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ESTIMATES FOR PARTIAL DERIVATIVES OF VECTOR-VALUED FUNCTIONS

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ABSTRACT. An upper bound for $\|D^\beta u\|_q$ in terms of other similar norms $\|D^\alpha u\|_p$ is derived for vector-valued test functions $u \in C_c^\infty(\mathbf{R}^n, X)$, where X is a Banach space with the UMD property. This gives a new proof and an extension of a classical result of Besov–Il'in–Nikol'skiĭ for scalar functions.

1. Introduction

There is a long history, dating back at least to S. L. Sobolev, of estimates of the form

$$\left\|D^\beta u\right\|_{\bar q} \le K \sum_{\lambda\in\Lambda} \|D^{\alpha_\lambda} u\|_{\bar p_\lambda} \qquad \text{for all } u \in C_c^\infty(\mathbf{R}^n). \tag{1.1}$$

Here and in the sequel, Λ is a finite index set, and we denote by $\beta = (\beta_1, \ldots, \beta_n)$, $\alpha_\lambda \in \mathbf{N}^n$, multi-indices, and

$$D^\beta := \frac{\partial^{\beta_1+\cdots+\beta_n}}{\partial x_1^{\beta_1}\ldots\partial x_n^{\beta_n}};$$

moreover $\bar q = (q_1, \ldots, q_n)$ and $\bar p_\lambda$ are multiexponents, and

$$\|u\|_{\bar q} := \left\{\int_{-\infty}^{\infty}\left[\ldots\left(\int_{-\infty}^{\infty}|u(x)|^{q_1}\,\mathrm{d}\,x_1\right)^{q_2/q_1}\ldots\right]^{q_n/q_{n-1}}\mathrm{d}\,x_n\right\}^{1/q_n}$$

is the norm of the mixed-norm space

$$L^{\bar q}(\mathbf{R}^n) := L^{q_n}(\mathbf{R}, L^{q_{n-1}}(\mathbf{R}, \ldots L^{q_2}(\mathbf{R}, L^{q_1}(\mathbf{R}))\ldots)).$$

For $\bar q = q\cdot\bar 1$ (where $\bar 1 := (1, \ldots, 1)$), this is just the usual Lebesgue space $L^q(\mathbf{R}^n)$ by Fubini's theorem.

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After several special cases of (1.1) had been obtained by various authors, O. V. Besov, V. P. Il'in and S. M. Nikol'skiĭ [1] established this inequality in the maximal range of the parameters for the reflexive range of the exponents $\bar{p}_\lambda,\ \bar{q} \in\,]1,\infty[^n$, a restriction that we will also make in this paper. Denoting by $\triangle^\Lambda$ the collection of convex coefficients with the index set Λ,

$$\triangle^\Lambda := \Big\{(t_\lambda)_{\lambda\in\Lambda} : t_\lambda \geq 0,\ \sum_{\lambda\in\Lambda} t_\lambda = 1\Big\},$$

it was shown by Besov et al. (see [1], Theorem 15.7) that (1.1) holds provided that

$$\beta - \frac{1}{\bar{q}} = \sum_{\lambda\in\Lambda} t_\lambda\left(\alpha_\lambda - \frac{1}{\bar{p}_\lambda}\right) \text{ and } \frac{1}{\bar{q}} \leq \sum_{\lambda\in\Lambda}\frac{t_\lambda}{\bar{p}_\lambda} \text{ for some } (t_\lambda)\in\triangle^\Lambda. \tag{1.2}$$

Here and below, the reciprocal of a multiexponent is interpreted in the usual way as $1/\bar{q} := (1/q_1,\ldots,1/q_n)$, and an inequality between vectors of $\mathbf{R}^n$ is understood as the corresponding inequality between all the respective coordinates. Actually, Besov et al. even proved that (1.2) already implies the formally stronger (by the inequality between arithmetic and geometric means) multiplicative estimate

$$\left\|D^\beta u\right\|_{\bar{q}} \leq K\prod_{\lambda\in\Lambda}\|D^{\alpha_\lambda}u\|_{\bar{p}_\lambda}^{t_\lambda} \quad \text{for some } (t_\lambda)\in\triangle^\Lambda,\ \text{all } u\in C_c^\infty(\mathbf{R}^n), \tag{1.3}$$

in fact, with the same (t_λ) as in (1.2). That the algebraic condition (1.2) is also necessary for (1.1) and (1.3) is probably known to experts, but it is not explicitly stated in [1]; we give a proof in Appendix A for the convenience of the reader.

The Besov–Il'in–Nikol'skiĭ theorem contains (as described in Appendix B), e.g., the reflexive range of the well-known Gagliardo–Nirenberg inequalities (cf. [1], Theorem 15.1)

$$\left\|D^\beta u\right\|_q \leq K\,\|u\|_{p_0}^{1-\theta}\left(\sum_{|\alpha|=\ell}\|D^\alpha u\|_{p_1}\right)^\theta, \qquad u\in C_c^\infty(\mathbf{R}^n), \tag{1.4}$$

for

$$0\leq\theta\leq 1, \quad |\beta|\leq\theta\ell, \quad \text{and} \quad \frac{1}{q}-\frac{|\beta|}{n} = (1-\theta)\frac{1}{p_0}+\theta\left(\frac{1}{p_1}-\frac{\ell}{n}\right).$$

A further specialization to $\theta = 1$ and $\beta = 0$ gives the classical Sobolev inequality.

The main purpose of this paper is to show that the implication (1.2) $\Rightarrow$ (1.3) (and then the equivalence (1.1) $\Leftrightarrow$ (1.2) $\Leftrightarrow$ (1.3)) remains true when $C_c^\infty(\mathbf{R}^n)$ is replaced by the space $C_c^\infty(\mathbf{R}^n, X)$ of functions with values in a (complex) Banach space X, which has the so-called UMD property introduced by D. L. Burkholder [4]. This probabilistic notion means, by definition, that all *martingale difference sequences* in $L^p(\Omega, X)$, where $p \in]1,\infty[$ and Ω is any probability space, are unconditionally convergent (see [4], [5] for details). However, we will only be concerned with the more analytic characterization of these spaces provided by the vector-valued extension of the classical Mihlin multiplier theorem:

Bourgain's multiplier theorem ([3]). *A Banach space X is UMD if and only if the mapping $f \mapsto \mathcal{F}^{-1}(m\,\mathcal{F}f)$, where $\mathcal{F}$ is the Fourier transform, defines a bounded operator on $L^p(\mathbf{R}, X)$ whenever $p \in]1,\infty[$ and the multiplier $m \in L^\infty(\mathbf{R})$ is differentiable away from 0 and $\pm 2^k$, $k \in \mathbf{Z}$, and satisfies*

$$|m(\xi)| + |\xi m'(\xi)| \leq C, \qquad \xi \in \mathbf{R} \setminus \{0, 2^k, -2^k : k \in \mathbf{Z}\}. \tag{1.5}$$

Besides the above result on Fourier multipliers, the class of UMD spaces has proven to be the ultimate setting for the vector-valued generalization of several further results from classical harmonic analysis, its applications, and related topics; due to the large amount of relevant research papers, we refer the reader to the recent survey lectures [12] for further information. However, the inequalities (1.1) and (1.3) have so far not been established in this generality, although some results of related nature, but imposing additional assumptions besides UMD, have been obtained by V. S. Guliev [8], [9].

Guliev proves several cases of (1.1) for $u \in C_c^\infty(\mathbf{R}^n, X)$ under the assumption that X is a UMD space with an *unconditional basis* [8] or that X is a *Banach lattice* with the UMD property [9]. He makes use of D. L. Fernandez' [7] extension to such spaces of the *Marcinkiewicz–Lizorkin multiplier theorem*, which guarantees the $L^p(\mathbf{R}^n, X)$-boundedness of $f \mapsto \mathcal{F}^{-1}(m\,\mathcal{F}f)$ under assumptions of the type

$$|\xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n} D^\alpha m(\xi)| \leq C, \qquad \alpha \in \{0,1\}^n.$$

As pointed out in [8] (after Theorem 4), these methods could be carried out in a somewhat greater generality in view of Zimmermann's [17] proof of the vector-valued Marcinkiewicz–Lizorkin theorem in UMD spaces with Pisier's *property* (α) from [15]. (We define and discuss this property further in the following section.) One could also see directly, by careful inspection of the original proof in [1], that the Besov–Il'in –Nikol'skiĭ theorem remains valid for $u \in C_c^\infty(\mathbf{R}^n, X)$ if the Marcinkiewicz–Lizorkin multiplier theorem holds in $L^p(\mathbf{R}^n, X)$. But it was already known to Zimmermann [17] that this is not the case in all UMD spaces, and G. Lancien [13] has later shown that the Marcinkiewicz–Lizorkin theorem holds precisely in UMD spaces with the additional property (α).

While this is the situation with the main analytic ingredient of the known proofs of (1.2) $\Rightarrow$ (1.3) (or (1.2) $\Rightarrow$ (1.1)), we are going to show that, despite the n-dimensional character of the theorem, it is possible to devise a proof which only makes use of the 1-dimensional Mihlin multiplier theorem plus the special algebraic structure of the multipliers involved, and thus we obtain the following extension of the Besov–Il'in–Nikol'skiĭ result:

Theorem. *Let X be a UMD space and $\bar{p}_\lambda, \bar{q} \in \,]1,\infty[^n$. Then the algebraic condition* (1.2) *implies* (1.3) *with $C_c^\infty(\mathbf{R}^n, X)$ in place of $C_c^\infty(\mathbf{R}^n)$.*

As a byproduct, we also obtain a new proof for the scalar-valued result. However, if one takes into account the standard reformulations in some parts of the argument that one can make in the scalar case, the differences become rather small as compared to the proof in [1]; our principal contribution is really the vector-valued inequality, whose proof does not admit such short-cuts. The main new ingredient of the present proof is a Hölder-type estimate for coefficients of vector-valued multiple random series, which we prove in the following section. As discussed in more detail there, these random series reduce to (non-random) quadratic expressions in the scalar-valued case, and then this key estimate of our reasoning becomes essentially identical with [1], Lemma 15.4. It is perhaps surprising that proving the analogue of such an innocent-looking Hölder inequality in our vector-valued situation makes use of a complex interpolation argument, and this point was the key step in the discovery of our proof.

While the present efforts were aimed at eliminating all assumptions on the Banach space X other than UMD, it should be mentioned that various special cases of (1.1) or (1.3) may be proved by different methods involving no conditions on X at all. In particular, H.-J. Schmeißer and W. Sickel [16] have established the important Gagliardo–Nirenberg inequalities (1.4) for $u \in C_c^\infty(\mathbf{R}^n, X)$, at least when $\theta = |\beta|/\ell \in \,]0,1[$, where X is an arbitrary Banach space. Also vector-valued versions of some well-known limiting embedding theorems, where the quantity on the left of (1.1) is replaced by a Besov norm or an Orlicz–Lorentz norm of u, have been obtained in the same generality of all Banach spaces by M. Krbec and H.-J. Schmeißer [11], and by A. Pełczyński and M. Wojciechowski [14]. We also wish to point out the classical results of J. Boman [2] on the cases of (1.1) valid when $\bar{q} = \bar{p}_\lambda = \infty \cdot \bar{1}$, since his techniques readily generalize to an arbitrary Banach space -valued situation. This is perhaps not so surprising, as the scalar-valued $L^\infty(\mathbf{R}^n)$ is already a very "bad" space, where only limited methods of Harmonic Analysis are available.

Notwithstanding the above-mentioned results in general Banach spaces, however, it seems that the Mihlin-type multiplier estimates, and thus the UMD property, are unavoidable to get the full Besov–Il'in–Nikol'skiĭ inequality (1.3); cf. [8], Remark after Theorem 3. In any case, the above Theorem represents the most general situation in which the estimate (1.3) has been proved so far.

2. An estimate for multiple random series

Let $n \in \{1, 2, \dots\}$ be arbitrary but fixed. By $(\zeta_{j,k})_{k=-\infty}^{\infty}$, $j = 1, \dots, n$, we denote n doubly infinite sequences of independent random variables on some probability space $(\Omega, \Sigma, \mathsf{P})$, having uniform (with respect to the Lebesgue measure) distribution on the complex unit-circle (i.e., so-called Steinhaus random variables). We write E for the mathematical expectation on $(\Omega, \Sigma, \mathsf{P})$.

We note that the real version of these random variables, $\varepsilon_{j,k}$ with uniform distribution on $\{-1, +1\}$, have been used more often in the literature, but the properties of these systems are essentially identical, and in fact

$$(2.1)\qquad \left(\mathsf{E}\left|\sum_{k\in\mathbf{Z}}\varepsilon_k x_k\right|_X^p\right)^{1/p} \eqsim \left(\mathsf{E}\left|\sum_{k\in\mathbf{Z}}\zeta_k x_k\right|_X^p\right)^{1/p}$$

with absolute constants of comparison independent of the Banach space X and $p \in [1, \infty[$. (In writing such sums, we make the implicit assumption that the series are absolutely convergent.) The estimate (2.1) is a quick consequence of *Kahane's contraction principle* (see [10], Theorem 2.5, or [6], 12.2)

$$(2.2)\qquad \left(\mathsf{E}\left|\sum_{k\in\mathbf{Z}}\varepsilon_k \lambda_k x_k\right|_X^p\right)^{1/p} \leq c\left(\mathsf{E}\left|\sum_{k\in\mathbf{Z}}\varepsilon_k x_k\right|_X^p\right)^{1/p}, \qquad |\lambda_k| \leq 1,$$

where one may take $c = 1$ for real coefficients $\lambda_k \in [-1, 1]$, and $c = 2$ (by splitting into real and imaginary parts) in general. To prove (2.1) (which is also implicit in [10], Theorem 2.6), one observes that the series $(\zeta_k)_{k\in\mathbf{Z}}$ on Ω and $(\varepsilon_k(\omega_1)\zeta_k(\omega_2))_{k\in\mathbf{Z}}$ on $\Omega\times\Omega$ are equidistributed, and then one may use the contraction principle to "pull out" the numbers $\zeta_k(\omega_2)$, or their inverses, from the ε_k-sum for any fixed $\omega_2 \in \Omega_2$. It also easily follows that the contraction principle itself is valid with ζ_k in place of ε_k.

Let us further recall that the expressions in (2.1) for two different values of $p \in [1, \infty[$ are also comparable to each other, with constants of comparison only depending on the exponents p_1 and p_2, say. This is usually called Kahane's inequality, or the Hinčin–Kahane inequality (see [10], Eq. $(*)$ on p. 282, or [6], Theorem 11.1). In particular, when $X = \mathbf{C}$, or more generally when X is a Hilbert space, the choice $p = 2$ (or the classical Hinčin inequality) shows that the randomized sums in (2.1) are comparable to the quadratic expression

$$\left(\sum_{k\in\mathbf{Z}}|x_k|^2\right)^{1/2}.$$

Indeed, the original scalar-valued proof of Besov–Il'in–Nikol'skiĭ mainly uses such "square functions" in place of (2.1), avoiding the present probabilistic framework, but even in the scalar case it is occasionally useful to resort to the linearization of the quadratic expressions provided by their equivalence to (2.1) (cf. [1], p. 315).

Of importance to us will be n-fold products of the Steinhaus variables of the form

$$\bar{\zeta}_{\bar{k}} := \prod_{j=1}^{n} \zeta_{j,k_j}, \qquad \bar{k} = (k_1, \ldots, k_n).$$

The random variables $\bar{\zeta}_{\bar{k}}$, $\bar{k} \in \mathbf{Z}^n$, still have the Steinhaus distribution individually, but are no longer independent. The following lemma is our key tool for getting away from the use of the Marcinkiewicz–Lizorkin multiplier theorem in the proof of our main result.

Lemma. *Let Λ be a finite index set and $\bar{a}_\lambda : \mathbf{Z}^n \to]0,\infty[$, $\lambda \in \Lambda$, be functions of the form*

$$\bar{a}_\lambda(\bar{k}) = \prod_{j=1}^{n} a_{\lambda,j}(k_j), \qquad a_{\lambda,j} : \mathbf{Z} \to]0,\infty[\,. \tag{2.3}$$

Let $(t_\lambda) \in \triangle^\Lambda$. Then there holds

$$\left(\mathsf{E} \left| \sum_{\bar{k}\in\mathbf{Z}^n} \bar{\zeta}_{\bar{k}} \prod_{\lambda\in\Lambda} \bar{a}_\lambda(\bar{k})^{t_\lambda} x_{\bar{k}} \right|_X^p \right)^{1/p} \leq \prod_{\lambda\in\Lambda} \left(\mathsf{E} \left| \sum_{\bar{k}\in\mathbf{Z}^n} \bar{\zeta}_{\bar{k}} \bar{a}_\lambda(\bar{k}) x_{\bar{k}} \right|_X^p \right)^{t_\lambda/p}$$

for any $p \in [1,\infty[$, any Banach space X, and all $(x_{\bar{k}})_{\bar{k}\in\mathbf{Z}^n} \subset X$.

Proof. By induction on the size of Λ, it suffices to consider the case $\Lambda = \{0,1\}$; hence we have to prove that

$$\begin{aligned} &\left(\mathsf{E} \left| \sum \bar{\zeta}_{\bar{k}} \bar{a}_0(\bar{k})^{1-\theta} \bar{a}_1(\bar{k})^{\theta} x_{\bar{k}} \right|_X^p \right)^{1/p} \\ &\quad \leq \left(\mathsf{E} \left| \sum \bar{\zeta}_{\bar{k}} \bar{a}_0(\bar{k}) x_{\bar{k}} \right|_X^p \right)^{(1-\theta)/p} \left(\mathsf{E} \left| \sum \bar{\zeta}_{\bar{k}} \bar{a}_1(\bar{k}) x_{\bar{k}} \right|_X^p \right)^{\theta/p}, \end{aligned} \tag{2.4}$$

where $0 < \theta < 1$ (the boundary cases being trivial).

We define the holomorphic function

$$F : z \in \mathbf{C} \mapsto \sum_{\bar{k}\in\mathbf{Z}^n} \bar{\zeta}_{\bar{k}} \bar{a}_0(\bar{k})^{1-z} \bar{a}_1(\bar{k})^{z} x_{\bar{k}} \in L^p(\Omega, X),$$

so that the left side of (2.4) is $\|F(\theta)\|$. By the Three Lines Theorem,

$$\|F(\theta)\| \leq \sup_{t\in\mathbf{R}} \|F(\mathbf{i}t)\|^{1-\theta} \times \sup_{t'\in\mathbf{R}} \|F(1+\mathbf{i}t')\|^{\theta}, \tag{2.5}$$

and for $\sigma \in \{0,1\}$,

$$F(\sigma + \mathbf{i}t) = \sum \bar{\zeta}_{\bar{k}} \bar{a}_0(\bar{k})^{-\mathbf{i}t} \bar{a}_1(\bar{k})^{\mathbf{i}t} \bar{a}_\sigma(\bar{k}) x_{\bar{k}}.$$

For each fixed $t \in \mathbf{R}$,

$$\bar{\zeta}_{\bar{k}} \bar{a}_0(\bar{k})^{-\mathbf{i}t} \bar{a}_1(\bar{k})^{\mathbf{i}t} = \prod_{j=1}^{n} (\zeta_{j,k_j} a_{0,j}(k_j)^{-\mathbf{i}t} a_{1,j}(k_j)^{\mathbf{i}t}),$$

and $(\zeta_{j,k_j} a_{0,j}(k_j)^{-\mathbf{i}t} a_{1,j}(k_j)^{\mathbf{i}t})$, $k_j \in \mathbf{Z}$, $j = 1, \ldots, n$, is equidistributed with (ζ_{j,k_j}). This shows that also the right side of (2.4) coincides with the right side of (2.5), and we are done. □

REMARK. Let us now recall the definition of Pisier's property (α) from [15], and explain why things would be easier in its presence. As we already pointed out the equivalence (2.1), we also make this definition using the Steinhaus variables ζ_k, although the Rademacher variables ε_k were originally used. A Banach space X is said to have property (α) if the (non-independent) products $\bar{\zeta}_{\bar{k}}$ of Steinhaus variables are equivalent to independent variables in the sense of the following two-sided estimate, with constants of comparison only depending on $n = 2, 3, \ldots$ and $p \in [1, \infty[$:

$$\left(\mathsf{E} \left| \sum_{\bar{k} \in \mathbf{Z}^n} \bar{\zeta}_{\bar{k}} x_{\bar{k}} \right|_X^p \right)^{1/p} \eqsim \left(\mathsf{E} \left| \sum_{\bar{k} \in \mathbf{Z}^n} \zeta_{\bar{k}} x_{\bar{k}} \right|_X^p \right)^{1/p}, \tag{2.6}$$

where $(\zeta_{\bar{k}})_{\bar{k} \in \mathbf{Z}^n}$ on the right side is an independent Steinhaus sequence indexed by $\mathbf{Z}^n$.

Next we observe that, for $n = 1$, the Lemma can be proved in a more elementary way using only Kahane's contraction principle (2.2). The crucial step (2.4) follows from $a_0(k)^{1-\theta} a_1(k)^\theta \leq (1-\theta) a_0(k) + \theta a_1(k)$, which implies

$$\begin{aligned} &\left(\mathsf{E} \left| \sum_{k \in \mathbf{Z}} \zeta_k a_0(k)^{1-\theta} a_1(k)^\theta x_k \right|_X^p \right)^{1/p} \\ &\qquad \leq \left(\mathsf{E} \left| \sum_{k \in \mathbf{Z}} \zeta_k [(1-\theta) a_0(k) + \theta a_1(k)] x_k \right|_X^p \right)^{1/p}; \end{aligned}$$

then we use the triangle inequality, and a scaling argument which permits the assumption that the two factors on the right of (2.4) are equal to 1. But if X enjoys property (α), the case of a general n effectively reduces to $n = 1$ by (2.6), and we get the conclusion of the Lemma, even for arbitrary positive functions $\bar{k} \mapsto \bar{a}_\lambda(\bar{k})$ which need not split into the product form (2.3), only with an additional multiplicative constant C on the right.

Conversely, if this modified conclusion of the Lemma holds, then the space X has property (α). In fact, let $A \subseteq \mathbf{Z}^n$ be an arbitrary subset, let $\bar{a}_0(\bar{k}) = 1_A(\bar{k}) + \delta 1_{A^c}(\bar{k})$ and $\bar{a}_1(\bar{k}) \equiv 1$. In the limit $\delta \downarrow 0$, the estimate (2.4) becomes, after simplification,

$$\left(\mathsf{E} \left| \sum_{\bar{k} \in A} \bar{\zeta}_{\bar{k}} x_{\bar{k}} \right|_X^p \right)^{1/p} \leq C \left(\mathsf{E} \left| \sum_{\bar{k} \in \mathbf{Z}^n} \bar{\zeta}_{\bar{k}} x_{\bar{k}} \right|_X^p \right)^{1/p}, \qquad A \subset \mathbf{Z}^n,$$

which is easily seen to be equivalent with (2.6).

3. Proof of the Theorem

Despite the novel features at certain key points, the structure of the proof that follows is still very much borrowed from the original argument of Besov et al. [1]. To avoid any problems of well-definedness of the Fourier multipliers in the following, we may, instead of $u \in C_c^\infty(\mathbf{R}^n, X)$, operate on functions from the test-function class

$$\mathcal{F}^{-1}C_c^\infty(\mathbf{R}\setminus\{0\}) \otimes \cdots \otimes \mathcal{F}^{-1}C_c^\infty(\mathbf{R}\setminus\{0\}) \otimes X,$$

where $\mathcal{F}$ is the Fourier transform, and resort to density arguments with respect to the norms appearing in (1.3) in the end.

Another approach, used in [1], proceeds via estimates for periodic functions: By scaling, we may assume that $\operatorname{supp} u \subset \,]-1/2, 1/2[^n$ and extend u to a smooth function with period 1 in each variable. Then one could readily modify the argument given below to prove the analogous estimates on the periodic $L^{\bar{p}}$ spaces using multipliers of Fourier series instead of those of Fourier integrals, though some care would be needed when treating the zero frequencies which cannot be ignored in the periodic case.

Now, let $\beta, \bar{q}$ and $\alpha_\lambda, \bar{p}_\lambda, t_\lambda$, for $\lambda \in \Lambda$, be as in (1.2). Let us define α and $\bar{p}$ by the conditions

$$\alpha := \sum_{\lambda\in\Lambda} t_\lambda \alpha_\lambda, \qquad \frac{1}{\bar{p}} := \sum_{\lambda\in\Lambda} \frac{t_\lambda}{\bar{p}_\lambda};$$

hence $\beta - 1/\bar{q} = \alpha - 1/\bar{p}$ and $1/\bar{q} \le 1/\bar{p}$, thus also $\beta \le \alpha$.

Note that the α defined above may fail to have integer components, but we may define D^α for arbitrary $\alpha \in \mathbf{R}^n$ to be the Fourier multiplier operator with symbol

$$(\mathbf{i}\xi)^\alpha = \prod_{j=1}^n |\xi_j|^{\alpha_j} \exp\left(\mathbf{i}\frac{\pi}{2}\alpha_j \operatorname{sgn}\xi_j\right).$$

This definition coincides with the usual one when $\alpha \in \mathbf{N}^n$. We also introduce the operators of fractional integration, $I^\alpha = I_1^{\alpha_1}\ldots I_n^{\alpha_n} := D^{-\alpha}$, which are well-defined on our test-function class.

The proof of (1.2) $\Rightarrow$ (1.3) will consist of showing the two inequalities

$$\left\|D^\beta u\right\|_{\bar{q}} \le A \left\|D^\alpha u\right\|_{\bar{p}}, \quad \beta - \frac{1}{\bar{q}} = \alpha - \frac{1}{\bar{p}}, \quad \bar{0} < \frac{1}{\bar{q}} \le \frac{1}{\bar{p}} < \bar{1}, \tag{3.1}$$

and

$$\left\|D^\alpha u\right\|_{\bar{p}} \le B \prod_{\lambda\in\Lambda} \left\|D^{\alpha_\lambda} u\right\|_{\bar{p}_\lambda}^{t_\lambda}, \quad \alpha = \sum_{\lambda\in\Lambda} t_\lambda\alpha_\lambda, \quad \frac{1}{\bar{p}} = \sum_{\lambda\in\Lambda}\frac{t_\lambda}{\bar{p}_\lambda}. \tag{3.2}$$

The estimate (3.1) is valid for an arbitrary Banach space X, and is a consequence of the following chain of inequalities, where $\sigma := \alpha - \beta \ge \bar{0}$ and

$v := D^\alpha u$:

$$\begin{aligned}
\|I^\sigma v\|_{\bar q} &= \|I_1^{\sigma_1} I_2^{\sigma_2} \dots I_n^{\sigma_n} v\|_{L^{q_n}(\mathbf{R},\dots,L^{q_1}(\mathbf{R},X))} \\
&\le A_1 \|I_2^{\sigma_2} \dots I_n^{\sigma_n} v\|_{L^{q_n}(\mathbf{R},\dots,L^{p_1}(\mathbf{R},X))} \le \cdots \\
&\le A_1 \dots A_{k-1} \left\| I_k^{\sigma_k} I_{k+1}^{\sigma_{k+1}} \dots I_n^{\sigma_n} v \right\|_{L^{q_n}(\mathbf{R},\dots,L^{q_k}(\mathbf{R},\dots,L^{p_1}(\mathbf{R},X)))} \\
&\le A_1 \dots A_{k-1} A_k \left\| I_{k+1}^{\sigma_{k+1}} \dots I_n^{\sigma_n} v \right\|_{L^{q_n}(\mathbf{R},\dots,L^{p_k}(\mathbf{R},\dots,L^{p_1}(\mathbf{R},X)))} \\
&\le \dots \le A_1 \dots A_n \|v\|_{L^{p_n}(\mathbf{R},\dots,L^{p_1}(\mathbf{R},X))} \\
&= A_1 \dots A_n \|v\|_{\bar p},
\end{aligned}$$

where the kth majorization is an application of the classical Hardy–Littlewood theorem on fractional integration,

$$\|I^{\sigma_k} w\|_{q_k} \le A_k \|w\|_{p_k}, \qquad \sigma_k = \frac{1}{p_k} - \frac{1}{q_k}, \quad 1 < p_k \le q_k < \infty, \tag{3.3}$$

to the $L^{p_{k-1}}(\mathbf{R}, \dots, L^{p_1}(\mathbf{R}, X))$-valued real-variable function

$$w(x) := I_{k+1}^{\sigma_{k+1}} \dots I_n^{\sigma_n} v(\cdot, \dots, \cdot, x, x_{k+1}, \dots, x_n),$$

for every fixed choice of $x_{k+1}, \dots, x_n \in \mathbf{R}$. (The range space is understood as X if $k = 1$.) Note that the inequality (3.3) remains valid for functions with values in an arbitrary Banach space, and with the same constant, since the fractional integrals I^σ are positive linear operators.

Thus we are left with proving (3.2). We keep working in each coordinate direction at a time. Given $a \in \mathbf{R}$ and arbitrary complex numbers ζ_k of unit length, $k \in \mathbf{Z}$, one easily verifies that

$$m(\xi) := \sum_{k \in \mathbf{Z}} \zeta_k \frac{2^{ka}}{(\mathbf{i}\xi)^a} 1_{\{2^{k-1} < |\xi| \le 2^k\}}$$

and $1/m(\xi)$ both satisfy the Mihlin estimate (1.5) with C independent of the ζ_k.

Let us denote by $\Delta_{\bar k}$, $\bar k \in \mathbf{Z}^n$, the Fourier multiplier operator on $\mathbf{R}^n$ with symbol

$$\prod_{j=1}^n 1_{\{2^{k_j - 1} < |\xi_j| \le 2^{k_j}\}}.$$

An n-fold application of Bourgain's multiplier theorem, combined with averaging over all the complex unit vectors appearing in the multipliers above, shows that

$$\|D^\alpha u\|_{\bar p} \eqsim \mathsf{E} \left\| \sum_{\bar k \in \mathbf{Z}^n} \bar\zeta_{\bar k} 2^{\bar k \cdot \alpha} \Delta_{\bar k} u \right\|_{\bar p} \eqsim \left\| \mathsf{E} \left| \sum_{\bar k \in \mathbf{Z}^n} \bar\zeta_{\bar k} 2^{\bar k \cdot \alpha} \Delta_{\bar k} u \right|_X \right\|_{\bar p}, \tag{3.4}$$

where the last equivalence follows from Kahane's inequality (i.e., the equivalence of the Steinhaus-randomized norms (2.1) for different $p \in [1, \infty[$) and

Fubini's theorem. Here, of course, the $\bar{\zeta}_{\bar{k}}$ denote products of Steinhaus variables as in the previous section. This n-fold use of Mihlin's theorem replaces the application of the (now generally invalid) Marcinkiewicz–Lizorkin theorem in [1].

The Theorem now follows from

$$\mathsf{E}\left|\sum_{\bar{k}\in\mathbf{Z}^n}\bar{\zeta}_{\bar{k}}2^{\bar{k}\cdot\alpha}\Delta_{\bar{k}}u\right|_X \leq \prod_{\lambda\in\Lambda}\left(\mathsf{E}\left|\sum_{\bar{k}\in\mathbf{Z}^n}\bar{\zeta}_{\bar{k}}2^{\bar{k}\cdot\alpha_\lambda}\Delta_{\bar{k}}u\right|_X\right)^{t_\lambda},$$

which is an application of our Lemma with $\bar{a}_\lambda(\bar{k}) = 2^{\bar{k}\cdot\alpha_\lambda}$, and

$$\left\|\prod_{\lambda\in\Lambda}\left(\mathsf{E}\left|\sum_{\bar{k}\in\mathbf{Z}^n}\bar{\zeta}_{\bar{k}}2^{\bar{k}\cdot\alpha_\lambda}\Delta_{\bar{k}}u\right|_X\right)^{t_\lambda}\right\|_{\bar{p}} \leq \prod_{\lambda\in\Lambda}\left(\left\|\mathsf{E}\left|\sum_{\bar{k}\in\mathbf{Z}^n}\bar{\zeta}_{\bar{k}}2^{\bar{k}\cdot\alpha_\lambda}\Delta_{\bar{k}}u\right|_X\right\|_{\bar{p}_\lambda}\right)^{t_\lambda},$$

which is Hölder's inequality for mixed norms. This completes the proof.

Appendix A. The necessity of the algebraic condition

We argue by contradiction to show that (1.1) $\Rightarrow$ (1.2). Let us fix a non-zero $u \in C_c^\infty\left([-\frac{1}{2},\frac{1}{2}]^n\right)$ and write (1.1) for all the functions

$$\zeta_{\tau\mu}(x) := \sum_{\kappa\in\mathbf{N}^n:1\leq\kappa_j\leq 2^{\mu_j}\ \ \forall j} u(2^{\tau_1}x_1-\kappa_1,\ldots,2^{\tau_n}x_n-\kappa_n),$$

where $\tau \in \mathbf{R}^n$ and $\mu \in \mathbf{N}^n$. Using the disjointness of the supports of the summands above and simple changes of variables, it is easily seen that $\left\|D^\beta\zeta_{\tau\mu}\right\|_{\bar{q}} = 2^{\tau\cdot(\beta-1/\bar{q})}2^{\mu\cdot 1/\bar{q}}\|u\|_{\bar{q}}$. Thus (1.1) implies the family of inequalities

$$2^{\tau\cdot(\beta-1/\bar{q})}2^{\mu\cdot 1/\bar{q}} \leq K'\sum_{\lambda\in\Lambda}2^{\tau\cdot(\alpha_\lambda-1/\bar{p}_\lambda)}2^{\mu\cdot 1/\bar{p}_\lambda}, \tag{A.1}$$

which continue to hold for $\tau\in\mathbf{R}^n$ and $\mu\in\bar{\mathbf{R}}_+^n := [0,\infty[^n$ (instead of $\mu\in\mathbf{N}^n$), possibly adjusting the constant. On the other hand, (1.2) means the inclusion

$$\left(\beta-\frac{1}{\bar{q}},\frac{1}{\bar{q}}\right)\in\operatorname{conv}\left(\left\{\left(\alpha_\lambda-\frac{1}{\bar{p}_\lambda},\frac{1}{\bar{p}_\lambda}\right):\lambda\in\Lambda\right\}-\{0\}\times\bar{\mathbf{R}}_0^n\right),$$

where conv stands for the convex hull, which in this case is seen to be automatically closed, too. If this last inclusion does not hold, then the geometric version of the Hahn–Banach theorem guarantees the existence of a separating hyperplane, i.e., for some $(\tau,\mu)\in\mathbf{R}^{n+n}$ and $r\in\mathbf{R}$, there holds

$$\tau\cdot\left(\beta-\frac{1}{\bar{q}}\right)+\mu\cdot\frac{1}{\bar{q}} > r \geq \tau\cdot\left(\alpha_\lambda-\frac{1}{\bar{p}_\lambda}\right)+\mu\cdot\frac{1}{\bar{p}_\lambda}-\mu\cdot\nu \tag{A.2}$$

for all $\lambda\in\Lambda$ and $\nu\in\bar{\mathbf{R}}_+^n$. If some component of μ, say μ_j, is negative, then $\nu = Ne_j$ contradicts the previous displayed line as $N\to\infty$, so we must have $\mu\in\bar{\mathbf{R}}_+^n$. But this means that these particular (τ,μ), and also their multiples $(N\tau, N\mu)$, are admissible values of the variables in (A.1). However it is clear that (A.2), with $\nu = 0$, and (A.1), with $(N\tau, N\mu)$ in place of (τ, μ) and $N \to \infty$, contradict each other, and we have proved that (1.2) cannot fail if (1.1) holds.

Let us note that if we assume the stronger inequality (1.3) instead of (1.1), then by similar considerations we obtain, instead of (A.1),

$$2^{\mu\cdot\bar{q}}2^{\tau\cdot(\beta-1/\bar{q})} \le K' 2^{\mu\cdot\sum_{\lambda\in\Lambda} t_\lambda/\bar{p}_\lambda} 2^{\tau\cdot\sum_{\lambda\in\Lambda} t_\lambda(\alpha_\lambda - 1/\bar{p}_\lambda)}$$

for all $\tau \in \mathbf{R}^n$ and $\mu \in \bar{\mathbf{R}}^n_+$. From this the necessity of (1.2) is immediate, with the same (t_λ) as in (1.3).

Appendix B. The Gagliardo–Nirenberg inequalities

We want to show that the algebraic condition of the Gagliardo–Nirenberg inequalities (1.4) is a special case of (1.2). This may also serve as a model for identifying other estimates for partial derivatives as special cases of (1.3).

Let us denote

$$C_\ell := \operatorname{conv}\{\ell e_i : i = 1, \dots, n\} \subseteq \operatorname{conv}\{\alpha \in \mathbf{N}^n : |\alpha| = \ell\},$$

where e_i is the ith standard unit-vector, and conv stands for the convex hull. We first note that any non-zero multi-index $\beta \in \mathbf{N}^n$ and $\ell \in \mathbf{Z}_+$ satisfy

$$\beta = \sum_{i=1}^n \beta_i e_i = \frac{|\beta|}{\ell} \sum_{i=1}^n \frac{\beta_i}{|\beta|} \ell e_i \in \frac{|\beta|}{\ell} C_\ell. \tag{B.1}$$

By inspection, the first condition in (1.2) follows from the assumption of (1.4) if we can find $(t_i)_{i=1}^n \in \triangle^{\{1,\dots,n\}}$ such that

$$\beta - \theta \sum_{i=1}^n t_i\, \ell e_i = \left(\frac{|\beta|}{n} - \theta\frac{\ell}{n}\right)\bar{1}. \tag{B.2}$$

But by (B.1), we have

$$\beta + \frac{\theta\ell - |\beta|}{n}\bar{1} \in \frac{|\beta|}{\ell} C_\ell + \frac{\theta\ell - |\beta|}{n}\frac{n}{\ell} C_\ell = \theta C_\ell,$$

and this is precisely (B.2). We leave it to the reader to check (B.2) in the easy cases when the previous argument involves divisions by zero.

The second condition in (1.2) reduces in the present case to $1/q \le (1-\theta)/p_0 + \theta/p_1$, and this is immediate from the assumptions of (1.4).

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