}$$

This ends the proof of Theorem 4.2. □

Now we are ready to prove a global two-weight Poincaré inequality for A-harmonic tensors on manifolds.

THEOREM 4.3. *Let $u \in D'(M, \wedge^l)$ be an A-harmonic tensors on a manifold M and $du \in L^s(M, \wedge^{l+1})$, $l = 0, 1, \ldots, n$. Assume that $0 < \alpha \leq \lambda$, $1 + \alpha(r-1)/\lambda < s < \infty$ and $(w_1(x), w_2(x)) \in A_{r,\lambda}(M)$ for some $\lambda \geq 1$ and $1 < r < \infty$. Then*

$$\|u - u_M\|_{s,M,w_1^\alpha} \leq C \operatorname{diam}(M)\|du\|_{s,M,w_2^\alpha}. \tag{4.4}$$

Here C is a constant independent of u.

Proof. Using (1.6), (1.8) and Theorem 4.2, we have

$$\begin{aligned} \|u - u_M\|_{s,M,w_1^\alpha} &= \|u - d(Tu)\|_{s,M,w_1^\alpha} \\ &= \|T(du)\|_{s,M,w_1^\alpha} \\ &\leq C_1 \operatorname{diam}(M)\|du\|_{s,M,w_2^\alpha}. \end{aligned}$$

The proof of Theorem 4.3 has been completed. □

Finally, we prove the following global two-weight Poincaré inequality for the projection operator applied to differential forms on manifolds.

THEOREM 4.4. *Let $u \in D'(M, \wedge^l)$ be an A-harmonic tensors on a manifold M, $du \in L^s(M, \wedge^{l+1})$, $l = 0, 1, \ldots, n$, and let H be the projection operator. Assume that $0 < \alpha \leq \lambda$, $1 + (\alpha(r-1)\lambda) < s < \infty$ and $(w_1(x), w_2(x)) \in A_{r,\lambda}(M)$ for some $\lambda \geq 1$ and $1 < r < \infty$. Then*

$$\|H(u) - (H(u))_M\|_{s,M,w_1^\alpha} \leq C \operatorname{diam}(M)\|du\|_{s,M,w_2^\alpha}. \tag{4.5}$$

Here C is a constant independent of u.

Proof. Applying Lemma 3.1 and the same method as in the proof of Theorem 4.2, we obtain

$$\|\triangle G(u) - (\triangle G(u))_M\|_{s,M,w_1^\alpha} \leq C_1 \operatorname{diam}(M)\|du\|_{s,M,w_2^\alpha}. \tag{4.6}$$

Using (3.1), (4.4) and (4.6), we conclude that

$$\begin{aligned}\|H(u)-(H(u))_M\|_{s,M,w_1^\alpha} &= \|(u-\triangle G(u))-(u-\triangle G(u))_M\|_{s,M,w_1^\alpha}\\ &\le \|u-u_M\|_{s,M,w_1^\alpha}+\|\triangle G(u)-(\triangle G(u))_M\|_{s,M,w_1^\alpha}\\ &\le C_3\,\mathrm{diam}(M)\|du\|_{s,M,w_2^\alpha}.\end{aligned}$$

The proof of Theorem 4.4 has been completed. □

References

[1] G. Bao, *$A_r(\lambda)$-weighted integral inequalities for A-harmonic tensors*, J. Math. Anal. Appl. **247** (2000), 466–477. MR 1769089 (2001g:58001)

[2] S. Ding, *Weighted Hardy-Littlewood inequality for A-harmonic tensors*, Proc. Amer. Math. Soc. **125** (1997), 1727–1735. MR 1372027 (97i:30030)

[3] ______, *Integral estimates for the Laplace-Beltrami and Green's operators applied to differential forms on manifolds*, Z. Anal. Anwendungen **22** (2003), 939–957. MR 2036938 (2005b:31009)

[4] S. Ding and C. A. Nolder, *Weighted Poincaré inequalities for solutions to A-harmonic equations*, Illinois J. Math. **46** (2002), 199–205. MR 1936085 (2003m:35069)

[5] S. Ding and P. Shi, *Weighted Poincaré-type inequalities for differential forms in $L^s(\mu)$-averaging domains*, J. Math. Anal. Appl. **227** (1998), 200–215. MR 1652939 (99m:46077)

[6] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1993, Oxford Science Publications. MR 1207810 (94e:31003)

[7] T. Iwaniec and A. Lutoborski, *Integral estimates for null Lagrangians*, Arch. Rational Mech. Anal. **125** (1993), 25–79. MR 1241286 (95c:58054)

[8] C. J. Neugebauer, *Inserting A_p-weights*, Proc. Amer. Math. Soc. **87** (1983), 644–648. MR 687633 (84d:42026)

[9] C. A. Nolder, *Hardy-Littlewood theorems for A-harmonic tensors*, Illinois J. Math. **43** (1999), 613–632. MR 1712513 (2001a:35195)

[10] F. W. Warner, *Foundations of differentiable manifolds and Lie groups*, Graduate Texts in Mathematics, vol. 94, Springer-Verlag, New York, 1983, Corrected reprint of the 1971 edition. MR 722297 (84k:58001)

[11] X. Yuming, *Weighted integral inequalities for solutions of the A-harmonic equation*, J. Math. Anal. Appl. **279** (2003), 350–363. MR 1970511 (2004b:35123)

Department of Mathematics, Harbin Institute of Technology, Harbin, 150001, P.R. China

E-mail address, Gejun Bao: baogj@hit.edu.cn

E-mail address, Yuming Xing: xyuming@hit.edu.cn