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THE JOIN OF ALGEBRAIC CURVES

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Abstract. An effective description of the join of algebraic curves in the complex projective space $\mathbb{P}^n$ is given.

1. Introduction

Let $\mathbb{P}^n$ be the n-dimensional projective space over $\mathbb{C}$. Denote by $G(1,\mathbb{P}^n)$ the Grassmannian of all projective lines in $\mathbb{P}^n$. By the Plücker embedding $G(1,\mathbb{P}^n) \hookrightarrow \mathbb{P}^{\binom{n+1}{2}-1}$ the Grassmannian is an algebraic subset of $\mathbb{P}^{\binom{n+1}{2}-1}$. For any projective line $L \subset \mathbb{P}^n$ we will denote by $[L]$ the corresponding point of $G(1,\mathbb{P}^n)$, and for any $P, Q \in \mathbb{P}^n$, $P \neq Q$, we will denote by $\overline{PQ}$ the unique projective line in $\mathbb{P}^n$ spanned by P and Q. Likewise, for any projective subspaces $L, K \subset \mathbb{P}^n$ we will denote by $\operatorname{Span}(L,K)$ the unique projective subspace in $\mathbb{P}^n$ spanned by L and K.

If X is an algebraic subset of $\mathbb{P}^n$ then $\operatorname{Sing}(X)$ is the set of singular points of X. For $P \in X - \operatorname{Sing}(X)$ we denote by $T_P X \subset \mathbb{P}^n$ the embedded tangent space to X at P.

Let $X, Y \subset \mathbb{P}^n$ be two varieties in $\mathbb{P}^n$, i.e., irreducible algebraic subsets of $\mathbb{P}^n$. The definition of the join of X and Y is as follows (see [H, p. 88], [Z, p. 15], [FOV, Def. 1.3.5]). Define the subsets of the Grassmannian

$$\mathcal{J}^0(X,Y) := \{[\overline{PQ}] \in G(1,\mathbb{P}^n) : P \in X,\, Q \in Y,\, P \neq Q\},$$
$$\mathcal{J}(X,Y) := \overline{\mathcal{J}^0(X,Y)} \text{ – the closure of } \mathcal{J}^0(X,Y) \text{ in } G(1,\mathbb{P}^n),$$

and the corresponding subsets of the projective space

$$J^0(X,Y) := \bigcup_{[L]\in\mathcal{J}^0(X,Y)} L,$$
$$J(X,Y) := \bigcup_{[L]\in\mathcal{J}(X,Y)} L.$$

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$\mathcal{J}(X,Y)$ and $J(X,Y)$ are algebraic subsets of $G(1,\mathbb{P}^n)$ and $\mathbb{P}^n$, respectively. $\mathcal{J}(X,Y)$ is called *the variety of lines joining X and Y*, and $J(X,Y)$ is called *the join of X and Y*. In the case $X=Y$ the set $J(X,Y)$ is called *the secant variety of X* and is denoted by $\mathrm{Sec}(X)$ or X^2.

If $X\cap Y=\emptyset$ then we have $\mathcal{J}(X,Y)=\mathcal{J}^0(X,Y)$. In the case $X\cap Y\neq\emptyset$, the inclusion $\mathcal{J}^0(X,Y)\subset\mathcal{J}(X,Y)$ is, in general, strict. Thus there arises the following question: Which additional projective lines besides those containing points $P\in X$, $Q\in Y$, $P\neq Q$, are in $\mathcal{J}(X,Y)$? In this paper we give a complete solution of this problem in the case when X and Y are arbitrary projective curves (in particular for $X=Y$).

The key notion in the solution is the relative tangent cone $C_P(X,Y)$ to a pair of algebraic or analytic sets X,Y in a given common point $P\in X\cap Y$. (In [FOV, Section 2.5] this cone is denoted by $\mathrm{LJoin}_P(X,Y)$.) It is a generalization of one of the Whitney cones, namely $C_5(V,P)$ ([W1, p. 212], [W3, p. 211]), to the case of a pair of sets. The cone $C_P(X,Y)$ was introduced by Achilles, Tworzewski and Winiarski [ATW] in the analytic case when X and Y meet at a point. This notion was used in the new improper intersection theory in algebraic and analytic geometry ([FOV], [T], [CKT], [Cy]). It is easy to show (see Proposition 4.1) that for varieties $X,Y\subset\mathbb{P}^n$

$$J(X,Y)=J^0(X,Y)\cup\bigcup_{P\in X\cap Y}C_P(X,Y).$$

Thus the question is reduced to the problem of describing $C_P(X,Y)$. If P is an isolated intersection point of two analytic curves X and Y, Ciesielska [C] proved that the cone $C_P(X,Y)$ is a finite sum of two-dimensional hyperplanes. (In the case $X=Y$ this was proved by Briançon, Galligo and Granger [BGG].) In Theorem 3.4 we give an effective formula for the relative tangent cone $C_P(X,Y)$ in the general case when X,Y are arbitrary analytic curves and $P\in X\cap Y$ (and even in the case $X=Y$). This formula is expressed in terms of local parametrizations of X and Y at P.

In the last section we summarize all results in Theorem 4.2, which gives a detailed description of the join of algebraic curves.

2. Relative tangent cones to analytic sets

Since the relative tangent cone is a local notion, we will work in $\mathbb{C}^n$ and in the case when X,Y are analytic sets. First we consider the case when the point P is the origin, i.e., $P=\mathbf{0}$. We start with the notion of the ordinary tangent cone to an analytic set.

Let X be an analytic set in a neighbourhood U of $\mathbf{0}\in\mathbb{C}^n$ such that $\mathbf{0}\in X$. *The tangent cone $C_0(X)$ of X at $\mathbf{0}$* is defined to be the set of $\mathbf{v}\in\mathbb{C}^n$ with the following property: There exist sequences $(\mathbf{x}_\nu)_{\nu\in\mathbb{N}}$ of points of X and $(\lambda_\nu)_{\nu\in\mathbb{N}}$

of complex numbers such that

$$\mathbf{x}_\nu \to 0 \text{ and } \lambda_\nu \mathbf{x}_\nu \to \mathbf{v} \text{ when } \nu \to \infty.$$

One can find properties of the tangent cones to analytic sets in [W2], [W3], and [Ch]. The tangent cone is an algebraic cone in $\mathbb{C}^n$ of dimension $\dim_0 X$.

Let X, Y be analytic subsets of a neighbourhood U of $\mathbf{0} \in \mathbb{C}^n$ such that $\mathbf{0} \in X \cap Y$. *The relative tangent cone $C_0(X,Y)$ of X and Y at $\mathbf{0}$* is defined to be the set of $\mathbf{v} \in \mathbb{C}^n$ with the following property: There exist sequences $(\mathbf{x}_\nu)_{\nu\in\mathbb{N}}$ of points of X, $(\mathbf{y}_\nu)_{\nu\in\mathbb{N}}$ of points of Y and $(\lambda_\nu)_{\nu\in\mathbb{N}}$ of complex numbers such that

$$\mathbf{x}_\nu \to 0, \quad \mathbf{y}_\nu \to 0, \quad \lambda_\nu(\mathbf{y}_\nu - \mathbf{x}_\nu) \to \mathbf{v} \text{ when } \nu \to \infty.$$

Immediately from the definition we obtain:

(1) $C_0(X,Y)$ is a cone with vertex at $\mathbf{0}$.
(2) If $Y = \{\mathbf{0}\}$, then $C_0(X,Y) = C_0(X),$.
(3) $C_0(X,Y) = C_0(Y,X)$.
(4) $C_0(X,Y)$ depends only on the germs of X and Y at $\mathbf{0}$.
(5) $C_0(X_1 \cup X_2, Y) = C_0(X_1,Y) \cup C_0(X_2,Y)$ if X_1, X_2 are analytic sets containing $\mathbf{0}$.

The following two propositions are known. Since we will use facts from the proofs, we give simple and elementary proofs of these propositions in the analytic case. We will assume in the remainder of this section that X, Y are analytic subsets of a neighbourhood U of $\mathbf{0} \in \mathbb{C}^n$ such that $\mathbf{0} \in X \cap Y$.

Proposition 2.1 ([ATW, Property 2.9] in the case $X \cap Y = \{0\}$). *The cone $C_0(X,Y)$ is an algebraic cone in $\mathbb{C}^n$.*

Proof. By the Chow theorem it suffices to prove that $C_0(X,Y)$ is an analytic subset of $\mathbb{C}^n$. We will apply the elementary Whitney method ([W1, Th. 5.1], used there in the case $X = Y$), although one can also use the method of blowing-ups. Define the holomorphic functions

$$\alpha_{jk} : \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}, \quad j,k = 1,\dots,n,$$

$$\alpha_{jk}(\mathbf{x},\mathbf{y},\mathbf{v}) := \begin{vmatrix} y_j - x_j & y_k - x_k \\ v_j & v_k \end{vmatrix},$$

where $\mathbf{x} = (x_1,\dots,x_n)$, $\mathbf{y} = (y_1,\dots,y_n)$ and $\mathbf{v} = (v_1,\dots,v_n)$.

The functions α_{jk} all vanish if and only if $\mathbf{x} = \mathbf{y}$ or $\mathbf{v}$ is a multiple of $\mathbf{y} - \mathbf{x}$. Set

$$B := \{(\mathbf{x},\mathbf{y},\mathbf{v}) : \mathbf{x},\mathbf{y} \in U,\ \alpha_{jk}(\mathbf{x},\mathbf{y},\mathbf{v}) = 0,\ j,k = 1,\dots,n\}.$$

This is an analytic subset of $U \times U \times \mathbb{C}^n$, and hence so is

$$B' := B \cap (X \times Y \times \mathbb{C}^n).$$

The set $\Delta := \{(\mathbf{x},\mathbf{x}) : \mathbf{x} \in X \cap Y\} \subset U \times U$ is also analytic. Thus

$$B'' := \overline{(B' - (\Delta \times \mathbb{C}^n))} \cap (U \times U \times \mathbb{C}^n)$$

is an analytic set in $U \times U \times \mathbb{C}^n$. Therefore

$$C'_0(X,Y) := B'' \cap (\{(\mathbf{0},\mathbf{0})\} \times \mathbb{C}^n)$$

is analytic in $U \times U \times \mathbb{C}^n$. Since $\mathbf{v} \in C_0(X,Y)$ if and only if $(\mathbf{0},\mathbf{0},\mathbf{v}) \in C'_0(X,Y)$, it follows that $C_0(X,Y)$ is an analytic subset of $\mathbb{C}^n$. □

PROPOSITION 2.2 (cf. [FOV, Prop. 2.5.5]). $\dim C_0(X,Y) \leqslant \dim_0 X + \dim_0 Y$.

Proof. Since $C_0(X,Y)$ depends only on the germs of X and Y at $\mathbf{0}$, we may assume that $\dim X = \dim_0 X$ and $\dim Y = \dim_0 Y$. Consider the analytic set $B'' \subset U \times U \times \mathbb{C}^n$, defined in the proof of the previous proposition. If we denote by π the projection $U \times U \times \mathbb{C}^n \to U \times U$, then $\pi(B'') \subset X \times Y$ and over each point $(\mathbf{x},\mathbf{y}) \in (X \times Y) - \Delta$ we have $(\pi|B'')^{-1}(\mathbf{x},\mathbf{y}) = \{(\mathbf{x},\mathbf{y},\lambda(\mathbf{y}-\mathbf{x})) : \lambda \in \mathbb{C}\}$ and hence $\dim(\pi|B'')^{-1}(\mathbf{x},\mathbf{y}) = 1$. Since

$$B'' = \overline{(\pi|B'')^{-1}(X \times Y - \Delta)}, \tag{1}$$

we have

$$\dim B'' = \dim X + \dim Y + 1.$$

By the same equality (1) no irreducible component of B'' is contained in $\Delta \times \mathbb{C}^n$, and in particular in $(\mathbf{0},\mathbf{0}) \times \mathbb{C}^n$. Hence

$$\dim C'_0(X,Y) = \dim(B'' \cap (\{(\mathbf{0},\mathbf{0})\} \times \mathbb{C}^n)) \leqslant \dim X + \dim Y. \qquad \square$$

REMARK 2.3. Under some additional assumptions on X and Y the above inequality becomes an equality. Namely, in [ATW] it was proved that if $X \cap Y = \{\mathbf{0}\}$ then $\dim C_0(X,Y) = \dim_0 X + \dim_0 Y$. Of course, this is no longer true in the general case.

Before stating the next proposition we make precise some notions concerning analytic curves. By *an analytic curve* we mean an analytic set Γ of pure dimension 1 in an open set $U \subset \mathbb{C}^n$. For $P \in \Gamma$ we denote by $(\Gamma)_P$ the germ of Γ at P and by $\operatorname{mult}_P \Gamma$ *the multiplicity of* Γ *at* P. *A parametrization of* Γ *at* P is a holomorphic homeomorphism $\Phi : K(r) \to U$ (where $K(r) := \{z \in \mathbb{C} : |z| < r\}$ is an open disc) such that $\Phi(0) = P$ and $\Phi(K(r)) = \Gamma \cap U'$ (where $U' \subset U$ is an open neighbourhood of P). Then any superposition $\Phi(t^k)$, $k \in \mathbb{N}$, is called *a description of* X *at* P. It is known that any analytic curve Γ such that $(\Gamma)_P$ is irreducible has a parametrization. If $\mathbf{0} \neq \Phi = (\varphi_1, \dots, \varphi_n)$, and $\Phi(0) = \mathbf{0}$, then we define

$$\operatorname{ord} \Phi := \min(\operatorname{ord} \varphi_1, \dots, \operatorname{ord} \varphi_n).$$

If Φ is a parametrization of Γ at $\mathbf{0}$ then we have

$$\operatorname{mult}_0 \Gamma = \operatorname{ord} \Phi.$$

It is well known that if Γ is an analytic curve in a neighbourhood U of $\mathbf{0} \in \mathbb{C}^n$ and Φ is its parametrization at $\mathbf{0}$ then $C_0(\Gamma)$ is a line $\mathbb{C}\mathbf{v}$, where

$$\mathbf{v} = \lim_{t\to 0} \frac{\Phi(t)}{t^{\operatorname{ord}\Phi}}.$$

We will shortly denote this property by

$$\Phi(t) \underset{t\to 0}{\rightsquigarrow} \mathbf{v},$$

or in the more condensed form $\Phi(t) \rightsquigarrow \mathbf{v}$. Note that for any vector $\mathbf{w} \in \mathbb{C}\mathbf{v}$ there exists a change of parameter $t \to \alpha t$, $\alpha \in \mathbb{C}$, such that $\Phi(\alpha t) \rightsquigarrow \mathbf{w}$. Thus Φ gives the whole line $\mathbb{C}\mathbf{v}$ instead of just the vector $\mathbf{v}$. Therefore we will also use the notation $\Phi(t) \rightsquigarrow \mathbf{w}$ for any $\mathbf{w} \in \mathbb{C}\mathbf{v}$.

Proposition 2.4. *Assume that* $\dim_0(X \cup Y) > 0$. *For any vector* $\mathbf{0} \neq \mathbf{v} \in C_0(X,Y)$ *there exists an analytic curve* $\Gamma \subset X \times Y$ *having a parametrization* $\Phi = (\Phi_X, \Phi_Y) : K(r) \to X \times Y$ *at* $(\mathbf{0},\mathbf{0})$ *such that*

$$\Phi_Y(t) - \Phi_X(t) \rightsquigarrow \mathbf{v}.$$

Proof. Consider the analytic set $B'' \subset U \times U \times \mathbb{C}^n$ defined in the proof of Proposition 2.1. We have $P := (\mathbf{0},\mathbf{0},\mathbf{v}) \in B''$. Since this point lies in the closure of $B' - (\Delta \times \mathbb{C}^n)$, there exists an analytic curve $\Gamma' \subset B''$ passing through P such that $\Gamma' - \{P\} \subset B' - (\Delta \times \mathbb{C}^n)$. Take a parametrization $(\Phi_X(t), \Phi_Y(t), \mathbf{v}(t))$, $t \in K(r)$, at P of one irreducible component of $(\Gamma')_P$. We have $(\Phi_X(0), \Phi_Y(0), \mathbf{v}(0)) = (\mathbf{0},\mathbf{0},\mathbf{v})$. Since for any $t \in K(r)$, $\Phi_Y(t) - \Phi_X(t)$ and $\mathbf{v}(t)$ are linearly dependent and $\mathbf{v}(t) \to \mathbf{v}$ when $t \to 0$ we have $\Phi_Y(t) - \Phi_X(t) \rightsquigarrow \mathbf{v}$. □

Proposition 2.5 ([ATW, Prop. 2.10] in the case $X \cap Y = \{\mathbf{0}\}$). $C_0(X) + C_0(Y) \subset C_0(X,Y)$.

Proof. Let $\mathbf{0} \neq \mathbf{v} \in C_0(X)$, $\mathbf{0} \neq \mathbf{w} \in C_0(Y)$. Since $C_0(X)$ is a cone, we have $-\mathbf{v} \in C_0(X)$. Take analytic curves $\Gamma \subset X$ and $\Gamma' \subset Y$ having parametrizations $\Phi(t)$ and $\Psi(t)$ at $\mathbf{0}$, $t \in K(r)$, such that $\Phi(t) \rightsquigarrow -\mathbf{v}$ and $\Psi(t) \rightsquigarrow \mathbf{w}$. Since $\Phi(t^{\operatorname{ord}\Psi}) \in X$ and $\Psi(t^{\operatorname{ord}\Phi}) \in Y$ for sufficiently small t and

$$\Psi(t^{\operatorname{ord}\Phi}) - \Phi(t^{\operatorname{ord}\Psi}) \rightsquigarrow \mathbf{v} + \mathbf{w},$$

we conclude $\mathbf{v} + \mathbf{w} \in C_0(X,Y)$. □

We will need in the sequel the following proposition which was proved in [ATW, Prop. 2.10]. For completeness we shall give another proof of it using Proposition 2.4.

Proposition 2.6. *If* $C_0(X) \cap C_0(Y) = \{\mathbf{0}\}$ *then*

$$C_0(X,Y) = C_0(X) + C_0(Y).$$

Proof. It suffices to prove

$$C_0(X,Y) \subset C_0(X) + C_0(Y).$$

Take $\mathbf{0} \neq \mathbf{w} \in C_0(X,Y)$. We may assume that $\mathbf{w} \notin C_0(X) \cup C_0(Y)$. By Proposition 2.4 there exists an analytic curve $\Gamma \subset X \times Y$ having a parametrization $\Phi = (\Phi_X, \Phi_Y) : K(r) \to X \times Y$ at $(\mathbf{0}, \mathbf{0})$ such that

$$\Phi_Y(t) - \Phi_X(t) \rightsquigarrow \mathbf{w}.$$

Since $\mathbf{w} \notin C_0(X)$ and $\mathbf{w} \notin C_0(Y)$, we have

$$\operatorname{ord} \Phi_Y = \operatorname{ord} \Phi_X < +\infty. \tag{2}$$

Let

$$\Phi_X(t) \rightsquigarrow \mathbf{v}_1, \quad \mathbf{0} \neq \mathbf{v}_1 \in C_0(X),$$
$$\Phi_Y(t) \rightsquigarrow \mathbf{v}_2, \quad \mathbf{0} \neq \mathbf{v}_2 \in C_0(Y).$$

Since $C_0(X) \cap C_0(Y) = \{\mathbf{0}\}$, $\mathbf{v}_1$and $\mathbf{v}_2$ are linearly independent. Hence, using (2), we have

$$\Phi_Y(t) - \Phi_X(t) \rightsquigarrow \mathbf{v}_2 - \mathbf{v}_1.$$

Thus $\mathbf{w} = \mathbf{v}_2 - \mathbf{v}_1 \in C_0(X) + C_0(Y)$. □

Let now X, Y be analytic subsets of a neighbourhood U of a point $P \in \mathbb{C}^n$ such that $P \in X \cap Y$. We define the *relative tangent cone* $C_P(X,Y)$ *of* X *and* Y *at* P by

$$C_P(X,Y) := P + C_0(X - P, Y - P).$$

3. Relative tangent cones to analytic curves

In the case X, Y are analytic curves we can give a more detailed description of $C_0(X,Y)$. The aim of this section is to give an effective formula for $C_0(X,Y)$ in terms of local parametrizations of X and Y.

First, we formulate a useful lemma which is a simple generalization of Proposition 2.4.

LEMMA 3.1. *Let* X, Y *be analytic curves in a neighbourhood of* $\mathbf{0} \in \mathbb{C}^n$ *such that* $\mathbf{0} \in X \cap Y$ *and the germs* $(X)_\mathbf{0}, (Y)_\mathbf{0}$ *are irreducible. Let* $\Phi(t)$ *and* $\Psi(\tau)$, $t, \tau \in K(r)$, *be parametrizations of* X *and* Y *at* $\mathbf{0}$. *Then for any* $\mathbf{v} \in C_0(X,Y)$ *there exists an analytic curve* $\Gamma \subset K(r) \times K(r)$ *having a parametrization* $\Theta(s) = (t(s), \tau(s)) : K(r') \to K(r) \times K(r)$ *at* $(\mathbf{0}, \mathbf{0})$ *such that*

$$\Phi(t(s)) - \Psi(\tau(s)) \rightsquigarrow \mathbf{v}.$$

Moreover, we have the same result if Φ *and* Ψ *are only descriptions of* X *and* Y *at* $\mathbf{0}$.

Proof. The result follows from Proposition 2.4 and the fact that the mapping (Φ,Ψ) is an analytic cover. □

Now we prove a key proposition for a description of relative tangent cones. This proposition was proved by Ciesielska [C] in the case $X \cap Y = \{\mathbf{0}\}$, but the idea of her proof can be used in the more general case $\mathbf{0} \in X \cap Y$.

Proposition 3.2. *Let X, Y be analytic curves in a neighbourhood of $\mathbf{0} \in \mathbb{C}^n$ such that $\mathbf{0} \in X \cap Y$. Then*

$$C_0(X, Y) + C_0(X) = C_0(X, Y).$$

Proof. We may assume that the germs $(X)_\mathbf{0}, (Y)_\mathbf{0}$ are irreducible. It suffices to prove that

$$C_0(X, Y) + C_0(X) \subset C_0(X, Y). \tag{3}$$

Since X, Y are analytic curves and $(X)_\mathbf{0}, (Y)_\mathbf{0}$ are irreducible at $\mathbf{0}$, we have two possible cases:

Case 1. $C_0(X) \cap C_0(Y) = \{\mathbf{0}\}$. Then, by Proposition 2.6, $C_0(X, Y) = C_0(X) + C_0(Y)$. Hence we get (3).

Case 2. $C_0(X) = C_0(Y)$. After a linear change of coordinates in $\mathbb{C}^n$ we may assume that $C_0(X) = \mathbb{C}\mathbf{e}_1$, where $\mathbf{e}_1 := (1, 0, \ldots, 0)$. Put $k := \operatorname{mult}_0 X$, $l := \operatorname{mult}_0 Y$. Let Φ and Ψ be parametrizations of X and Y at $\mathbf{0}$, respectively. Since $C_0(X) = C_0(Y) = \mathbb{C}\mathbf{e}_1$, we may assume that

$$\Phi(t) = (t^k, \phi_2(t), \ldots, \phi_n(t)),\ t \in K(r),\ \operatorname{ord} \phi_i > k, \quad i = 2, \ldots, n, \tag{4}$$

$$\Psi(\tau) = (\tau^l, \psi_2(\tau), \ldots, \psi_n(\tau)),\ \tau \in K(r),\ \operatorname{ord} \psi_i > l, \quad i = 2, \ldots, n. \tag{5}$$

Consider the descriptions of X and Y

$$\tilde{\Phi}(t) := \Phi(t^l) = (t^{kl}, \phi_2(t^l), \ldots, \phi_n(t^l)),\ \ t \in K(\tilde{r}),$$
$$\tilde{\Psi}(\tau) := \Psi(\tau^k) = (\tau^{kl}, \psi_2(\tau^k), \ldots, \psi_n(\tau^k)),\ \ \tau \in K(\tilde{r}),$$

where $\tilde{r}$ is a sufficiently small positive number.

Take now $\mathbf{0} \neq \mathbf{v} = (v_1, \ldots, v_n) \in C_0(X, Y)$ and $\mathbf{w} = (w, 0, \ldots, 0) \in C_0(X)$. From Lemma 3.1 there is an analytic curve $\Gamma \subset K(\tilde{r}) \times K(\tilde{r})$ having a parametrization $\Theta(s) = (t(s), \tau(s)) : K(r') \to K(\tilde{r}) \times K(\tilde{r})$ at $(\mathbf{0}, \mathbf{0})$ such that

$$\tilde{\Phi}(t(s)) - \tilde{\Psi}(\tau(s)) \rightsquigarrow \mathbf{v}.$$

Define

$$N := \operatorname{ord}(\tilde{\Phi}(t(s)) - \tilde{\Psi}(\tau(s))).$$

Then

$$\mathbf{v} = \lim_{s \to 0} \frac{\tilde{\Phi}(t(s)) - \tilde{\Psi}(\tau(s))}{s^N}.$$

Since Θ is a parametrization of a curve we have that $t(s)$ or $\tau(s)$ is not identically zero. Without loss of generality, we may assume that $t(s) \not\equiv 0$

and $\operatorname{ord} t(s) \leqslant \operatorname{ord} \tau(s)$. Put $p := \operatorname{ord} t(s)$. Hence $N \geqslant pkl$. Without loss of generality, we may assume that $t(s) = s^p$. We define

$$\tilde{t}(s) := s^p + \frac{w}{kl} s^{p+N-pkl}.$$

We claim that

$$\tilde{\Phi}(\tilde{t}(s)) - \tilde{\Psi}(\tau(s)) \rightsquigarrow \mathbf{v} + \mathbf{w}.$$

In fact, for the first coordinate we have

$$\lim_{s\to 0} \frac{(\tilde{t}(s))^{kl} - (\tau(s))^{kl}}{s^N}$$
$$= \lim_{s\to 0} \frac{(\tilde{t}(s))^{kl} - (t(s))^{kl} + (t(s))^{kl} - (\tau(s))^{kl}}{s^N} = w + v_1$$

and for $i = 2, \ldots, n$

$$\lim_{s\to 0} \frac{(\phi_i(\tilde{t}(s)^l) - \psi_i(\tau(s)^k)}{s^N}$$
$$= \lim_{s\to 0} \frac{\phi_i(\tilde{t}(s)^l) - \phi_i(t(s)^l) + \phi_i(t(s)^l) - \psi_i(\tau(s)^k)}{s^N} = v_i. \qquad \square$$

From this proposition we obtain the first description of relative tangent cones to analytic curves (cf. [BGG, Prop. IV.1], [C, Cor. 3.2]).

Corollary 3.3. *Let X, Y be analytic curves in a neighbourhood of $\mathbf{0} \in \mathbb{C}^n$ such that $\mathbf{0} \in X \cap Y$, and let $(X)_\mathbf{0}, (Y)_\mathbf{0}$ be irreducible germs at $\mathbf{0}$. Then one of the following two cases may occur:*

1. *$C_0(X,Y) = C_0(X) = C_0(Y)$.*
2. *$C_0(X,Y)$ is a finite union of two-dimensional hyperplanes.*

Proof. If $C_0(X) \cap C_0(Y) = \{\mathbf{0}\}$, then, by Proposition 2.6, $C_0(X,Y) = C_0(X) + C_0(Y)$ is a two-dimensional hyperplane. If $C_0(X) = C_0(Y)$, then taking an $(n-1)$-dimensional hyperplane H through $\mathbf{0}$, transversal to $C_0(X)$, we easily obtain from Proposition 3.2 that

$$C_0(X,Y) = C_0(X,Y) \cap H + C_0(X). \tag{6}$$

Since, by Proposition 2.2, $\dim C_0(X,Y) \leqslant 2$, we have by (6) $\dim C_0(X,Y) \cap H \leqslant 1$. But $C_0(X,Y) \cap H$ is also an algebraic cone. Hence $C_0(X,Y) \cap H$ is either $\{\mathbf{0}\}$ or a finite number of lines. Thus, by (6), $C_0(X,Y)$ is equal to $C_0(X)$ in the first case, and is a finite sum of two-dimensional hyperplanes in the second case. $\square$

Now we give a formula for the $C_0(X,Y)$ in terms of parametrizations of X and Y (cf. the proof of Proposition IV.1 in [BGG]). First we fix some notations. By $\mathbf{e}_1, \ldots, \mathbf{e}_n$ we denote the standard basis of $\mathbb{C}^n$. For vectors $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$ we denote by $\operatorname{Lin}(\mathbf{v}, \mathbf{w})$ the hyperplane in $\mathbb{C}^n$ generated by $\mathbf{v}$ and $\mathbf{w}$. Given a power series $\chi(s) \not\equiv 0$, we denote by $\operatorname{in}(\chi(s))$ its *initial form*;

i.e., if $\chi(s) = \beta_p s^p + \cdots$ with $\beta_p \neq 0$, then $\mathrm{in}(\chi(s)) = \beta_p s^p$. (We also put $\mathrm{in}(0) := 0$.)

THEOREM 3.4. *Let X, Y be analytic curves in a neighbourhood U of the point $\mathbf{0} \in \mathbb{C}^n$ such that $\mathbf{0} \in X \cap Y$, and let $(X)_0$, $(Y)_0$ be irreducible germs. Let*

$$\Phi(t) = (t^k, \phi_2(t), \ldots, \phi_n(t)),\ t \in K(r),\ \mathrm{ord}\,\phi_i > k,\ i = 2, \ldots, n, \tag{7}$$

$$\Psi(\tau) = (\tau^l, \psi_2(\tau), \ldots, \psi_n(\tau)),\ \tau \in K(r),\ \mathrm{ord}\,\psi_i > l,\ i = 2, \ldots, n, \tag{8}$$

be parametrizations of X and Y at $\mathbf{0}$. Assume that $l \leqslant k$. Let $\varepsilon_1, \ldots, \varepsilon_l$ be the roots of unity of degree l. For $i = 1, \ldots, l$ we define

$$n_i := \begin{cases} \mathrm{ord}(\Phi(t^l) - \Psi(\varepsilon_i t^k)) & \text{if } \Phi(t^l) - \Psi(\varepsilon_i t^k) \not\equiv 0, \\ 0 & \text{if } \Phi(t^l) - \Psi(\varepsilon_i t^k) \equiv 0, \end{cases} \tag{9}$$

$$\mathbf{v}_i := \lim_{t \to 0} \frac{\Phi(t^l) - \Psi(\varepsilon_i t^k)}{t^{n_i}}.$$

Then

$$C_0(X, Y) = \mathrm{Lin}(\mathbf{v}_1, \mathbf{e}_1) \cup \cdots \cup \mathrm{Lin}(\mathbf{v}_l, \mathbf{e}_1).$$

Proof. Instead of the parametrizations Φ and Ψ, we shall use descriptions of X and Y. Define

$$\tilde{\Phi}(t) := \Phi(t^l) = (t^{kl}, \phi_2(t^l), \ldots, \phi_n(t^l)), \quad t \in K(r^{1/l}),$$
$$\tilde{\Psi}(\tau) := \Psi(\tau^k) = (\tau^{kl}, \psi_2(\tau^k), \ldots, \psi_n(\tau^k)), \quad \tau \in K(r^{1/k}).$$

Obviously, $(\tilde{\Phi}(K(r^{1/l}))_0 = (X)_0$, $(\tilde{\Psi}(K(r^{1/k}))_0 = (Y)_0$. From the form of $\tilde{\Phi}$ and $\tilde{\Psi}$ we see that

$$C_0(X) = C_0(Y) = \mathbb{C}\mathbf{e}_1.$$

Take the hyperplane

$$H := \{(x_1, \ldots, x_n) \in \mathbb{C}^n : x_1 = 0\},$$

which is transversal to $C_0(X) = C_0(Y)$. From Proposition 3.2 we easily obtain

$$C_0(X, Y) = C_0(X, Y) \cap H + C_0(X).$$

Since $C_0(X, Y)$ is an analytic cone in $\mathbb{C}^n$ of dimension $\leqslant 2$, it follows from this equality that $C_0(X, Y) \cap H$ is either $\{\mathbf{0}\}$ or a finite system of lines. Thus it suffices to prove that

$$C_0(X, Y) \cap H = \bigcup_{i=1}^{l} \mathbb{C}\mathbf{v}_i.$$

By the definition of $\mathbf{v}_i$ we have obviously

$$\bigcup_{i=1}^{l} \mathbb{C}\mathbf{v}_i \subset C_0(X, Y) \cap H.$$

Take now any vector $\mathbf{0} \neq \mathbf{w} \in C_0(X,Y) \cap H$. By Lemma 3.1 there exists an analytic curve $\Gamma \subset K(r^{1/l}) \times K(r^{1/k})$ having a parametrization $\Theta(s) = (t(s), \tau(s)) : K(r') \to K(r^{1/l}) \times K(r^{1/k})$ at $(\mathbf{0}, \mathbf{0})$ such that

$$\left(\tilde{\Phi}(t(s)) - \tilde{\Psi}(\tau(s))\right) \rightsquigarrow \mathbf{w} \text{ when } s \to 0,$$

i.e., such that

$$(t(s)^{kl} - \tau(s)^{kl}, \phi_2(t(s)^l) - \psi_2(\tau(s)^k), \ldots, \phi_n(t(s)^l) - \psi_n(\tau(s)^k)) \rightsquigarrow \mathbf{w}$$

when $s \to 0$. Since $t(s) \not\equiv 0$ or $\tau(s) \not\equiv 0$ we may assume that $t(s) \not\equiv 0$. Changing the parameter s we may further assume that

$$t(s) = s^p, \ p \in \mathbb{N}.$$

Thus

$$(s^{pkl} - \tau(s)^{kl}, \phi_2(s^{pl}) - \psi_2(\tau(s)^k), \ldots, \phi_n(s^{pl}) - \psi_n(\tau(s)^k)) \rightsquigarrow \mathbf{w}$$

when $s \to 0$. Since $\mathbf{w} = (0, w_2, \ldots, w_n) \neq 0$, there exists $j \in \{2, \ldots, n\}$ such that

$$\operatorname{ord}(\phi_j(s^{pl}) - \psi_j(\tau(s)^k) < \operatorname{ord}(s^{pkl} - \tau(s)^{kl}). \tag{10}$$

Denote by J the set of $j \in \{2, \ldots, n\}$ for which the above inequality holds. Since $\operatorname{ord} \phi_j > k$ and $\operatorname{ord} \psi_j > l$, we obtain from this inequality that $\tau(s)$ has the form

$$\tau(s) = \alpha_p s^p + \alpha_{p+1} s^{p+1} + \cdots, \ \ \alpha_p^{kl} = 1.$$

Hence $\alpha_p^k = \varepsilon_{i_0}$ for some $i_0 \in \{1, \ldots, l\}$. We shall show that $\mathbf{w} = \mathbf{v}_{i_0}$. We consider the following cases:

Case 1. The coefficients α_r all vanish for $r > p$, i.e., $\tau(s) = \alpha_p s^p$. Then $\tau(s)^k = \alpha_p^k s^{pk} = \varepsilon_{i_0} s^{pk}$. Hence we have $\mathbf{w} = \mathbf{v}_{i_0}$.

Case 2. Not all the coefficients α_r vanish for $r > p$. Let m be the smallest positive integer such that $\alpha_{p+m} \neq 0$. Then

$$\tau(s) = \alpha_p s^p + \alpha_{p+m} s^{p+m} + \cdots,$$

$$\tau(s)^k = \varepsilon_{i_0} s^{pk} + \alpha s^{pk+m} + \cdots, \ \ \alpha \neq 0, \tag{11}$$

$$\operatorname{ord}(s^{pkl} - \tau(s)^{kl}) = pkl + m, \tag{12}$$

$$\operatorname{ord}(\phi_j(s^{pl}) - \psi_j(\tau(s)^k) < pkl + m \quad \text{for} \quad j \in J, \tag{13}$$

$$\operatorname{ord}(\phi_j(s^{pl}) - \psi_j(\tau(s)^k) \geqslant pkl + m \quad \text{for} \quad j \notin J. \tag{14}$$

Let us first note that for $j \in \{2, \ldots, n\}$ we have from (11) and the fact that $\operatorname{ord} \psi_j > l$

$$\operatorname{ord}(\psi_j(\tau(s)^k) - \psi_j(\varepsilon_{i_0} s^{pk})) \geqslant pkl + m. \tag{15}$$

From this and (13) we deduce for $j \in J$

$$\begin{aligned} (16) \qquad & \operatorname{in}\left(\phi_j(s^{pl}) - \psi_j(\tau(s)^k\right) \\ &= \operatorname{in}\left(\phi_j(s^{pl}) - \psi_j(\varepsilon_{i_0}s^{pk}) + \psi_j(\varepsilon_{i_0}s^{pk}) - \psi_j(\tau(s)^k\right) \\ &= \operatorname{in}\left(\phi_j(s^{pl}) - \psi_j(\varepsilon_{i_0}s^{pk})\right), \end{aligned}$$

and for $j \notin J$ we get from (14)

$$\begin{aligned} (17) \qquad & \operatorname{ord}\left(\phi_j(s^{pl}) - \psi_j(\varepsilon_{i_0}s^{pk})\right) \\ &= \operatorname{ord}\left(\phi_j(s^{pl}) - \psi_j(\tau(s)^k) + \psi_j(\tau(s)^k) - \psi_j(\varepsilon_{i_0}s^{pk})\right) \\ &\geqslant pkl + m. \end{aligned}$$

Hence

$$(18) \qquad \operatorname{ord}\left(\Phi(s^{pl}) - \Psi(\tau(s)^k\right) = \operatorname{ord}\left(\Phi(s^{pl}) - \Psi(\varepsilon_{i_0}s^{pk})\right) = pn_{i_0}.$$

Now, we have

$$\begin{aligned} \mathbf{v}_{i_0} &= \lim_{t\to 0} t^{-n_{i_0}}\left(\Phi(t^l) - \Psi(\varepsilon_{i_0}t^k)\right) \\ &= \lim_{s\to 0} s^{-pn_{i_0}}\left(\Phi(s^{pl}) - \Psi(\varepsilon_{i_0}s^{pk})\right) \\ &= \lim_{s\to 0} s^{-pn_{i_0}}(0, \phi_2(s^{pl}) - \psi_2(\varepsilon_{i_0}s^{pk}), \ldots, \phi_n(s^{pl}) - \psi_n(\varepsilon_{i_0}s^{pk})) \\ &= \lim_{s\to 0} s^{-pn_{i_0}}\left(0, \operatorname{in}\left(\phi_2(s^{pl}) - \psi_2(\varepsilon_{i_0}s^{pk})\right), \ldots, \operatorname{in}\left(\phi_n(s^{pl}) - \psi_n(\varepsilon_{i_0}s^{pk})\right)\right). \end{aligned}$$

On the other hand, from definition of $\mathbf{w}$ and (18) we have

$$\begin{aligned} \mathbf{w} &= \lim_{s\to 0} \frac{\left(\Phi(s^{pl}) - \Psi(\tau(s)^k)\right)}{s^{\operatorname{ord}(\Phi(s^{pl}) - \Psi(\tau(s)^k))}} \\ &= \lim_{s\to 0} s^{-pn_{i_0}}\left(\Phi(s^{pl}) - \Psi(\tau(s)^k)\right) \\ &= \lim_{s\to 0} s^{-pn_{i_0}}\left(s^{pkl} - \tau(s)^k, \phi_2(s^{pl}) - \psi_2(\tau(s)^k), \ldots, \phi_n(s^{pl}) - \psi_n(\tau(s)^k)\right) \\ &= \lim_{s\to 0} s^{-pn_{i_0}}\left(\operatorname{in}\left(s^{pkl} - \tau(s)^k\right), \operatorname{in}\left(\phi_2(s^{pl}) - \psi_2(\tau(s)^k)\right), \right. \\ &\qquad \left. \ldots, \operatorname{in}\left(\phi_n(s^{pl}) - \psi_n(\tau(s)^k)\right)\right). \end{aligned}$$

Using (12), (16), (14), (17) we finally obtain

$$\mathbf{v}_{i_0} = \mathbf{w}.$$

This completes the proof. □

REMARK 3.5. From the forms (7) and (8) of the parametrizations it follows that $C_0(X) = C_0(Y) = \mathbb{C}\mathbf{e}_1$. By Proposition 2.6 we see that only this case is interesting. Moreover, the assumption on the form of the parametrizations is not restrictive, because it is well-known that for any analytic curve X with irreducible germ at $\mathbf{0}$ there exists a linear change of coordinates in $\mathbb{C}^n$ such that in the new coordinates $C_0(X) = \mathbb{C}\mathbf{e}_1$ and there exists a parametrization of X at $\mathbf{0}$ of the form (7).

From the above theorem it follows that under the same assumptions on X and Y, the cone $C_0(X, Y)$ is the union of at most $\min(\operatorname{mult}_0 X, \operatorname{mult}_0 Y)$ two-dimensional hyperplanes. It is easy to improve this result.

Corollary 3.6. *Let X, Y be analytic curves in a neighbourhood U of the point $\mathbf{0} \in \mathbb{C}^n$ such that $\mathbf{0} \in X \cap Y$ and $(X)_0$, $(Y)_0$ are irreducible germs. Then:*

1. *If $(X)_0 = (Y)_0$ and this germ is nonsingular at $\mathbf{0}$, then*
$$C_0(X, X) = C_0(X) = T_P X.$$
2. *In the remaining cases $C_0(X, Y)$ is the union of r two-dimensional hyperplanes, where*

$$1 \leqslant r \leqslant \gcd(\operatorname{mult}_0 X, \operatorname{mult}_0 Y). \tag{19}$$

Proof. Using a linear change of coordinates in $\mathbb{C}^n$ we may assume that X and Y satisfy all assumptions of Theorem 3.4.

The first part follows immediately from Theorem 3.4 because in this case $k = l = 1$, $\Phi(t) = \Psi(t) = (t, \varphi_2(t), \ldots, \varphi_n(t))$, $\operatorname{ord} \varphi_i > 1$, $i = 1, \ldots, n$, and $\mathbf{v}_1 = \mathbf{0}$.

We now prove the second part. From Theorem 3.4 we obtain

$$C_0(X, Y) = \operatorname{Lin}(\mathbf{v}_1, \mathbf{e}_1) \cup \cdots \cup \operatorname{Lin}(\mathbf{v}_l, \mathbf{e}_1),$$

where

$$\mathbf{v}_i = \lim_{t \to 0} \frac{\Phi(t^l) - \Psi(\varepsilon_i t^k)}{t^{n_i}}, \quad i = 1, \ldots, l,$$

$\Phi(t), \Psi(t)$ are parametrizations of X and Y at $\mathbf{0}$ of the form (7) and (8), $l \leqslant k$, ε_i, $i = 1, \ldots, l$, are the roots of unity of degree l, and the numbers n_i are given by (9).

By analysing this formula in the two possible cases in this part, i.e., the case when $(X)_0 \neq (Y)_0$ and the case when $(X)_0 = (Y)_0$ and $\mathbf{0}$ is a singular point of X, we easily obtain that $r \geqslant 1$.

Now, let $D := \gcd(\operatorname{mult}_0 X, \operatorname{mult}_0 Y) = \gcd(k, l)$ and let $\eta_1, \ldots, \eta_D$ be the roots of unity of degree D. It is easy to see that for any ε_i there exists ε_j such that $\varepsilon_i \varepsilon_j^k = \eta_p$ for some $p \in \{1, \ldots, D\}$. Then by the substitution $t \mapsto \varepsilon_j t$ we obtain

$$\mathbf{v}_i = \lim_{t \to 0} \frac{\Phi((\varepsilon_j t)^l) - \Psi(\varepsilon_i (\varepsilon_j t)^k)}{(\varepsilon_j t)^{n_i}} = \varepsilon_j{}^{-n_i} \lim_{t \to 0} \frac{\Phi(t^l) - \Psi(\eta_p t^k)}{t^{n_i}}.$$

Thus there are at most D different lines among $\mathbb{C}\mathbf{v}_1, \ldots, \mathbb{C}\mathbf{v}_l$. Hence $r \leqslant D$. □

EXAMPLE 3.7.

1. The estimation from above in (19) is strict since for

$$X := \{(t^2, t^3, 0) : t \in \mathbb{C}\} \subset \mathbb{C}^3,$$
$$Y := \{(\tau^2, 0, \tau^3) : \tau \in \mathbb{C}\} \subset \mathbb{C}^3.$$

we have by Theorem 3.4 $k = l = 2$ and $\mathbf{v}_1 = [0, 1, 1]$, $\mathbf{v}_2 = [0, 1, -1]$. Hence $r = 2$ and

$$C_0(X, Y) = \mathrm{Lin}(\mathbf{v}_1, \mathbf{e}_1) \cup \mathrm{Lin}(\mathbf{v}_2, \mathbf{e}_1) = \{(x, y, z) \in \mathbb{C}^3 : y^2 - z^2 = 0\}.$$

2. The inequality in the upper bound for r in (19) is not an equality in general since for

$$X := \{(t^2, t^5, 0) : t \in \mathbb{C}\} \subset \mathbb{C}^3,$$
$$Y := \{(\tau^2, 0, \tau^3) : \tau \in \mathbb{C}\} \subset \mathbb{C}^3.$$

we have by Theorem 3.4 $k = l = 2$ and $\mathbf{v}_1 = \mathbf{v}_2 = [0, 0, 1]$. Hence $r = 1$.

4. The join of algebraic curves

In this section we answer the question posed in the introduction: Which additional projective lines besides those containing points $P \in X, Q \in Y, P \neq Q$, are in $\mathcal{J}(X, Y)$ in the case when X and Y are algebraic curves? First, we give a relation between the join of arbitrary varieties and relative tangent cones.

Let X, Y be arbitrary algebraic subsets of $\mathbb{P}^n$ and $P \in X \cap Y$. Let $U \subset \mathbb{P}^n$ be a canonical affine part of $\mathbb{P}^n$ such that $P \in U$, and let $\varphi : U \to \mathbb{C}^n$ the corresponding canonical map. Then we define the relative tangent cone $C_P(X, Y)$ to X and Y at P by

$$C_P(X, Y) := \overline{\varphi^{-1}(C_{\varphi(P)}(\varphi(X \cap U), \varphi(Y \cap U))}.$$

One can easily check that this definition does not depend on the choice of the canonical affine part U of $\mathbb{P}^n$. (In [FOV, Def. 4.3.6] there is another equivalent definition of $C_P(X, Y)$ using the affine cones $\hat{X}, \hat{Y} \subset \mathbb{C}^{n+1}$ generated by X and Y.)

Since $C_P(X, Y)$ is a union of projective lines passing through P we may define

$$\mathcal{C}_P(X, Y) := \{[L] \in G(1, \mathbb{P}^n) : L \subset C_P(X, Y) \text{ and } P \in L\}.$$

PROPOSITION 4.1. *Let X, Y be arbitrary algebraic subsets of $\mathbb{P}^n$. Then*

$$\mathcal{J}(X, Y) = \mathcal{J}^0(X, Y) \cup \bigcup_{P \in X \cap Y} \mathcal{C}_P(X, Y),$$
$$J(X, Y) = J^0(X, Y) \cup \bigcup_{P \in X \cap Y} C_P(X, Y).$$

Proof. Note that the topology in $G(1,\mathbb{P}^n)$ can be described in the following elementary way: If $[L],[L_i] \in G(1,\mathbb{P}^n)$, $i=1,2,\dots$, then $[L_i] \to [L]$ when $i \to \infty$ in $G(1,\mathbb{P}^n)$ if and only if there exist points $P_i, Q_i \in L_i$, $i=1,2,\dots$, $P_i \neq Q_i$, $P,Q \in L$, $P \neq Q$, with homogeneous coordinates $P_i = (x_0^i : \cdots : x_n^i)$, $Q_i = (y_0^i : \cdots : y_n^i)$, $P = (x_0 : \cdots : x_n)$, $Q = (y_0 : \cdots : y_n)$ such that $x_j^i \to x_j$ and $y_j^i \to y_j$ when $i \to \infty$ in $\mathbb{C}$ for $j = 0,1,\dots,n$.

Take $[L] \in \mathcal{J}(X,Y) - \mathcal{J}^0(X,Y)$. Then there exist $[\overline{P_iQ_i}] \in G(1,\mathbb{P}^n)$, $i = 1,2,\dots$, $P_i \in X$, $Q_i \in Y$, $P_i \neq Q_i$, such that $[\overline{P_iQ_i}] \to [L]$ when $i \to \infty$. Since X,Y are compact sets we may assume that $P_i \to P \in X$ and $Q_i \to Q \in Y$. Since $[L] \notin \mathcal{J}^0(X,Y)$, we have $P = Q$. Hence $P \in X \cap Y$. Of course, $P \in L$. From the above description of the topology in $G(1,\mathbb{P}^n)$ we easily obtain that $L \subset C_P(X,Y)$.

The opposite inclusion $\bigcup_{P \in X\cap Y} \mathcal{C}_P(X,Y) \subset \mathcal{J}(X,Y)$ is obvious. □

From the above proposition and the previous results we obtain the full description of the join of algebraic curves in $\mathbb{P}^n$.

THEOREM 4.2. *Let X,Y be irreducible curves in $\mathbb{P}^n$. Then:*

1. *If $X = Y$ then*

$$\mathcal{J}(X,X) = \mathcal{J}^0(X,X) \cup \bigcup_{P\in \operatorname{Sing}(X)} \mathcal{C}_P(X,X) \cup \bigcup_{P \in X - \operatorname{Sing}(X)} [T_P(X)],$$

$$J(X,X) = J^0(X,X) \cup \bigcup_{P\in \operatorname{Sing}(X)} C_P(X,X) \cup \bigcup_{P \in X - \operatorname{Sing}(X)} T_P(X).$$

2. *If $X \neq Y$ and $X \cap Y = \{P_1,\dots,P_k\}$ then*

$$\mathcal{J}(X,Y) = \mathcal{J}^0(X,Y) \cup \bigcup_{i=1}^{k} \mathcal{C}_{P_i}(X,Y),$$

$$J(X,Y) = J^0(X,Y) \cup \bigcup_{i=1}^{k} C_{P_i}(X,Y).$$

Moreover, in both cases each $C_P(X,Y)$ is a finite sum of projective two-dimensional hyperplanes passing through P. They are effectively described as follows: For a given point $P \in X \cap Y$ if $X \neq Y$, or for a singular point P of X if $X = Y$, we decompose $(X)_P = (X_1)_P \cup \cdots \cup (X_r)_P$, $(Y)_P = (Y_1)_P \cup \cdots \cup (Y_s)_P$ into irreducible curve-germs. Then

$$C_P(X,Y) = \bigcup_{i,j} C_P(X_i,Y_j).$$

Each $C_P(X_i,Y_j)$ is described in the following way:

(i) *If $(X_i)_P = (Y_j)_P$ and this germ is nonsingular, then*

$$C_P(X_i,Y_j) = C_P(X_i) = C_P(Y_j) = T_PX_i = T_PY_j.$$

(ii) *If* $(X_i)_P \neq (Y_j)_P$ *or one of these germs is singular, then:*
(1) *If* $C_P(X_i) \cap C_P(Y_j) = \{P\}$ *then*

$$C_P(X_i, Y_j) = \operatorname{Span}(C_P(X_i), C_P(Y_j)).$$

(2) *If* $C_P(X_i) = C_P(Y_j)$ *then*

$$C_P(X_i, Y_j) = \bigcup_{l=1}^{m} \operatorname{Span}(C_P(X_i), \overline{PQ_l}),$$
$$1 \leqslant m \leqslant \gcd(\operatorname{mult}_P X_i, \operatorname{mult}_P Y_j),$$

where $Q_l := \varphi^{-1}(\varphi(P) + \mathbf{v}_l)$ *(where* $\varphi : U \to \mathbb{C}^n$ *is a canonical map of* $\mathbb{P}^n$ *such that* $P \in U$*) and the vectors* $\mathbf{v}_l$ *are calculated from the local parametrization of the curves* $\varphi(X_i) - \varphi(P)$ *and* $\varphi(Y_j) - \varphi(P)$ *at* $\mathbf{0}$*, as described in Theorem 3.4 (after a linear change of coordinates in* $\mathbb{C}^n$*).*

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