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EXPANSION OF SOLUTIONS OF PARAMETERIZED EQUATIONS AND ACCELERATION OF NUMERICAL METHODS

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In the memory of J. Doob

Abstract. A general scheme of parameterized families of equations is considered, and abstract results on the expansion of the solutions and on the acceleration of their convergence in terms of the parameter are presented. These results are applied to fractional step approximations for linear parabolic PDEs, systems of linear PDEs, and for nonlinear ordinary differential equations. Applications to accelerating the convergence of finite difference schemes for these equations will be presented in a subsequent paper.

1. Introduction

We consider for every 'parameter' $\tau \in [0,1]$ a pair of equations

$$v = \varphi + A_0\Theta_0(Lv + f), \tag{1.1}$$

$$w = \varphi + \sum_{k=1}^{m} A_k\Theta_k(L_k w + f_k) \tag{1.2}$$

in a separable Banach space W, where

$$\varphi = \varphi(\tau), \quad f_k = f_k(\tau), \quad f = f(\tau)$$

are given elements of W, and

$$L = L(\tau), \quad L_k = L_k(\tau), \quad A_k = A_k(\tau), \quad A_0 = A_0(\tau),$$
$$\Theta_k = \Theta_k(\tau), \quad \Theta_0 = \Theta_0(\tau)$$

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are given linear operators in W for every τ and $k = 1, 2, \ldots, m$. The operators L, L_k may be unbounded. We assume that

$$L = L_1 + L_2 + \cdots + L_m, \quad f = f_1 + f_2 + \cdots + f_m. \tag{1.3}$$

Together with equations (1.1) and (1.2) a subset $W^*(\tau)$ of the dual space W^* is also assumed to be given for every τ.

Our aim is to investigate the dependence on τ of $w - v$, the difference of the solutions of (1.1) and (1.2). Under general conditions we obtain an expansion for $\langle w^*, w\rangle - \langle w^*, v\rangle$ in terms of powers of τ, where $\langle w^*, w\rangle$ denotes the duality product of $w = w(\tau)$ and $w^* = w^*(\tau) \in W^*(\tau)$. When A_0, Θ_0, L, and f are independent of τ, this result reads as follows:

For any integer $k \geq 0$ there exists $v_0 := v, v_1, v_2, \ldots, v_k \in W$, independent of τ, such that

$$\langle w^*, w\rangle = \sum_{i=0}^{k} \tau^i \langle w^*, v_i\rangle + O(\tau^{k+1})$$

for all $\tau \in (0, 1]$ and $w^*(\tau) \in W^*(\tau)$, where

$$|O(\tau^{k+1})| \leq N\tau^{k+1}\|w^*\|$$

with a constant N independent of τ. This is what Theorems 2.14 and 2.18 below are about. Hence we easily get that under the conditions of this result there exist some constants $\lambda_0, \ldots, \lambda_k$, depending only on k, such that

$$\left| \sum_{j=0}^{k} \lambda_j \langle w^*, w_j\rangle - \langle w^*, v\rangle \right| \leq N\tau^{k+1}\|w^*\|$$

for every $\tau \in (0, 1]$, $w^* \in \bigcap_{j=0}^{k} W^*(\tau_j)$, where $\tau_j := 2^{-j}\tau$, $w_j = w(\tau_j)$, and N is a constant, independent of τ (see Theorem 2.15 below).

These results are motivated by applications to numerical methods of solving ordinary and partial differential equations. We apply them in the present paper to accelerating splitting-up approximations for a class of linear PDEs and also to nonlinear ODEs, and we indicate further applications to finite difference schemes. We will present applications to accelerating the convergence of finite difference schemes for linear PDEs in a subsequent paper.

The splitting-up method appears first in the context of semigroups as Trotter's formula [14], which can be formulated as follows:

$$\lim_{n\to\infty} \left(e^{tL_m/n} \ldots e^{tL_1/n}\right)^n = e^{tL}\varphi, \quad \forall \varphi \in \mathbb{B},$$

where $L := L_1 + L_2 + \cdots + L_m$ and L_k are infinitesimal generators of C_0-semigroups of contractions $\{e^{tL} : t \geq 0\}$ and $\{e^{tL_k} : t \geq 0\}$ on a Banach space $\mathbb{B}$, such that the intersection of the domains of the generators L_k is dense in

$\mathbb{B}$. Clearly, in the context of Cauchy problems Trotter's formula states the convergence of the *splitting-up approximations*, defined by $w(t) = \mathbb{S}^n(t/n)v_0$,

$$\mathbb{S}(\tau) := \mathbb{P}_\tau^{(m)} \dots \mathbb{P}_\tau^{(2)}\mathbb{P}_\tau^{(1)}, \quad \tau > 0, \tag{1.4}$$

to the solution of the abstract Cauchy problem

$$\frac{d}{dt}v(t) = Lv(t), \quad v(0) = v_0,$$

where $\mathbb{P}_\tau^{(k)}\varphi$ is the solution at τ of

$$\frac{d}{dt}u(t) = L_k u(t), \quad u(0) = \varphi.$$

In Section 4 we will see how the splitting-up given by (1.4) can be obtained from our abstract scheme (1.1)–(1.2).

Under certain conditions one knows that for every fixed $T > 0$

$$\max_{t\in T_\tau} \|\mathbb{S}^{t/\tau}(\tau)v_0 - v(t)\|_{\mathbb{B}} \leq N\tau, \quad \forall \tau \in (0, 1],$$

where N is a constant independent of τ and

$$T_\tau := \{i\tau : i = 0, 1, 2, \dots\} \cap [0, T]. \tag{1.5}$$

In other words, the error of the *splitting-up method* $\mathbb{S}$ given by (1.4) is proportional to the step size τ. There are splitting-up methods which are more accurate. A celebrated example is Strang's method

$$\mathbb{S}(\tau) := \mathbb{P}_{\tau/2}^{(1)}\mathbb{P}_{\tau/2}^{(2)} \dots \mathbb{P}_{\tau/2}^{(m)}\mathbb{P}_{\tau/2}^{(m)} \dots \mathbb{P}_{\tau/2}^{(2)}\mathbb{P}_{\tau/2}^{(1)},$$

introduced in [10], whose error is proportional to τ^2. Inspired by this example, for given $k \geq 2$ one looks for splitting-up methods of the form

$$\mathbb{S}(\tau) := \mathbb{P}_{s_\xi\tau}^{k_\xi} \dots \mathbb{P}_{s_2\tau}^{k_2}\mathbb{P}_{s_1\tau}^{k_1} \tag{1.6}$$

with some integer $\xi \geq m$, real numbers $s_1, \dots, s_\xi$ and integers $k_1, \dots, k_\xi \in \{1, 2, \dots, m\}$ such that the error of the methods is proportional to τ^k. Such methods, called methods of (at least) *order* k, are obtained for Hamiltonian systems and for linear and nonlinear equations by variants of the Trotter and Baker-Campbell-Hausdorff formulas, and by an adaptation of the method of *rooted trees* from the theory of Runge-Kutte approximations (see, e.g., [9] [7], [12], [15], [17], [8] and the references therein). By [11] and [16], however, the numbers s_i in each method (1.6) of order $k \geq 3$ cannot be all non-negative. Thus, by [11] and [16] the above methods of order greater than or equal to 3 cannot be used to approximate the solution of partial differential equations of parabolic type.

Therefore it is natural to ask if there exists, in the case of parabolic equations, a method different from the multiplicative one to accelerate the convergence of splitting-up approximations to a higher order.

In [4] we show, inspired by Richardson's idea, that using a step size of order τ, but organizing the computations differently, one can achieve an accuracy of order τ^k for any k, even if L_r are (degenerate) elliptic operators with coefficients depending on time. Namely, we show that each time when one has *any* algorithm of implementing a splitting-up method to approximating the solutions of Cauchy problems for parabolic equations with sufficiently smooth coefficients and free terms, one can improve the rate of convergence to any degree, by mixing the splitting-up approximations of different step sizes. Since we believe that usually in practice one computes several approximations with different step sizes, we prove that computing, for instance, approximations with three different step sizes, each of which is of accuracy τ, and just taking a linear combination of the results, one gets an approximation of accuracy τ^3.

In the present paper we show that the method of [4] is much more universal in the sense that it covers very many situations in which approximations, depending on a parameter τ, for the solution of an equation can be embedded into the solutions of a family of equations satisfying certain properties.

The paper is organized as follows. In the next section we introduce our general setting, illustrating it by simple examples, and formulate our main results, Theorems 2.14, 2.15, and 2.18. We remark that these theorems are presented without proofs in [5]. Theorem 2.15 follows easily from Theorem 2.14. Theorem 2.14 is a simple consequence of Theorem 2.18, which we prove in Section 3. The rest of the paper is dedicated to applications. In Section 4 we apply the abstract scheme and the theorems of Section 2 to splitting-up approximations of the solutions of parabolic PDEs. In particular, we obtain the results of [4] in the time independent case. In Section 5 similar applications to systems of PDEs, in particular to symmetric systems of first order hyperbolic PDEs, are given. In Section 6 we formulate some applications of the general scheme to splitting-up approximations for ordinary (nonlinear) differential equations. The results of this section are proved in [5].

In conclusion we introduce some notation used everywhere below. Throughout the paper $d, m \geq 1$ are fixed positive integers, K, T are fixed finite positive constants, $\mathbb{R}^d$ is a d-dimensional Euclidean space of points $x = (x^1, \dots, x^d)$ and

$$D_i := \partial/\partial x^i, \quad D_{ij} := \partial^2/\partial x^i \partial x^j, \quad D_t := \partial/\partial t.$$

Unless otherwise indicated, we use the summation convention with respect to repeated indices. We also use the notation δ_{ij}, the 'Kronecker delta', which is 1 if $i = j$ and 0 if $i \neq j$.

2. General setting and an illustration

In this section we present three theorems in a very abstract setting. In order not to lose connection to real things and give the reader some justification of our assumptions we interrupt a few times the main stream of the section with discussions of a simple looking example. The reader will understand much better also the proofs of our main result in Section 3 if he keeps applying abstract constructions to Example 2.2, which by the way has nothing to do with the splitting-up method.

It is probably hard to appreciate Theorems 2.14 and 2.15 looking only at Example 2.2. We reiterate that the goal of this example is to give the reader a feeling of what is behind quite abstract assumptions and objects. Later we will see much more serious applications of our abstract results.

Fix an integer $l \geq 1$ and assume that we have a sequence of Banach spaces

$$W_0, W_1, W_2, \ldots, W_l$$

such that W_i is continuously embedded into W_{i-1}, for every $i = 1, 2, \ldots, l$, and W_1 is dense in W_0.

For each number $\tau \in (0, 1]$ we consider the pair of equations (1.1), (1.2) for $v = v(\tau)$ and $w = w(\tau)$, respectively, where $L = L(\tau)$, $L_k = L_k(\tau)$, $A_r = A_r(\tau)$, $\Theta_r = \Theta_r(\tau)$ are certain linear operators and $f = f(\tau)$, $f_k = f_k(\tau)$, $\varphi = \varphi(\tau)$ are elements from W_l, for all $r = 0, 1, \ldots, m$ and $k = 1, 2, \ldots, m$. Almost everywhere below in the article we drop the argument τ.

Assumption 2.1.

(i) For all $i = 0, \ldots, l$ the operators A_r, Θ_r are bounded operators from W_i to W_i such that

$$\|\Theta_r u\|_i \leq K\|u\|_i, \quad \|A_r u\|_i \leq K\|u\|_i$$

for $r = 0, \ldots, m$ and $u \in W_i$.

(ii) For all $i = 0, \ldots, l-1$ the operators L, L_k are bounded operators from W_{i+1} to W_i such that

$$\|Lu\|_i \leq K\|u\|_{i+1}, \quad \|L_k u\|_i \leq K\|u\|_{i+1}.$$

for $k = 1, \ldots, m$ and $u \in W_{i+1}$.

(iii) $\|\varphi\|_i \leq K$, $\|f_k\|_i \leq K$ for all $i = 1, 2, \ldots, l$ and $k = 1, 2, \ldots, m$.

(iv) Equations (1.3) hold.

Example 2.2. Let

$$W_0 = \cdots = W_l = D([0, T], \mathbb{R}^d) \tag{2.1}$$

be the space of $\mathbb{R}^d$-valued bounded functions on $[0, T]$ having right limits on $[0, T)$ and left limits on $(0, T]$. We provide these spaces with the uniform norm.

Let $m = 1$, $a_0(t) = t$, $a_1(t) = \tau[t/\tau]$, and define the operators A_k by

$$(A_k u)(t) = \int_{(0,t]} u(s)\, da_k(s). \tag{2.2}$$

Next, take a $d \times d$-matrix valued cadlag function $L(t)$, $t \in [0,T]$, and define the operators L, L_1 by

$$(Lu)(t) = (L_1 u)(t) = L(t)u(t).$$

Finally, take a function $\varphi \in \mathbb{R}^d$ and consider the two equations

$$v(t) = \varphi + \int_0^t L(s)v(s)\,ds, \tag{2.3}$$

$$w(t) = \varphi + \int_{(0,t]} L(s-)w(s-)\,da_1(s), \tag{2.4}$$

which in our notation can be written as (1.1) and (1.2), respectively, with $m = 1$, i.e.,

$$v = \varphi + A_0\Theta_0 Lv, \quad w = \varphi + A_1\Theta_1 Lw,$$

where Θ_0 is the unit operator and Θ_1 is the operator defined by

$$(\Theta_1 u)(t) = u(t-) \quad t \in (0,T], \quad (\Theta_1 u)(0) = 0. \tag{2.5}$$

Our goal is to compare w and v. □

Assumption 2.3. For each $k = 0, \dots, m$ there is a bounded linear operator $\mathcal{R}_k : W_0 \to W_0$ such that

(i) we have $\mathcal{R}_k : W_i \to W_i$ for all $i = 0, \dots, l$ and

$$\|\mathcal{R}_k g\|_i \le K\|g\|_i, \quad g \in W_i, \tag{2.6}$$

(ii) (existence) for any $g \in W_1$ the function $u = \mathcal{R}_k g$ satisfies

$$u = A_0\Theta_0 Lu + A_k g, \tag{2.7}$$

(iii) (uniqueness) if $g_k \in W_0$, $k = 0, \dots, m$, $u \in W_1$ and

$$u = A_0\Theta_0 Lu + \sum_{k=0}^m A_k g_k,$$

then

$$u = \sum_{k=0}^m \mathcal{R}_k g_k.$$

Remark 2.4. Assumption 2.3 is satisfied in Example 2.2. To see this it suffices to notice that for $\bar{u} = u - A_k g$ equation (2.7) becomes

$$\bar{u} = A_0(L\bar{u} + h), \quad \frac{d\bar{u}}{dt} = L\bar{u} + h,$$

where $h = LA_k g$.

In order to be able to compare the solutions v and w of equations (1.1) and (1.2) we need to relate not only the operators L, L_k (see (iv) in Assumption 2.1), but also the operators A_0, A_k and Θ_0, Θ_k. To formulate the corresponding conditions we need to introduce some further objects.

We call a sequence of numbers $\alpha = \alpha_1\alpha_2\dots\alpha_i$ a multi-number of length $|\alpha| := i$, if $\alpha_j \in \{0, 1, 2, \dots, m\}$. The reader should notice the difference between multi-numbers and multi-indices. The set of all multi-numbers is denoted by $\mathcal{N}$.

For each $\tau \in (0, 1]$ and $\alpha \in \mathcal{N}$ let $b^+_\alpha = b^+_\alpha(\tau)$, $b^-_\alpha = b^-_\alpha(\tau)$ be linear operators on W_0, let $c_\alpha = c_\alpha(\tau)$ be a real number, and let $B_\alpha = B_\alpha(\tau)$ be a linear operator introduced by

$$\begin{aligned} \tau B_\alpha &= A_\alpha \Theta_\alpha - A_0\Theta_0, \quad |\alpha| = 1, \\ \tau B_{\alpha k} &= A_k b^-_\alpha \Theta_k - c_{\alpha k} A_0 \Theta_0, \quad k = 0, \dots, m. \end{aligned} \tag{2.8}$$

We impose the following assumptions, in which $K_\alpha \geq 0$, $\alpha \in \mathcal{N}$, are some fixed finite constants, independent of τ.

Assumption 2.5. For all $i = 0, \dots, l$ the operators b^+_α, b^-_α are bounded operators from W_i to W_i and

$$\|b^+_\alpha u\|_i \leq K_\alpha \|u\|_i, \quad \|b^-_\alpha u\|_i \leq K_\alpha \|u\|_i \tag{2.9}$$

for all $\alpha \in \mathcal{N}$ and $u \in W_i$.

Assumption 2.6. For any $\alpha \in \mathcal{N}$ and $k = 0, \dots, m$

$$B_\alpha A_k = b^+_\alpha A_k - A_k b^-_\alpha, \quad A_0\Theta_0 b^+_\alpha = A_0 b^-_\alpha \Theta_0. \tag{2.10}$$

Assumption 2.7. For any $\alpha \in \mathcal{N}$ and $k = 1, \dots, m$ and $r = 0, \dots, m$

$$\begin{aligned} &L_k\Theta_r = \Theta_r L_k, \quad L_k b^\pm_\alpha = b^\pm_\alpha L_k, \quad A_r L_k = L_k A_r, \\ &B_\alpha \varphi = b^+_\alpha \varphi, \quad B_\alpha f_k = b^+_\alpha f_k. \end{aligned}$$

Definition 2.8. We say that $g \in W_0$ is time independent if $B_\alpha g = b^+_\alpha g$ for all $\alpha \in \mathcal{N}$.

Remark 2.9. Since $L = \sum_k L_k$, the operator L commutes with Θ_r, $b^\pm_\alpha$, and A_r as well. Also it follows from the definition of B_α and Assumption 2.7 that B_α commutes with L, L_k for all α and k.

Remark 2.10. In Example 2.2 the requirement that $A_0 L = L A_0$ means that $L(t)$ is independent of t. We want to introduce $b^\pm_\alpha$, B_α, and c_α in this example by formulas ready for use later on. In more general situations along with Θ_k we also need operators $\bar{\Theta}_k$ and $\bar{\Theta}_\alpha$, which we define in Example 2.2 to be the identity operators. So we let k vary in $\{0, 1\}$ and for $\alpha \in \mathcal{N}$ define recursively

$$b_k(t) = \frac{1}{\tau}[a_k(t) - a_0(t)], \quad c_{\alpha k} = \frac{1}{\tau}\int_{(0,\tau]} \bar{\Theta}_\alpha b_\alpha(s)\,da_k(s),$$
$$b_{\alpha k}(t) = \frac{1}{\tau}\left(\int_{(0,t]} \bar{\Theta}_\alpha b_\alpha(s)\,da_k(s) - c_{\alpha k}a_0(t)\right). \tag{2.11}$$

It is easy to prove (see, however, Lemma 2.13 in a more general setting) that c_α are independent of τ, $b_\alpha(t)$ are τ-periodic in t and $b_\alpha(i\tau) = 0$ for integers $i \geq 0$.

Next, introduce $b_\alpha^\pm$ as the operator of multiplying by the function b_α and define B_α by the formula

$$(B_\alpha u)(t) = \int_{(0,t]} u(s-)\,db_\alpha(s).$$

These definitions are consistent with what is done in the general scheme. Indeed, (2.8) holds obviously, as does the second relation in (2.10). The first relation is a consequence of the well known fact that for two right-continuous functions of bounded variation

$$d(b(t)a(t)) = a(t-)\,db(t) + b(t)\,da(t), \tag{2.12}$$

so that

$$d\left(b_\alpha(t)\int_{(0,t]} u(s)\,da_k(s)\right) = \int_{(0,t)} u(s)\,da_k(s)\,db_\alpha(t) + b_\alpha(t)u(t)\,da_k(t).$$

REMARK 2.11. If we modify the definition of Θ_1 in (2.5) to

$$(\Theta_1 u)(t) = \vartheta u(t) + (1-\vartheta)u(t-) \tag{2.13}$$

with a fixed constant $\vartheta \in \mathbb{R}$, then for $\vartheta \neq 0$ the operators b_α^+ and b_α^- which we need to use are not equal. We show this in the following generalization of Example 2.2.

EXAMPLE 2.12. Consider Example 2.2 with L independent of t, and with Θ_1 defined by (2.13) in place of (2.5), so that if $\vartheta = 0$ we just have the same situation as in Example 2.2. Interestingly enough, even if below $\vartheta = 0$, this time we take the operators $\bar{\Theta}_\alpha$ different from the identity. As in Example 2.2 we define Θ_0 to be the identity operator and introduce the operators A_k as before by (2.2). Then clearly Assumptions 2.1 and 2.3 still hold. For future use we introduce further notation. We set $\vartheta_0 = 1$ and $\vartheta_1 = \vartheta$ and let k vary in $\{0,1\}$. We define the operators $\bar{\Theta}_k$ by

$$(\bar{\Theta}_k u)(t) = (1-\vartheta_k)u(t) + \vartheta_k u(t-),$$

and set for $\alpha = \alpha_1 \dots \alpha_j \in \mathcal{N}$

$$\Theta_\alpha = \Theta_{\alpha_j}, \quad \bar{\Theta}_\alpha = \bar{\Theta}_{\alpha_j}.$$

Notice that for right-continuous functions of bounded variation, say a and b, we have by (2.12) that

$$d(b(t)a(t)) = \Theta_\alpha a(t)\,db(t) + \bar{\Theta}_\alpha b(t)\,da(t). \tag{2.14}$$

Next, we use formulas (2.11) to define the functions b_α and the numbers c_α. Observe that by Lemma 2.13 below the numbers c_α do not depend on τ. Define for every $\alpha \in \mathcal{N}$ the operator B_α by

$$(B_\alpha u)(t) = \int_{(0,t]} \Theta_\alpha u(s)\,db_\alpha(s),$$

and let b_α^- be the operator of multiplying by the function $\bar{\Theta}_\alpha b_\alpha$. Then this definition of the operator B_α is the same as the general definition of B_α given by (2.8), by virtue of the above definition of b_α. Using (2.14) with $b = b_\alpha$ and $a = A_k u$ we get

$$d\left(b_\alpha(t)\int_{(0,t]} u(s)\,da_k(s)\right) = d(B_\alpha A_k u)(t) + d(A_k b_\alpha^- u)(t).$$

Thus, defining the operator b_α^+ as the multiplication by b_α, we have

$$b_\alpha^+ A_k = B_\alpha A_k + A_k b_\alpha^-,$$

i.e., the first identity in Assumption 2.6. Notice that $b_\alpha^+ \neq b_\alpha^-$ if $\vartheta \neq 0$ in (2.13). Clearly, the second identity in Assumption 2.6 and Assumption 2.7 hold also for this example. □

Next we formulate a lemma which ensures that for a large class of applications of the general scheme the numbers c_α are independent of the parameter τ.

Let $H_0, H_1, \ldots, H_m$ be right-continuous functions on $\mathbb{R}$ which have finite variation on every finite interval. Assume that

$$H_r(0) = 0, \quad H_r(t+1) - H_r(t) = H_r(1) = 1, \quad \forall t \in \mathbb{R}, \quad r = 0, 1, \ldots, m.$$

For each $\tau \in (0,1]$ we define the functions

$$a_r(t) = \tau H_r(t/\tau), \quad t \geq 0, \quad r = 0, 1, \ldots, m.$$

Let $\Lambda_\alpha(\tau)$ be an operator for every $\alpha \in \mathcal{N}$ and $\tau \in (0,1)$, mapping $\mathbb{B}_\tau(\mathbb{R}_+)$ into itself, where $\mathbb{B}_\tau(\mathbb{R}_+)$ denotes the class of τ-periodic bounded functions on $\mathbb{R}_+ = [0,\infty)$ having left and right limits at every $t \in (0,\infty)$. We assume that $(\Lambda_\alpha(\tau)u)(t\tau)$, $t \geq 0$, is independent of τ for every $\alpha \in \mathcal{N}$ and every $u \in B_\tau(\mathbb{R}_+)$.

For every $\alpha \in \mathcal{N}$ we define a function $b_\alpha : [0,\infty) \to \mathbb{R}$ and a number c_α recursively, starting as follows:

$$b_\gamma = \tau^{-1}(a_\gamma - a_0), \quad c_\gamma = 0 \qquad \text{for } \gamma = 0, 1, 2, \ldots, m. \tag{2.15}$$

If for every multi-number $\beta = \beta_1 \dots \beta_i$ of length i the function b_β and the number c_β are defined, then we set

$$c_{\alpha\gamma} = \frac{1}{\tau} \int_0^\tau \Lambda_\alpha b_\alpha(t)\, da_\gamma(t), \tag{2.16}$$

$$b_{\alpha\gamma}(t) = \frac{1}{\tau} \left(\int_0^t \Lambda_\alpha b_\alpha(s)\, da_\gamma(s) - c_{\alpha\gamma} a_0(t) \right). \tag{2.17}$$

LEMMA 2.13. *For every $\alpha \in \mathcal{N}$ the function b_α is τ-periodic, i.e., $b_\alpha(t+\tau) = b_\alpha(t)$ for all $t \geq 0$, and $b_\alpha(i\tau) = 0$ for all integers $i \geq 0$. Moreover, the numbers c_α, the functions $C_\alpha(t) := b_\alpha(\tau t)$, and*

$$\sup_{t\geq 0} |b_\alpha(t)| = \sup_{t \geq 0} |C_\alpha(t)|$$

are finite and do not depend on τ.

Proof. Clearly

$$\tau^{-1}(a_r(\tau) - a_0(\tau)) = H_r(1) - H_0(1) = 0.$$

Since $H_r(t+1) = H_r(t) + H_r(1)$,

$$\begin{aligned} b_r(t+\tau) &= \tau^{-1}(a_r(t+\tau) - a_0(t+\tau)) \\ &= H_r\left(\frac{t}{\tau} + 1\right) - H_0\left(\frac{t}{\tau}+1\right) = H_r\left(\frac{t}{\tau}\right) - H_0\left(\frac{t}{\tau}\right) = b_r(t), \end{aligned}$$

i.e., b_r is τ-periodic, and $C_r(s) = b_r(t\tau) = H_r(t) - H_0(t)$ is independent of τ. Consequently, the assertions of the lemma hold for $\alpha = 0, \dots, m$. Assume now that the statements of the lemma are true for $\alpha = \beta$, where β is a multi-number. Then for every $\gamma = 0, 1, 2, \dots, m$

$$c_{\beta\gamma} = \int_0^\tau \Lambda_\beta b_\beta(s)\, dH_\gamma(s/\tau) = \int_0^1 (\Lambda_\beta b_\beta)(\tau s)\, dH_\gamma(s).$$

Thus $c_{\beta\gamma}$, and hence

$$\begin{aligned} C_{\beta\gamma}(t) &= \int_0^{\tau t} \Lambda_\beta b_\beta(s)\, dH_\gamma(s/\tau) - c_{\beta\gamma} H_0(t) \\ &= \int_0^t (\Lambda_\beta b_\beta)(\tau s)\, dH_\gamma(s) - c_{\beta\gamma} H_0(t) \end{aligned}$$

are independent of τ. Moreover, by the definition of $c_{\beta\gamma}$,

$$\begin{aligned} C_{\beta\gamma}(t+1) &= \int_0^{t+1} (\Lambda_\beta b_\beta)(\tau s)\, dH_\gamma(s) - c_{\beta\gamma} H_0(t+1) \\ &= \int_1^{t+1} (\Lambda_\beta b_\beta)(\tau s)\, dH_\gamma(s) - c_{\beta\gamma}(H_0(t+1) - H_0(1)) \\ &= \int_0^{t} (\Lambda_\beta b_\beta)(\tau (s+1))\, dH_\gamma(s+1) - c_{\beta\gamma} H_0(t) \\ &= \int_0^{t} (\Lambda_\beta b_\beta)(\tau s)\, dH_\gamma(s) - c_{\beta\gamma} H_0(t) = C_{\beta\gamma}(t), \end{aligned}$$

i.e., $C_{\beta\gamma}$ is 1-periodic, and hence $b_{\beta\gamma}$ is τ-periodic. Thus induction on the length $|\alpha|$ finishes the proof of the lemma. □

THEOREM 2.14. *Let $k \geq 0$ be an integer and let Assumptions 2.1, 2.3, 2.5, 2.6 and 2.7 hold with $l \geq 2k+2$. Assume that (for a given $\tau \in (0,1]$) equations* (1.1) *and* (1.2) *have solutions $v \in W_l$ and $w \in W_l$, respectively, such that $\|w\|_l \leq K$. Then for any continuous linear functional w^* on W_0, such that $w^* b_\alpha^+ = 0$ for all $\alpha \in \mathcal{N}$, equation*

$$\langle w^*, w\rangle = \sum_{i=0}^{k} \tau^i \langle w^*, v_i\rangle + O(\tau^{k+1}), \tag{2.18}$$

holds, where $v_0 = v, v_i \in W_0$ are uniquely determined by A_0, Θ_0, L_r, f_r, k, and c_α, and

$$|O(\tau^{k+1})| \leq N\tau^{k+1}\|w^*\|,$$

with a constant N depending only on K_α, K, and l.

Theorem 2.14 follows immediately from Theorem 2.18 below.

Generally, the solutions of (1.2) and (1.1) depend on τ, i.e, $w = w(\tau)$, $v = v(\tau)$. However, if A_0, Θ_0, L_r, f_r, and c_α are independent of τ, then v and other v_i's in (2.18) are independent of τ as well (since they are uniquely determined by A_0, Θ_0, L_r, f_r, and c_α). In this situation we have the following result on 'acceleration'.

THEOREM 2.15. *Let $k \geq 0$ be an integer and let Assumptions 2.1, 2.3, 2.5, 2.6, and 2.7 hold with $l \geq 2k+2$. Let A_0, Θ_0, L_r, f_r, and c_α be independent of τ, and assume that* (1.1) *has a solution $v \in W_l$. Suppose that for a given $\tau_0 \in (0,1]$ for all $j = 0, 1, \ldots, k$ equation* (1.2) *with $\tau = \tau_j := \tau_0 2^{-j}$ has a solution $w = w_j$ such that $\|w_j\|_l \leq K$. Assume that an element $w^* \in W_0^*$ satisfies*

$$w^* b_\alpha^+(\tau_j) = 0, \quad \forall \alpha \in \mathcal{N},\ j = 0, 1, \ldots, k.$$

Then, for some constants $\lambda_0, \dots, \lambda_k$ depending only on k, we have

$$\left| \sum_{j=0}^{k} \lambda_j \langle w^*, w_j \rangle - \langle w^*, v \rangle \right| \le N \tau_0^{k+1} \|w^*\|,$$

where N depends only on K_α, K, and l.

Proof. By Theorem 2.14 we have

$$\langle w^*, w_j \rangle = \langle w^*, v \rangle + \sum_{i=1}^{k} 2^{-ji} \tau_0^i \langle w^*, v_i \rangle + R_j(w^*, \tau_0), \quad j = 0, 1, \dots, k,$$

with

$$|R_j(w^*, \tau_0)| \le 2^{-j(k+1)} N \|w^*\| \tau_0^{k+1}.$$

Let V denote the square matrix defined by $V^{ij} := 2^{-(i-1)(j-1)}$, $i, j = 1, \dots, k+1$. Notice that the determinant of V is the Vandermonde determinant, generated by $1, 2^{-1}, \dots, 2^{-k}$, and hence it is different from 0. Thus V is invertible. Define

$$(\lambda_0, \lambda_1, \dots, \lambda_k) = (1, 0, 0, \dots, 0) V^{-1}.$$

Then

$$\begin{aligned}\sum_{j=0}^{k} \lambda_j \langle w^*, w_j \rangle &= \left(\sum_{j=0}^{k} \lambda_j \right) \langle w^*, v \rangle + \sum_{j=0}^{k} \sum_{i=1}^{k} \lambda_j 2^{-ij} \tau_0^i \langle w^*, v_i \rangle \\ &\quad + \sum_{j=0}^{k} \lambda_j R_j(w^*, \tau) \\ &= \langle w^*, v \rangle + \sum_{i=1}^{k} \tau_0^i \langle w^*, v_i \rangle \sum_{j=0}^{k} \lambda_j 2^{ij} + O(\tau_0^{k+1}) \\ &= \langle w^*, v \rangle + O(\tau_0^{k+1}),\end{aligned}$$

since $\sum_{j=0}^{k} \lambda_j = 1$ and $\sum_{j=0}^{k} \lambda_j 2^{ij} = 0$ for $i = 1, 2, \dots, k$ by the definition of $(\lambda_0, \dots, \lambda_k)$, and

$$\left| \sum_{j=0}^{k} \lambda_j R_j(w^*, \tau) \right| \le \sum_{j=0}^{k} N 2^{-j} |\lambda_j| \|w^*\| \tau_0^{k+1} \le C \|w^*\| \tau_0^{k+1},$$

with constants N and C depending only on K_α, K, and l. □

REMARK 2.16. In Example 2.2 assume that $L(t)$ is independent of t. Then by Remark 2.10 the assumptions of Theorem 2.14 are satisfied for any k with appropriate l, K, and K_α. Also, since $b_\alpha(j\tau) = 0$ for all $j = 0, 1, \dots,$ as w^*

in Theorem 2.14 one can take the restriction of elements in $D([0,T],\mathbb{R}^d)$ to any of the times in (1.5).

From Theorem 2.14 we now conclude that there exist $\mathbb{R}^d$-valued functions $v_i = v_i(t)$, $i = 0, 1, \dots$, $t \in [0, T]$, independent of τ, with $v_0 = v$ such that

$$\sup_{t\in T_\tau} \left| w(\tau,t) - \sum_{i=0}^{k} \tau^i v_i(t) \right| \le N\tau^{k+1}, \tag{2.19}$$

where N depends only on T, k, $|L|$, and $|\varphi|$.

By the way, under the above time independence assumption we have

$$v(t) = e^{Lt}\varphi.$$

Also equation (2.4) amounts to saying that

$$\begin{gathered} w(0) = \varphi, \quad w(t) = w(j\tau) \quad \text{for} \quad t \in [j\tau, (j+1)\tau), \\ w((j+1)\tau) = w(j\tau) + Lw(j\tau)\tau, \quad j = 0, 1, \dots, \end{gathered}$$

which is just Euler's scheme for equation (2.3). It is also an explicit finite-difference scheme for the equation $v' = Lv$. It follows that

$$w(t) = w(j\tau) = (1+\tau L)^j \varphi \quad \text{for} \quad t \in [j\tau, (j+1)\tau), \quad j = 0, 1, \dots . \tag{2.20}$$

Hence (2.19) means that

$$\max_{j:j\tau\le T} \left| (1+\tau L)^j\varphi - \sum_{i=0}^{k} \tau^i v_i(j\tau) \right| \le N\tau^{k+1},$$

with N depending only on T, k, $|\varphi|$, and L, and v_i independent of τ with $v_0 = v$. In particular, for $\tau = 1/n, T = 1, j = n$ we get that as $n \to \infty$

$$(1+L/n)^n\varphi = e^L\varphi + \sum_{i=1}^{k} \frac{v_i}{n^i} + O(n^{-(k+1)}), \tag{2.21}$$

where v_i are some vectors. Theorem 2.15 applied to Example 2.2 says that, as $\tau \downarrow 0$,

$$\max_{j:j\tau\le T} \left| \sum_{i=0}^{k} \lambda_i (1+\tau 2^{-i} L)^{2^i j}\varphi - e^{Lj\tau}\varphi \right| = O(\tau^{k+1}). \tag{2.22}$$

To get a feeling of the acceleration let us play with the following trivial numerical example. Take $d = k = \varphi = L = T = 1$, so that $\lambda_0 = -1$, $\lambda_1 = 2$, and use formula (2.22) with $j = 1/\tau$, $\tau = 1, 1/2, 1/4$, to approximate e:

$$e \approx -e(\tau) + 2e(\tau/2), \quad e(\tau) := (1+\tau)^{1/\tau}.$$

Let us calculate this approximation rounded to four decimal places, and compare it with the approximation $e(\tau/2)$, the better one between the approximations $e(\tau)$ and $e(\tau/2)$ for e, since $e(\tau) \uparrow e$ as $\tau \downarrow 0$.

Case (i) $\tau = 1$. Then $e \approx -2 + 2(\frac{3}{2})^2 = 2.5$, and the error 0.2183 is more than 2.1 times smaller than $0.4683 = e - e(1/2) = e - (\frac{3}{2})^2$, the error of $e(1/2)$.

Case (ii) $\tau = 1/2$. Then $e \approx -(\frac{3}{2})^2 + 2(\frac{5}{4})^4 \approx 2.6328$ with error 0.0855, which is more than 3.2 times smaller than 0.2769 the error of $e(1/4) = (\frac{5}{4})^4$.

Case (iii) $\tau = 1/4$. Then $e \approx -(\frac{5}{4})^4 + 2(\frac{9}{8})^8 \approx 2.6902$, and the error, 0.0281, is more than 5.4 smaller than 0.1525, the error of $e(1/8) = (\frac{9}{8})^8$.

Take now $k = 2$ in this example. Then $\lambda_0 = \frac{1}{3}$, $\lambda_1 = -2$, $\lambda_2 = \frac{8}{3}$, and by virtue of the above formula we approximate e by

$$e \approx \frac{1}{3}e(\tau) - 2e(\tau/2) + \frac{8}{3}e(\tau/4).$$

For $\tau = 1$ we get $e \approx \frac{1}{3}2 - 2(\frac{3}{2})^2 + \frac{8}{3}(\frac{5}{4})^4 \approx 2.6771$. The error is 0.0412, which is more than 6.7 times smaller than that of $e(1/4) = (\frac{5}{4})^4$. For $\tau = 1/2$ we get $e \approx \frac{1}{3}(\frac{3}{2})^2 - 2(\frac{5}{4})^4 + \frac{8}{3}(\frac{9}{8})^8 \approx 2.7093$. The error is 0.0092, which is more than 16.5 times smaller than 0.1525, the error of $e(1/8) = (\frac{9}{8})^8$.

We illustrate some directions of further applications in the following example.

EXAMPLE 2.17 (*Splitting-up combined with finite differences*). For a $d \times d$-matrix L we want to approximate the solution, $v(t) = e^{Lt}\varphi$, of the equation

$$\frac{d}{dt}v(t) = Lv(t), \quad v(0) = \varphi \in \mathbb{R}^d, \tag{2.23}$$

on the grid (1.5), by splitting-up the equation into m equations

$$\frac{d}{dt}v(t) = L_k v(t), \quad k = 1, 2, \dots, m, \quad L = L_1 + L_2 + \cdots + L_m,$$

and solving them numerically on each fixed interval $[j\tau, (j+1)\tau]$, consecutively, by finite differences. Namely, for each k we take some $\vartheta_k \in \mathbb{R}$ and approximate the equation $dv(t) = L_k v(t)\, dt$ on each $[j\tau, (j+1)\tau)$ by the *θ-method* with $\theta = \bar{\vartheta}_k := 1 - \vartheta_k$, i.e., for its numerical solution u we take

$$u(t) = u(j\tau), \quad \text{for } t \in [j\tau, (j+1)\tau),$$
$$u((j+1)\tau) = u(j\tau) + \tau\bar{\vartheta}_k L_k u(j\tau) + \tau\vartheta_k L_k u((j+1)\tau).$$

For $\theta = 1/2$ this is the so-called Crank-Nicholson scheme. Thus, assuming that the matrix $I - \tau\vartheta_k L_k$ is invertible, we have the recursion

$$u((j+1)\tau) = (I - \tau\vartheta_k L_k)^{-1}(I + \tau\bar{\vartheta}_k L_k)u(j\tau).$$

Using this recursion for each $k = 1, 2, \dots, m$ consecutively on every interval $[j\tau, (j+1)\tau)$, for $j = 0, 1, 2 \dots, i-1$, we get the approximation

$$w(t_i) = w(\tau, t_i) = \left(\Pi_{k=1}^m (I - \tau\vartheta_k L_k)^{-1}(I + \tau\bar{\vartheta}_k L_k)\right)^i \varphi \tag{2.24}$$

for $v(t_i) = e^{t_i L}\varphi$, when $t_i = i\tau$.

Now we describe this approximation in terms of the general setting. In order to express the splitting-up algorithm, we introduce the absolutely continuous functions $h_1,\dots,h_m$ on $\mathbb{R}$, whose derivatives are periodic with period m, such that

$$\dot{h}_k(t) := 1_{[k-1,k)}(t) \quad t \in [0,m). \tag{2.25}$$

We define for each $\tau \in [0,1)$ the non-decreasing right-continuous functions

$$a_k(t) = \tau[h_k(mt/\tau)], \quad t \geq 0, \quad k = 1,2,\dots,m,$$

where, as before, $[c]$ denotes the integer part of c. Then the approximation w given by (2.24) coincides with the solution of the equation

$$dw(t) = \sum_{k=1}^{m} L_k \Theta_k w(t)\, da_k(t), \quad w(0) = \varphi, \tag{2.26}$$

at the points $t_i = i\tau \in T_\tau$, where

$$(\Theta_k w)(t) := \vartheta_k w(t) + (1 - \vartheta_k) w(t-).$$

Clearly, (2.26) can be written in the form (1.2) and equation (2.23) is of the form (1.1), if we take $f = f_k \equiv 0$ and introduce Θ_0 as the identity and A_0, $A_1,\dots, A_m$ as the integral operators on the spaces (2.1) defined, as before, by (2.2) for $k = 0,1,\dots,m$ with $a_0(t) \equiv t$.

Now introduce the operators Θ_α and $\bar{\Theta}_\alpha$ and define the functions b_α, the numbers c_α, and, finally, the operators $b^{\pm}_\alpha$ and B_α by the same formulas which were used in Example 2.12, allowing there k to vary in $\{0,1,\dots,m\}$.

Notice that by Lemma 2.13 the numbers c_α do not depend on τ and, as in Example 2.12, it is easy to check that all assumptions of the general scheme are satisfied. Furthermore, we have $b_\alpha(j\tau) = 0$ for all integers $j \geq 0$. Therefore we can apply Theorem 2.14 with w^*, the restriction of functions $u \in D([0,T],\mathbb{R}^d)$ to any $t_j \in T_\tau$. Then we obtain that there exist $v_0, v_1, \dots, v_k \in D([0,T],\mathbb{R}^d)$, independent of τ, with $v_0 = v$, such that

$$\max_{t\in T_\tau} \left| w(\tau,t) - \sum_{i=0}^{k} v_i(t)\tau^i \right| \leq N\tau^{k+1} \quad \text{for} \quad \tau \in (0,1], \tag{2.27}$$

where $w(\tau,\cdot) = w$ is the approximation defined by (2.24), and N is a constant depending only on T, k, m, $|L|$, $|\varphi|$, and $\vartheta_1,\dots,\vartheta_m$. From Theorem 2.15 we get

$$\max_{t\in T_\tau} \left| e^{Lt}\varphi - \sum_{i=0}^{k} \lambda_i w(2^{-i}\tau, t) \right| = O(\tau^{k+1}). \qquad \square$$

We will see that Theorem 2.14 follows from an expansion of w into a power series with respect to τ. To state the corresponding result we need more notation.

For $\gamma \in \mathcal{N}$ we define $f_\gamma \in W_0$ and a linear operator L_γ as follows:

$$L_0 = 0,\ f_0 = 0,\ L_\gamma = L_r,\ f_\gamma = f_r \ \text{ for } \ \gamma = r \in \{1,2,\dots,m\}, \tag{2.28}$$

$$L_{\gamma 0} = LL_\gamma, \quad L_{\gamma r} = -L_\gamma L_r, \quad f_{\gamma 0} = Lf_\gamma, \quad f_{\gamma r} = -L_\gamma f_r, \tag{2.29}$$

for $r = 1,2,\dots,m$, $\gamma \in \mathcal{N}$. Notice that L_α is a bounded linear operator from W_j into $W_{j-|\alpha|}$ if $|\alpha| \le j$ and $f_\alpha \in W_{l-|\alpha|+1}$ if $|\alpha| \le l+1$.

Observe that f_α are time independent due to Assumption 2.7, because by Remark 2.9 we have

$$B_\gamma L_k f_\alpha = L_k B_\gamma f_\alpha$$

and one can use induction on $|\alpha|$.

Let $\mathcal{M}$ denote the set of multi-numbers $\gamma_1\gamma_2\dots\gamma_i$ with $\gamma_j \in \{1,2,\dots,m\}$, $j = 1,2,\dots,i$, and integers $i \ge 1$. Observe that $\mathcal{M} \subset \mathcal{N}$ and in contrast with $\mathcal{N}$ the entries in $\gamma \in \mathcal{M}$ are not allowed to equal zero.

Next, we introduce sequences $\sigma = (\beta_1, \beta_2, \dots, \beta_i)$ of multi-numbers $\beta_j \in \mathcal{M}$, where $i \ge 1$ is any integer, and set

$$|\sigma| = |\beta_1| + |\beta_2| + \dots + |\beta_i|.$$

We consider also the 'empty sequence' e of length $|e| = 0$, and denote the set of all these sequences by $\mathcal{J}$. For $\sigma = (\beta_1, \beta_2, \dots, \beta_i)$, $i \ge 1$, we define

$$S_\sigma = \mathcal{R}L_{\beta_1} \cdot \dots \cdot \mathcal{R}L_{\beta_i}, \quad \text{where} \quad \mathcal{R} := \mathcal{R}_0\Theta_0, \tag{2.30}$$

and for $\sigma = e$ we set $S_e = \mathcal{R}$. Notice that S_σ is well-defined as bounded linear operator from $W_{j+|\sigma|}$ to W_j if $j + |\sigma| \le l$. If we have a collection of $g_\nu \in W_0$ indexed by a parameter ν taking values in a set A, then we use the notation

$$\sum_{\nu\in A}^{*} g_\nu \tag{2.31}$$

for any linear combination of g_ν with coefficients depending only on c_α, A, and ν. For instance,

$$\sum_{A}^{*} S_\sigma w_\gamma = \sum_{(\sigma,\gamma)\in A}^{*} S_\sigma w_\gamma = \sum_{(\sigma,\gamma)\in A} c(\sigma,\gamma) S_\sigma w_\gamma,$$

where $c(\sigma,\gamma)$ are certain functions of $c_\alpha, \alpha \in \mathcal{N}$, and $(\sigma,\gamma) \in A$. These functions are allowed to change from one occurrence to another.

For $\mu = 0,\dots,l$, $\kappa \ge 0$, and functions $u = u_\alpha(\tau)$ depending on the parameters $\alpha \in \mathcal{N}$ and τ we write

$$u = O_\mu(\tau^\kappa) \quad \text{if} \quad \|u_\alpha(\tau)\|_\mu \le N\tau^\kappa,$$

where the constant $N < \infty$ depends only on $\alpha, K_\beta, \beta \in \mathcal{N}$, μ, l, and K. Finally, we set

$$A(i) = \{(\sigma,\beta) : \sigma \in \mathcal{J}, \beta \in \mathcal{M}, |\sigma| + |\beta| \le i\}, \tag{2.32}$$

$$B^*(i,j) = \{(\alpha,\beta) : \alpha \in \mathcal{N}, \beta \in \mathcal{M}, |\alpha| \le i, |\beta| \le j\}, \tag{2.33}$$

and $v_\beta = L_\beta v + f_\beta$, $w_\beta = L_\beta w + f_\beta$.

THEOREM 2.18. *Under the assumptions of Theorem 2.14 we have*

$$w = v + \sum_{i=1}^{k} \tau^i {\sum_{A(2i)}}^{*} S_\sigma v_\beta + \sum_{i=1}^{k} \tau^i {\sum_{B^*(i,i+j)}}^{*} b^+_{\alpha_1} w_{\beta_1} + O_0(\tau^{k+1}). \tag{2.34}$$

Furthermore, if $k \geq 1$, then

$${\sum_{A(2)}}^{*} S_\sigma v_\beta = \sum_{i,j=1}^{m} (c_{ij} - c_{j0}) \mathcal{R} v_{ij}$$

in (2.34)*, so that it vanishes if $c_{ij} = c_{j0}$ for all $i, j = 1, \dots, m$.*

REMARK 2.19. In Example 2.17 with $m = 1$ and $\theta = 1/2$, that is, for the Crank-Nicholson scheme, we have $c_{11} = c_{10}$, which implies that the coefficient of τ in (2.27) vanishes and $w = v + O_0(\tau^2)$ on T_τ, and we rediscover the well-known fact that the scheme is of second order accuracy.

REMARK 2.20. If the coefficient of τ in the first sum in (2.34) is zero, then to accelerate to get order of accuracy τ^3 it suffices to mix *two* grids instead of three as in the general case because one can find two universal constants such that the expansion in powers of τ of the corresponding linear combination has first term proportional to τ^3 and thus has error of order τ^3.

Indeed, let $\tau_0 \in (0, 1]$ and assume that equation (1.2) with τ_0 and $\tau_1 := \tau_0/2$ has a solution w_0 and w_1, respectively. Then by virtue of Theorem 2.18, under the assumptions of Theorem 2.15, and if $c_{ij} = c_{j0}$ for all i, j, we have

$$\left| \frac{4}{3}\langle w_1, w^* \rangle - \frac{1}{3}\langle w_0, w^* \rangle - \langle v, w^* \rangle \right| \leq N \tau_0^3 \|w^*\| \tag{2.35}$$

for all $w^* \in W_0^*$, satisfying $w^* b^+_\alpha(\tau_j) = 0$ for all $\alpha \in \mathcal{N}$, $j = 0, 1$. We say more about such situations in Remark 4.5 below.

EXAMPLE 2.21. We apply the two previous remarks to Example 2.17 with $m = 1$ and $\theta = 1/2$, that is, to the Crank-Nicholson scheme. By (2.24) and (2.35) we have

$$\max_{t \in T_\tau} \Bigg| e^{tL}\varphi + \frac{1}{3}\left(\left(I - \frac{\tau}{2}L\right)^{-1}\left(I + \frac{\tau}{2}L\right)\right)^{t/\tau} - \frac{4}{3}\left(\left(I - \frac{\tau}{4}L\right)^{-1}\left(I + \frac{\tau}{4}L\right)\right)^{2t/\tau} \Bigg| = O(\tau^3).$$

Let us see what this acceleration does in the trivial numerical example $d = k = \varphi = L = T = 1$, chosing $\tau = 1, 1/2, 1/4$. Thus, we approximate the number e now by

$$e \approx -\frac{1}{3}\bar{e}(\tau) + \frac{4}{3}\bar{e}(\tau/2), \quad \bar{e}(\tau) := \left(\frac{2+\tau}{2-\tau}\right)^{1/\tau}, \quad \text{for } \tau = 1, \frac{1}{2}, \frac{1}{4}.$$

Rounding up to four decimal places we have

$$\bar{e}(1) = 3, \quad \bar{e}(1/2) = \left(\frac{5}{3}\right)^2 \approx 2.7778,$$

$$\bar{e}(1/4) = \left(\frac{9}{7}\right)^4 \approx 2.7326, \quad \bar{e}(1/8) = \left(\frac{17}{15}\right)^8 \approx 2.7218.$$

Case (i) $\tau = 1$. Then $e \approx -\frac{1}{3}\bar{e}(1) + \frac{4}{3}\bar{e}(1/2) \approx 2.7037$, and the error is 0.0146, which is more than 4 times smaller than 0.0595, the error of $\bar{e}(1/2)$.

Case (ii) $\tau = 1/2$. Then $e \approx -\frac{1}{3}\bar{e}(1/2) + \frac{4}{3}\bar{e}(1/4) \approx 2.7176$. The error is 0.0007, which is more than 20 times smaller than 0.00143, the error of $\bar{e}(1/4)$.

Case (iii) $\tau = 1/4$. Then $e \approx -\frac{1}{3}\bar{e}(1/4) + \frac{4}{3}\bar{e}(1/8) \approx 2.71823872$. Rounded to 10 decimal places the error is 0.000043108, which is more than 82 times smaller than the error of $\bar{e}(1/8)$.

3. Proof of Theorem 2.18

Recall that the operator $\mathcal{R}$ is defined in (2.30) and introduce

$$\mathcal{Q}_k = \frac{1}{\tau}(\mathcal{R}_k\Theta_k - \mathcal{R}), \quad \mathcal{Q}_{\alpha k} = \frac{1}{\tau}(\mathcal{R}_k b^-_\alpha \Theta_k - c_{\alpha k}\mathcal{R})$$

for $\alpha \in \mathcal{N}$ and $k = 0, \dots, m$.

LEMMA 3.1. *For any $\alpha \in \mathcal{N}$ we have:*

(i) *The operators*

$$\mathcal{Q}_\alpha : W_i \to W_i, \quad i = 0, \dots, l,$$

are bounded.

(ii) *If $g \in W_1$, then $u := \mathcal{Q}_\alpha g$ satisfies*

$$u = A_0\Theta_0 Lu + B_\alpha g. \tag{3.1}$$

(iii) *If C is a finite set of multi-numbers, $f, g_\alpha \in W_0, u \in W_1$ and*

$$u = A_0\Theta_0(Lu + f) + \sum_C B_\alpha g_\alpha,$$

then

$$u = \mathcal{R}f + \sum_C \mathcal{Q}_\alpha g_\alpha.$$

Proof. This lemma follows immediately from our definitions of B_α, Q_α and from Assumption 2.3 about $\mathcal{R}_k$. □

REMARK 3.2. For any $\alpha \in \mathcal{N}$, $k = 0, \dots, m$, and $g \in W_0$ we have

$$\mathcal{Q}_\alpha \mathcal{R}_k g = b_\alpha^+ \mathcal{R}_k g - \mathcal{R}_k b_\alpha^- g. \tag{3.2}$$

Indeed, since both parts are continuous functions on W_0 and W_1 is dense in W_0 we may assume that $g \in W_1$, so that $u := \mathcal{R}_k g \in W_1$ satisfies $u = A_0\Theta_0 Lu + A_k g$ and, since

$$b_\alpha^+ A_0\Theta_0 = B_\alpha A_0 \Theta_0 + A_0 b_\alpha^- \Theta_0 = B_\alpha A_0\Theta_0 + A_0\Theta_0 b_\alpha^+, \quad Lb_\alpha^+ = b_\alpha^+ L,$$
$$b_\alpha^+ A_k g = B_\alpha A_k g + A_k b_\alpha^- g,$$

it holds that

$$\begin{aligned} b_\alpha^+ u &= b_\alpha^+ A_0\Theta_0 Lu + B_\alpha A_k g + A_k b_\alpha^- g \\ &= B_\alpha (A_0\Theta_0 Lu + A_k g) + A_0\Theta_0 L(b_\alpha^+ u) + A_k b_\alpha^- g \\ &= A_0\Theta_0 L(b_\alpha^+ u) + B_\alpha u + A_k b_\alpha^- g. \end{aligned}$$

It follows that $b_\alpha^+ u = \mathcal{R}_k b_\alpha^- g + \mathcal{Q}_\alpha u$, and this is (3.2).

Formula (3.2) and assumptions (2.6) and (2.9) yield the following.

LEMMA 3.3. *For any $\alpha \in \mathcal{N}$, $k = 0, \dots, m$, $i = 0, \dots, l$, and $g \in W_i$ we have*

$$\|\mathcal{Q}_\alpha \mathcal{R}_k g\|_i \le 2KK_\alpha \|g\|_i. \tag{3.3}$$

REMARK 3.4. For any $\alpha \in \mathcal{N}$ and $g \in W_1$ we have

$$\mathcal{Q}_\alpha g = \mathcal{R} L B_\alpha g + B_\alpha g. \tag{3.4}$$

Indeed, since $u := \mathcal{Q}_\alpha g \in W_1$ satisfies $u = A_0\Theta_0 Lu + B_\alpha g$, $w := u - B_\alpha g \in W_1$ satisfies

$$w = A_0\Theta_0(Lw + h_\alpha)$$

with $h_\alpha := LB_\alpha g \in W_0$. Hence $w = \mathcal{R}LB_\alpha g$, and this is (3.4).

LEMMA 3.5. *Let $i = 1, \dots, l$, $g \in W_i$ be time independent, and $\alpha \in \mathcal{N}$. Then*

$$\|\mathcal{Q}_\alpha g\|_{i-1} \le K^2 K_\alpha \|g\|_i + K_\alpha \|g\|_{i-1}.$$

The lemma follows immediately from Remark 3.4 and our assumptions.

The following lemma exhibits our two main tools, which in the framework of differential equations translate to centering the integrand (assertion (i)) and integrating by parts (assertion (ii)).

LEMMA 3.6.

(i) *Let $g \in W_0$ and $\alpha \in \mathcal{N}$. Then*

$$\mathcal{R} b_\alpha^+ g = c_{\alpha 0} \mathcal{R} g + \tau \mathcal{Q}_{\alpha 0} g. \tag{3.5}$$

(ii) *Let $u \in W_1$, $g_0 = 0$, $g_1, \dots, g_m \in W_0$, h be time independent, and*

$$u = \sum_r A_r \Theta_r g_r + h.$$

Then for any $\alpha \in \mathcal{N}$

$$\mathcal{Q}_\alpha u = \mathcal{R}(c_{\alpha 0} Lu - c_{\alpha r} g_r) + \tau \mathcal{Q}_{\alpha 0} Lu - \tau \mathcal{Q}_{\alpha r} g_r + b_\alpha^+ u. \tag{3.6}$$

Proof. (i) Since both parts of (3.5) are continuous functions of g and W_1 is dense in W_0, it suffices to prove (3.5) for $g \in W_1$. In that case $w := \mathcal{R} b_\alpha^+ g \in W_1$ is the unique solution of $w = A_0 \Theta_0 (Lw + b_\alpha^+ g)$, which owing to our assumptions and definitions implying that

$$A_0 \Theta_0 b_\alpha^+ = A_0 b_\alpha^- \Theta_0 = \tau B_{\alpha 0} + c_{\alpha 0} A_0 \Theta_0,$$

can be written as

$$w = A_0 \Theta_0 (Lw + c_{\alpha 0} g) + \tau B_{\alpha 0} g,$$

and (3.5) follows by the definition of $\mathcal{R}$ and $\mathcal{Q}_{\alpha 0}$.

(ii) Observe that $p := \mathcal{Q}_\alpha u \in W_1$ satisfies $p = A_0 \Theta_0 L p + B_\alpha u$, where by (2.10) and (2.8)

$$\begin{aligned} B_\alpha u &= B_\alpha A_r \Theta_r g_r + b_\alpha^+ h = (b_\alpha^+ A_r - A_r b_\alpha^-) \Theta_r g_r + b_\alpha^+ h \\ &= b_\alpha^+ u - A_r b_\alpha^- \Theta_r g_r = b_\alpha^+ u - c_{\alpha r} A_0 \Theta_0 g_r - \tau B_{\alpha r} g_r. \end{aligned}$$

Also note that $Lu \in W_0$ and

$$A_0 \Theta_0 b_\alpha^+ Lu = A_0 b_\alpha^- \Theta_0 Lu = c_{\alpha 0} A_0 \Theta_0 Lu + \tau B_{\alpha 0} Lu.$$

Hence for $q := p - b_\alpha^+ u \in W_1$ we obtain

$$\begin{aligned} q &= A_0 \Theta_0 (Lq + b_\alpha^+ Lu) - c_{\alpha r} A_0 \Theta_0 g_r - \tau B_{\alpha r} g_r \\ &= A_0 \Theta_0 Lq + c_{\alpha 0} A_0 \Theta_0 Lu + \tau B_{\alpha 0} Lu - c_{\alpha r} A_0 \Theta_0 g_r - \tau B_{\alpha r} g_r, \end{aligned}$$

and (3.6) follows by Lemma 3.1 (iii). The lemma is proved. □

From now on the operators B_α will no longer be needed in our considerations.

We have the following statement regarding the function w from Theorem 2.14. Recall the notation $w_\beta = L_\beta w + f_\beta$ for $\beta \in \mathcal{N}$.

LEMMA 3.7. *Let $\alpha, \beta \in \mathcal{N}$ and $|\beta| + 1 \le l$. Then*

$$\mathcal{Q}_\alpha w_\beta = c_{\alpha r} \mathcal{R} w_{\beta r} + \tau \mathcal{Q}_{\alpha r} w_{\beta r} + b_\alpha^+ w_\beta. \tag{3.7}$$

Proof. Apply formula (3.6) to $u := w_\beta \in W_1$, after noting that

$$u = L_\beta \varphi + L_\beta A_r \Theta_r (L_r w + f_r) + f_\beta = A_r \Theta_r g_r + h,$$

where $g_r := L_\beta w_r$, and $h := L_\beta \varphi + f_\beta$ is time independent. Then the left-hand side of (3.6) equals

$$\mathcal{R}(c_{\alpha 0} L w_\beta - c_{\alpha r} L_\beta w_r) + \tau \mathcal{Q}_{\alpha 0} L w_\beta - \tau \mathcal{Q}_{\alpha r} L_\beta w_r + b_\alpha^+ w_\beta,$$

which is easily seen to be equal to the right-hand side of (3.7). □

We derive from (3.7) one of the most important formulas.

PROPOSITION 3.8. *Let $\kappa \geq 0$ be an integer and $l \geq \kappa + 1$. Then*

$$w = v + \sum_{i=1}^{\kappa} \tau^i \sum_{|\alpha|=i} b_\alpha^+ w_\alpha + \sum_{i=1}^{\kappa} \tau^i \sum_{|\alpha|=i+1} c_\alpha \mathcal{R} w_\alpha + \tau^{\kappa+1} r^{(\kappa+1)} \tag{3.8}$$

for all $t \in [0, T]$, where

$$r^{(\kappa+1)} = \sum_{|\alpha|=\kappa+1} \mathcal{Q}_\alpha w_\alpha.$$

Proof. First notice that for $u := w - v \in W_1$ we have

$$\begin{aligned} u &= A_r \Theta_r (L_r w + f_r) - A_0 \Theta_0 (Lv + f) \\ &= (A_r \Theta_r - A_0 \Theta_0) L_r w + A_0 \Theta_0 \left(\sum_r L_r w - Lv \right) \\ &\quad + (A_r \Theta_r - A_0 \Theta_0) f_r + A_0 \Theta_0 \left(\sum_r f_r - f \right) \\ &= A_0 \Theta_0 L u + \tau B_r w_r, \end{aligned}$$

which proves (3.8) for $\kappa = 0$. Next we fix some $\kappa \geq 1$ and transform $r^{(i)}$, for $i = 1, \ldots, \kappa$, by applying (3.7) with $\alpha = \beta$ and $|\alpha| = i$ when $|\alpha| + 1 \leq \kappa + 1 \leq l$. Then we get

$$\begin{aligned} r^{(i)} &= \sum_{|\alpha|=i, |\beta|=1} c_{\alpha\beta} \mathcal{R} w_{\alpha\beta} + \tau \sum_{|\alpha|=i, |\beta|=1} \mathcal{Q}_{\alpha\beta} w_{\alpha\beta} + \sum_{|\alpha|=i} b_\alpha^+ w_\alpha \\ &= \sum_{|\alpha|=i} b_\alpha^+ w_\alpha + \sum_{|\alpha|=i+1} c_\alpha \mathcal{R} w_\alpha + \tau r^{(i+1)}. \end{aligned}$$

This shows how $r^{(1)}, r^{(2)}, \ldots, r^{(\kappa+1)}$ are related to each other and proves the proposition. □

Our next step is to "solve" (3.8) with respect to w by the method of successive iterations, i.e., by substituting w given by (3.8) into the right-hand side of the same equation. In the process of doing so we encounter only one difficulty when the second term on the right is plugged into the third one and we have to develop expressions like $\mathcal{R}(b_\alpha^+ w)$ into power series in τ. We transform these terms by using (3.5) and (3.6). First we note the following:

LEMMA 3.9. *If $\kappa \geq 0$ is an integer and $\alpha, \beta \in \mathcal{N}$ and $|\beta| + \kappa \leq l$, then*

$$\mathcal{R}(b^+_\alpha w_\beta) = \sum_{i=0}^{\kappa} \tau^i \sum_{|\gamma|=i} c_{\alpha 0\gamma} \mathcal{R} w_{\beta\gamma} + \sum_{i=1}^{\kappa} \tau^i \sum_{|\gamma|=i-1} b^+_{\alpha 0\gamma} w_{\beta\gamma} + \tau^{\kappa+1} \sum_{|\gamma|=\kappa} \mathcal{Q}_{\alpha 0\gamma} w_{\beta\gamma}, \tag{3.9}$$

where for any multi-numbers μ, ν

$$\sum_{|\gamma|=0} c_{\nu\gamma} \mathcal{R} w_{\mu\gamma} := c_\nu \mathcal{R} w_\mu, \quad \sum_{|\gamma|=0} b^+_{\nu\gamma} w_{\mu\gamma} := b^+_\nu w_\mu,$$
$$\sum_{|\gamma|=0} \mathcal{Q}_{\nu\gamma} w_{\mu\gamma} : = \mathcal{Q}_\nu w_\mu.$$

Proof. If $\kappa = 0$, then $w_\beta \in W_0$ and (3.9) follows from (3.5). If $\kappa \geq 1$, we first claim that

$$\mathcal{Q}_\alpha w_\beta = \sum_{i=0}^{\kappa-1} \tau^i \sum_{|\gamma|=i+1} c_{\alpha\gamma} \mathcal{R} w_{\beta\gamma} + \sum_{i=0}^{\kappa-1} \tau^i \sum_{|\gamma|=i} b^+_{\alpha\gamma} w_{\beta\gamma} + \tau^\kappa \sum_{|\gamma|=\kappa} \mathcal{Q}_{\alpha\gamma} w_{\beta\gamma}. \tag{3.10}$$

Indeed, if $\kappa = 1$, formula (3.10) is just (3.7). If (3.10) is true for some $\kappa \geq 1$ and $|\beta| + \kappa + 1 \leq l$, then we use (3.7) with $\beta\gamma$ in place of β and for $|\gamma| = \kappa$ transform the last term in (3.10) to obtain

$$\mathcal{Q}_{\alpha\gamma} w_{\beta\gamma} = c_{\alpha\gamma r} \mathcal{R} w_{\beta\gamma r} + b^+_{\alpha\gamma} w_{\beta\gamma} + \tau \mathcal{Q}_{\alpha\gamma r} w_{\beta\gamma r}.$$

We substitute this result into (3.10) and see that induction on κ proves our claim. For $\kappa \geq 1$ we use (3.10) with $\alpha 0$ in place of α and finish the proof of the lemma by referring to (3.5). □

In the above sums with respect to γ this term was running through $\mathcal{N}$. It is more convenient to restrict γ to $\mathcal{M}$, introduced after Definition 2.8.

LEMMA 3.10. *The following statements hold.*

(i) *Let $\gamma = \gamma_1\gamma_2 \dots \gamma_i \in \mathcal{M}$, $|\gamma| \leq l$. Then on W_l we have*

$$L_\gamma = (-1)^{|\gamma|-1} L_{\gamma_1} \dots L_{\gamma_i} \quad \text{and} \quad f_\gamma = (-1)^{|\gamma|-1} L_{\gamma_1} \dots L_{\gamma_{i-1}} f_{\gamma_i},$$

(ii) *Let $\beta, \gamma \in \mathcal{M}$, $|\beta| + |\gamma| \leq l$. Then on W_l we have*

$$L_\beta L_\gamma = -L_{\beta\gamma} \quad \text{and} \quad L_\beta f_\gamma = -f_{\beta\gamma}.$$

(iii) *Let* $\alpha \in \mathcal{N}$, $|\alpha| \le l$. *Then there exist constants* $c(\gamma) = c(\alpha,\gamma) \in \{0, \pm 1\}$ *defined for all* $\gamma \in \mathcal{M}$ *with* $|\gamma| = |\alpha|$, *such that*

$$L_\alpha = \sum_{\gamma \in \mathcal{M}, |\gamma|=|\alpha|} c(\gamma) L_\gamma, \quad f_\alpha = \sum_{\gamma \in \mathcal{M}, |\gamma|=|\alpha|} c(\gamma) f_\gamma. \tag{3.11}$$

Proof. Part (i) follows immediately from the definition of L_γ, f_γ by induction on $|\gamma|$. Part (i) obviously implies Part (ii). Part (iii) clearly holds for $\alpha = 0$ and $\alpha = r \in \{1, \dots, m\}$. Assume that equations (3.11) hold for some $\alpha \in \mathcal{N}$, $|\alpha| < l$. Then

$$L_{\alpha r} = -L_\alpha L_r = -\sum_{\gamma \in \mathcal{M}, |\gamma|=|\alpha|} c(\gamma) L_\gamma L_r = \sum_{\gamma \in \mathcal{M}, |\gamma|=|\alpha|} c(\gamma) L_{\gamma r},$$
$$f_{\alpha r} = -L_\alpha f_r = -\sum_{\gamma \in \mathcal{M}, |\gamma|=|\alpha|} c(\gamma) L_\gamma f_r = \sum_{\gamma \in \mathcal{M}, |\gamma|=|\alpha|} c(\gamma) f_{\gamma r},$$

for $r \in \{1, 2 \dots, m\}$, and

$$L_{\alpha 0} = L L_\alpha = \sum_{r=1}^m L_r \sum_{\gamma \in \mathcal{M}, |\gamma|=|\alpha|} c(\gamma) L_\gamma = -\sum_{r=1}^m \sum_{\gamma \in \mathcal{M}, |\gamma|=|\alpha|} c(\gamma) L_{r\gamma},$$
$$f_{\alpha 0} = L f_\alpha = \sum_{r=1}^m L_r \sum_{\gamma \in \mathcal{M}, |\gamma|=|\alpha|} c(\gamma) f_\gamma = -\sum_{r=1}^m \sum_{\gamma \in \mathcal{M}, |\gamma|=|\alpha|} c(\gamma) f_{r\gamma},$$

which prove (iii) by induction on $|\alpha|$. □

LEMMA 3.11. *For any* $\alpha, \beta \in \mathcal{N}$ *with* $|\beta| + \mu + 1 \le l$ *we have*

$$\mathcal{Q}_\alpha w_\beta = O_\mu(1).$$

Proof. Observe that for $\bar{w} := w - \varphi$

$$L_\beta \bar{w} = A_k \Theta_k (L_\beta L_k w + L_\beta f_k) = A_0 \Theta_0 (L L_\beta \bar{w} - u) + A_k \Theta_k (L_\beta L_k w + L_\beta f_k),$$

where $u := L L_\beta \bar{w}$. Hence

$$L_\beta \bar{w} = -\mathcal{R} L L_\beta \bar{w} + \mathcal{R}_k \Theta_k (L_\beta L_k w + L_\beta f_k).$$

Now it only remains to recall that $w_\beta = L_\beta \bar{w} + f_\beta + \varphi_\beta$, where f_β, $\varphi_\beta := L_\beta \varphi$ are time independent, and to use Lemmas 3.3 and 3.5. The lemma is proved. □

Recall that the operators S_σ are defined in (2.30) and the sets $A(i)$ in (2.32), and denote

$$B(i,j) = \{(\alpha, \beta) : \alpha \in \mathcal{N}, \beta \in \mathcal{M}, |\alpha| = i, |\beta| \le j\}.$$

LEMMA 3.12. *Let $\kappa, \mu \geq 0$ be integers and $\alpha \in \mathcal{N}, \beta \in \mathcal{M}$, $\sigma \in \mathcal{J}$. Assume that*

$$|\sigma| + |\beta| + \kappa + \mu + 1 \leq l. \tag{3.12}$$

Then

$$S_\sigma(b^+_\alpha w_\beta) = \sum_{i=0}^{\kappa} \tau^i \sum^*_{A(|\sigma|+|\beta|+i)} S_{\sigma_1} w_{\beta_1} + \sum_{i=1}^{\kappa} \tau^i \sum^*_{B(|\alpha|+i,|\sigma|+|\beta|+i-1)} b^+_{\alpha_1} w_{\beta_1} + O_\mu(\tau^{\kappa+1}). \tag{3.13}$$

Proof. For $\sigma = e$, when $S_\sigma = \mathcal{R}$, equation (3.13) is just a different form of equation (3.9), which is applicable since $|\beta| + \kappa \leq l$. Indeed, owing to Lemma 3.10 (iii),

$$\sum_{\gamma \in \mathcal{N}, |\gamma| = i} c_{\alpha 0 \gamma} \mathcal{R} w_{\beta\gamma} = \sum^*_{A(|\sigma|+|\beta|+i)} S_{\sigma_1} w_{\beta_1},$$
$$\sum_{\gamma \in \mathcal{N}, |\gamma| = i-1} b^+_{\alpha 0 \gamma} w_{\beta\gamma} = \sum^*_{B(|\alpha|+i,|\sigma|+|\beta|+i-1)} b^+_{\alpha_1} w_{\beta_1}.$$

Furthermore, by Lemma 3.11 for $|\gamma| = \kappa$,

$$\mathcal{Q}_{\alpha 0 \gamma} w_{\beta\gamma} = O_\mu(1), \quad \text{since} \quad |\beta| + \kappa + \mu + 1 \leq l.$$

For $|\sigma| \geq 1$ we proceed by induction on the length $\ell(S_\sigma)$ of

$$S_\sigma = \mathcal{R}L_{\beta_1} \cdot \cdots \cdot \mathcal{R}L_{\beta_j},$$

which we define to be j. If $\ell(S_\sigma) = 1$, then $\sigma \in \mathcal{M}$, $S_\sigma = \mathcal{R}L_\sigma$, and it suffices to notice that for $\beta \in \mathcal{M}$

$$S_\sigma(b^+_\alpha w_\beta) = \mathcal{R}L_\sigma(b^+_\alpha w_\beta) = -\mathcal{R}(b^+_\alpha w_{\sigma\beta}) = -S_e(b^+_\alpha w_{\beta'}), \tag{3.14}$$

where $\beta' = \sigma\beta \in \mathcal{M}$ and

$$|\beta'| + \kappa + \mu + 1 = |\sigma| + |\beta| + \kappa + \mu + 1 \leq l.$$

Assume that (3.13) holds whenever $\ell(S_\sigma) = s$, and take an S_σ such that $\ell(S_\sigma) = s + 1$. Then $S_\sigma = \mathcal{R}L_\nu S_{\sigma'}$, where

$$\nu, \sigma' \in \mathcal{M}, \quad |\nu| + |\sigma'| = |\sigma|, \quad \ell(S_{\sigma'}) = s.$$

Furthermore,

$$|\sigma'| + |\beta| + \kappa + \mu' + 1 \leq l,$$

where $\mu' = \mu + |\nu|$. By the induction hypothesis

$$S_{\sigma'}(b^+_\alpha w_\beta) = \sum_{i=0}^{\kappa} \tau^i \sum\nolimits^*_{A(|\sigma'|+|\beta|+i)} S_{\sigma_1} w_{\beta_1} + \sum_{i=1}^{\kappa} \tau^i \sum\nolimits^*_{B(|\alpha|+i,|\sigma'|+|\beta|+i-1)} b^+_{\alpha_1} w_{\beta_1} + O_{\mu'}(\tau^{\kappa+1}).$$

We apply $\mathcal{R}L_\nu$ to both parts of this equality and take into account that $L_\nu w_{\beta_1} = -w_{\nu\beta_1}$ and $|\nu| + |\sigma'| = |\sigma|$. Then similarly to (3.14) we get that

$$S_\sigma(b^+_\alpha w_\beta) = \sum_{i=0}^{\kappa} \tau^i \sum\nolimits^*_{A(|\sigma|+|\beta|+i)} S_{\sigma_1} w_{\beta_1} + \sum_{i=1}^{\kappa} \tau^i \sum\nolimits^*_{B(|\alpha|+i,|\sigma|+|\beta|+i-1)} S_e(b^+_{\alpha_1} w_{\beta_1}) + O_\mu(\tau^{\kappa+1}). \tag{3.15}$$

Now we transform the second term on the right. Take

$$(\alpha_1, \beta_1) \in B(|\alpha| + i, |\sigma| + |\beta| + i - 1)$$

and notice that then $|\beta_1| \le |\sigma| + |\beta| + i - 1$. Hence by assumption (3.12)

$$|\beta_1| + \kappa - i + \mu + 1 < l.$$

Therefore, by the result for $\sigma = e$,

$$S_e(b^+_{\alpha_1} w_{\beta_1}) = \sum_{j=0}^{\kappa-i} \tau^j \sum\nolimits^*_{A(|\beta_1|+j)} S_{\sigma_2} w_{\beta_2} + \sum_{j=1}^{\kappa-i} \tau^j \sum\nolimits^*_{B(|\alpha_1|+j,|\beta_1|+j-1)} b^+_{\alpha_2} w_{\beta_2} + O_\mu(\tau^{\kappa-i+1}).$$

We substitute this result into (3.15) and obtain (3.13) after collecting the coefficients of τ^{i+j} and noticing that, if

$$(\alpha_1, \beta_1) \in B(|\alpha| + i, |\sigma| + |\beta| + i - 1)$$

and

$$(\alpha_2, \beta_2) \in B(|\alpha_1| + j, |\beta_1| + j - 1),$$

then

$$(\alpha_2, \beta_2) \in B(|\alpha| + i + j, |\sigma| + |\beta| + i + j - 1).$$

This justifies the induction and finishes the proof of the lemma. □

In the following proposition we use the fact that

$$B^*(i,j) = \bigcup_{i_1=1}^{i} B(i_1,j).$$

This proposition finishes the proof of Theorem 2.18, as is seen by taking $j = k$ in (3.16).

Proposition 3.13. *Let $k \geq 0$ be an integer such that $2k + 2 \leq l$. Then for any $j = 0, 1, \ldots, k$ we have*

$$w = v + \sum_{i=1}^{j} \tau^i {\sum_{A(2i)}}^{*} S_\sigma v_\beta + \sum_{i=j+1}^{k} \tau^i {\sum_{A(i+j+1)}}^{*} S_{\sigma_1} w_{\beta_1} + \sum_{i=1}^{k} \tau^i {\sum_{B^*(i,i+j)}}^{*} b^{+}_{\alpha_1} w_{\beta_1} + O_0(\tau^{k+1}), \tag{3.16}$$

where $v_\beta := L_\beta v + f_\beta$. Furthermore, if $j \geq 1$, then in (3.16) we have

$$ {\sum_{A(2)}}^{*} S_\sigma v_\beta = \sum_{i,j=1}^{m} (c_{ij} - c_{j0}) \mathcal{R} v_{ij}. \tag{3.17}$$

Proof. We prove formula (3.16) by induction on j. By Proposition 3.8 and Lemma 3.11 (where we use $k + 2 \leq l$) we have

$$w = v + \sum_{i=1}^{k} \tau^i \sum_{|\beta|=i} b^{+}_{\beta} w_\beta + \sum_{i=1}^{k} \tau^i \sum_{|\beta|=i+1} c_\beta \mathcal{R} w_\beta + O_0(\tau^{k+1}), \tag{3.18}$$

which means that (3.16) holds for $j = 0$, since by Lemma 3.10 (iii)

$$\sum_{|\beta|=i} b^{+}_{\beta} w_\beta = \sum_{|\beta|=i} b^{+}_{\beta} \sum_{\gamma\in\mathcal{M}, |\gamma|=i} c(\beta,\gamma) w_\gamma = {\sum_{B^*(i,i)}}^{*} b^{+}_{\alpha_1} w_{\beta_1},$$

$$\sum_{|\beta|=i+1} c_\beta \mathcal{R} w_\beta = \sum_{|\beta|=i+1} c_\beta \sum_{\gamma\in\mathcal{M}, |\gamma|=i+1} c(\beta,\gamma) \mathcal{R} w_\gamma = {\sum_{A(i+1)}}^{*} S_{\sigma_1} w_{\beta_1}.$$

Next, assume that $k \geq 1$ and (3.16) holds for some $j \in \{0, \ldots, k-1\}$. Transform the first term with $i = j + 1$ in the second sum on the right in (3.16) by using Lemma 3.12. To prepare the transformation take

$$(\sigma_1, \beta_1) \in A(2i) = A(i + j + 1)$$

so that $|\sigma_1|+|\beta_1| \le 2i$ and apply the operator $S_{\sigma_1} L_{\beta_1}$ to both parts of equation (3.8) with $k-i$ in place of κ. Then we obtain

$$\begin{aligned} S_{\sigma_1} w_{\beta_1} = S_{\sigma_1} v_{\beta_1} &+ \sum_{i_1=1}^{k-i} \tau^{i_1} \sum_{|\alpha_1|=i_1} S_{\sigma_1}(b^+_{\alpha_1} L_{\beta_1} w_{\alpha_1}) \\ &+ \sum_{i_1=1}^{k-i} \tau^{i_1} \sum_{|\alpha_1|=i_1+1} c_{\alpha_1} S_{\sigma_1} L_{\beta_1} \mathcal{R} w_{\alpha_1} + \tau^{k-i+1} r^{(k-i+1)}, \end{aligned}$$

where

$$r^{(k-i+1)} = \sum_{|\alpha|=k-i+1} S_{\sigma_1} L_{\beta_1} \mathcal{Q}_\alpha w_\alpha .$$

Since

$$\begin{aligned} l - (k - i + 1 + |\beta_1| + |\sigma_1|) &\ge l - (k + i + 1) \\ &\ge l - (2k+1) \ge 1, \end{aligned}$$

we have $r^{(k-i+1)} = O_0(1)$. We remark that this is the only place where we need $l \ge 2k+2$. Hence by Lemma 3.10 (iii)

$$\begin{aligned} S_{\sigma_1} w_{\beta_1} &= S_{\sigma_1} v_{\beta_1} + \sum_{i_1=1}^{k-i} \tau^{i_1} \sideset{}{^*}\sum_{(\alpha_2,\beta_2)\in B(i_1,|\beta_1|+i_1)} S_{\sigma_1}(b^+_{\alpha_2} w_{\beta_2}) \\ &\quad + \sum_{i_1=1}^{k-i} \tau^{i_1} \sideset{}{^*}\sum_{A(|\sigma_1|+|\beta_1|+i_1+1)} S_{\sigma_2} w_{\beta_2} + O_0(\tau^{k-i+1}) \\ &=: J_1 + \cdots + J_4. \end{aligned} \tag{3.19}$$

Now using Lemma 3.12 with $k - i - i_1$ in place of κ and 0 in place of μ we transform the terms of J_2. For $|\beta_2| \le |\beta_1| + i_1$ we have (recall that $(\sigma_1, \beta_1) \in A(2i)$)

$$\begin{aligned} |\sigma_1| + |\beta_2| + k - i - i_1 + 1 &\le |\sigma_1| + |\beta_1| + k - i + 1 \\ &\le i + k + 1 \le 2k + 1 < l. \end{aligned}$$

Therefore

$$\begin{aligned} S_{\sigma_1}(b^+_{\alpha_2} w_{\beta_2}) = &\sum_{i_2=0}^{k-i-i_1} \tau^{i_2} \sideset{}{^*}\sum_{A(|\sigma_1|+|\beta_2|+i_2)} S_{\sigma_3} w_{\beta_3} \\ &+ \sum_{i_2=1}^{k-i-i_1} \tau^{i_2} \sideset{}{^*}\sum_{B(|\alpha_2|+i_2,|\sigma_1|+|\beta_2|+i_2-1)} b^+_{\alpha_3} w_{\beta_3} + O_0(\tau^{k-i-i_1+1}). \end{aligned}$$

We plug this result into J_2. In order to collect the coefficients of $\tau^{i_1+i_2}$ notice that for

$$(\sigma_3,\beta_3)\in A(|\sigma_1|+|\beta_2|+i_2)$$

and

$$(\alpha_2,\beta_2)\in B(i_1,|\beta_1|+i_1)$$

we have

$$|\sigma_3|+|\beta_3|\le|\sigma_1|+|\beta_2|+i_2\le|\sigma_1|+|\beta_1|+i_1+i_2.$$

Furthermore, if

$$(\alpha_3,\beta_3)\in B(|\alpha_2|+i_2,|\sigma_1|+|\beta_2|+i_2-1),$$

then

$$|\alpha_3|=|\alpha_2|+i_2=i_1+i_2,\quad |\beta_3|\le|\sigma_1|+|\beta_2|+i_2-1<|\sigma_1|+|\beta_1|+i_1+i_2.$$

It follows that J_2 can be written as

$$\sum_{i_1=1}^{k-i}\tau^{i_1}\left(\sum_{A(|\sigma_1|+|\beta_1|+i_1)}^{*}S_{\sigma_2}w_{\beta_2}+\sum_{B(i_1,|\sigma_1|+|\beta_1|+i_1)}^{*}b^{+}_{\alpha_2}w_{\beta_2}\right)+O_0(\tau^{k-i+1}),$$

which just amounts to saying that visually in the definition of J_2 one can erase S_{σ_1}, carry all differentiations in it onto w_{β_2}, and still preserve (3.19). Of course, when speaking about "all differentiations" we mean the case that L_r are differential operators. Now from this new form of (3.19) for $S_{\sigma_1}w_{\beta_1}$ we see that

$$\begin{aligned}(3.20)\qquad \tau^{j+1}\sum_{A(2j+2)}^{*}S_{\sigma_1}w_{\beta_1}&=O_0(\tau^{k+1})+\tau^{j+1}\sum_{A(2j+2)}^{*}S_{\sigma_1}v_{\beta_1}\\&\quad+\sum_{i_1=1}^{k-j-1}\tau^{i_1+j+1}\left(\sum_{A(|\sigma_1|+|\beta_1|+i_1+1)}^{*}S_{\sigma_2}w_{\beta_2}\right.\\&\quad\left.+\sum_{B(i_1,|\sigma_1|+|\beta_1|+i_1)}^{*}b^{+}_{\alpha_2}w_{\beta_2}\right).\end{aligned}$$

Next we notice again that for $(\sigma_1,\beta_1)\in A(2j+2)$ and

$$|\sigma_2|+|\beta_2|\le|\sigma_1|+|\beta_1|+i_1+1$$

we have

$$|\sigma_2|+|\beta_2|\le j+2+i_1+j+1,$$

whereas if

$$|\beta_2|\le|\sigma_1|+|\beta_1|+i_1,$$

then

$$|\beta_2| \le j+1+i_1+j+1.$$

Therefore, after changing $i_1+j+1 \to i$ $(\ge j+2)$ we get

$$\tau^{j+1} \sum_{A(2j+2)}^{*} S_{\sigma_1} w_{\beta_1} = O_0(\tau^{k+1}) + \tau^{j+1} \sum_{A(2j+2)}^{*} S_{\sigma_1} v_{\beta_1} + \sum_{i=j+2}^{k} \tau^i \left(\sum_{A(i+j+2)}^{*} S_{\sigma_2} w_{\beta_2} + \sum_{B^*(i,i+j+1)}^{*} b_{\alpha_2}^{+} w_{\beta_2} \right).$$

This shows that the term with $i = j+1$ in the second sum on the right in (3.16) can be eliminated on the account of changing other terms with simultaneous shift $j \to j+1$, which gives formula (3.16) with $j+1$ in place of j. Thus the induction on j proves (3.16).

To prove (3.17) observe that, as follows from (3.20), the transformation of the first term with $i = j+1$ in the second sum the right in (3.16) does not affect the first j terms in the first sum in (3.16). Thus, once the first term in this sum appears, it remains unchanged as we move along. The first term appears when $j = 0$ and according to (3.18) we have to transform

$$\tau \sum_{|\beta|=2} c_\beta \mathcal{R} w_\beta,$$

which by (3.8) equals

$$\tau \sum_{|\beta|=2} c_\beta \mathcal{R} v_\beta = \tau \mathcal{R} \left(\sum_{i,j=1}^{m} c_{ij} v_{ij} + \sum_{i=1}^{m} c_{i0} v_{i0} + \sum_{j=0}^{m} c_{0j} v_{0j} \right) =: \tau \mathcal{R} P,$$

plus terms involving higher powers of τ. It only remains to observe that $L_0 = 0$,

$$\begin{aligned} v_{0j} &= L_{0j} v + f_{0j} = -L_0 L_j v - L_0 f_j = 0, \quad v_{i0} = L_{i0} v + f_{i0} = L L_i v + L f_i \\ &= \sum_{k=1}^{m} (L_k L_i v + L_k f_i) = -\sum_{k=1}^{m} v_{ki}, \quad i = 1, \dots, m, \end{aligned}$$

so that

$$P = \sum_{i,j=1}^{m} c_{ij} v_{ij} + \sum_{j=1}^{m} c_{j0} v_{j0} = \sum_{i,j=1}^{m} (c_{ij} - c_{j0}) v_{ij}.$$

This leads to (3.17) and the proof of the proposition is complete. □

4. An application to parabolic PDEs

Here we give an application of our general scheme to splitting-up for parabolic partial differential equations. We will see how to obtain part of the results in [4] in time-homogeneous case and derive further properties of these approximations. For $p > 1$ and integers $r \geq 1$ we denote by W^r_p the Sobolev space defined as the closure of C_0^∞ functions $\varphi : \mathbb{R}^d \to \mathbb{R}$ in the norm

$$\|\varphi\|_{r,p} := \left(\sum_{|\gamma| \leq r} \int_{\mathbb{R}^d} |D^\gamma \varphi(x)|^p \, dx \right)^{1/p},$$

where $D^\gamma := D_1^{\gamma_1} \dots D_d^{\gamma_d}$ for multi-indices $\gamma = (\gamma_1, \dots, \gamma_d)$ of length $|\gamma| := \gamma_1 + \gamma_2 + \cdots + \gamma_d$. In this section we fix a number $p \geq 2$.

We consider the problem

$$D_t v(t,x) = Lv(t,x) + f(x), \qquad t \in (0,T], \quad x \in \mathbb{R}^d, \tag{4.1}$$

$$v(0,x) = \varphi(x), \quad x \in \mathbb{R}^d, \tag{4.2}$$

where L is an operator of the form

$$L = a^{ij}(x) D_{ij} + a^i(x) D_i + a(x), \tag{4.3}$$

where a^{ij}, a^i, a, f, and φ are real-valued functions on $\mathbb{R}^d$.

Imagine that in order to solve (4.1)–(4.2) numerically we split equation (4.1) into the equations

$$D_t u(t,x) = L_r u(t,x) + f_r(x), \quad r = 1, 2, \dots, m \tag{4.4}$$

with

$$L_r = a_r^{ij}(x) D_{ij} + a_r^i(x) D_i + a_r(x), \quad L = \sum_{r=1}^m L_r, \quad f = \sum_{r=1}^m f_r,$$

such that these equations are more pleasant from the point of view of computing their solutions than the original one. This motivates the multi-stage splitting method, which we describe below.

We need some assumptions, in which $\nu \geq 2$ is a fixed number.

Assumption 4.1 (Ellipticity of L_r). For each $r = 1, 2, \dots, m$, $\lambda, x \in \mathbb{R}^d$,

$$a_r^{ij}(x) \lambda^i \lambda^j \geq 0.$$

Assumption 4.2 (Smoothness of data).

(i) For all multi-indices ρ satisfying $|\rho| \leq \nu$ the partial derivatives

$$D^\rho a_r^{ij}, \quad D^\rho a_r^i, \quad D^\rho a_r \quad \text{for } i,j = 1, 2, \dots, d, \ r = 1, 2, \dots, m$$

exist and are bounded in magnitude by K.

(ii) We have $\varphi, f_r \in W^\nu_p$ and

$$\|\varphi\|_{\nu,p} \le K, \quad \|f_r\|_{\nu,p} \le K, \quad r = 1, 2, \dots, m.$$

It is well known (see, for instance, Theorem 3.1 in [2] and recall that $\nu \ge 2$) that under the above conditions there is a unique W^ν_p-valued weakly continuous function $v(t)$, $t \ge 0$, such that

$$v(t) = \varphi + \int_0^t (Lv(s) + f)\, ds, \tag{4.5}$$

where one understands the integral as Bochner's weak (=strong) integral or, equivalently, one understands the equation in the sense of integral identity obtained by multiplying by test functions and integrating with respect to x. In the same sense we understand all differential equations in this section.

Hence under the above conditions equations (4.1) and (4.4) with initial conditions $v(0) = \varphi$, $u(0) = \varphi$ admit unique solutions v and u, respectively.

We want to approximate the solution v of (4.1)–(4.2) by using the splitting-up method, i.e., by solving equations (4.4) successively with appropriate initial conditions on appropriate time intervals. Namely, we take a number $\tau \in (0, 1]$, recall that the set T_τ is introduced in (1.5), and define an approximation v_τ at the points $t_i := i\tau \in T_\tau$ recursively as follows:

$$\begin{aligned} v_\tau(0) &= \varphi, \\ v_\tau(t_{i+1}) &= \mathbb{P}^{(m)}_\tau \dots \mathbb{P}^{(2)}_\tau \mathbb{P}^{(1)}_\tau v_\tau(t_i), \quad t_i, t_{i+1} \in T_\tau, \end{aligned} \tag{4.6}$$

where $\mathbb{P}^{(r)}_t \psi = u(t)$ denotes the solution of equation (4.4) for $t \ge 0$ with initial condition $u(0) = \psi$. Observe that if $f_r \equiv 0$, then (4.6) is essentially (1.4).

It is known that if Assumptions 4.1, 4.2 are satisfied with $\nu \ge \mu + 4$ for some integer $\mu \ge 0$, then

$$\max_{t \in T_\tau} \|v(t) - v_\tau(t)\|_{\mu,p} \le N\tau \tag{4.7}$$

for all $\tau \in (0, 1]$, where N depends only on ν, d, m, T, K, p, μ. Moreover, this rate of convergence is sharp (see [3], where this result is a special case of the rate of convergence estimates for stochastic PDEs). We remark that if $p = 2$, then (4.7) holds under a weaker restriction on ν, namely $\nu \ge 3 + \mu$ (see, for instance, [2]).

Applying the general results of Section 2 we show, in particular, that by suitable combinations of splitting-up approximations we can achieve as fast a convergence as we wish. (See Theorem 4.4 below.)

In order to apply Theorems 2.14 and 2.15 of the abstract setting, we first take $h_1, h_2, \dots, h_m$ from (2.25), introduce the absolutely continuous functions

$$a_r(t) = \tau h_r(mt/\tau), \quad t \ge 0,\ r = 1, 2, \dots, m,$$

and, for a fixed $\tau \in (0,1]$, consider the Cauchy problem

$$dw(t,x) = \sum_{r=1}^{m} (L_r w(t,x) + f_r)\, \dot{a}_r(t)\, dt, \quad w(0,x) = \varphi(x). \tag{4.8}$$

As we have pointed out, we understand this problem as in (4.5) and that due to [6] there is a unique W^{ν}_p-valued solution of (4.8).

From the structure of $\dot{a}_r$ it is easy to see that

$$w(t) = v_\tau(t) \quad \text{for all} \;\; t \in T_\tau. \tag{4.9}$$

This is our major technical observation, which allows us to treat splitting-up approximations by using tools from the standard theory of partial differential equations, which we translated into the general setting in the previous sections.

Next fix integers $\mu, k \geq 0$ and set $l = 2k+2$,

$$W_j = C_w([0,T], W^{\mu+2j}_p), \quad j = 0, 1, \ldots, l,$$

where $C_w([0,T], W^{\mu+2j}_p)$ denotes the Banach space of $W^{\mu+2j}_p$-valued weakly continuous functions f on $[0,T]$ with norm

$$\|f\|_j := \sup_{t\in[0,T]} \|f(t)\|_{\mu+2j,p}.$$

Then W_i is a separable Banach space which is continuously and densely embedded into W_{i-1} for every $i = 1, 2, \ldots, l$. Let $\Theta_0 = \Theta_1 = \cdots = \Theta_m$ be the identity operator on W_0, and define the operators A_k by

$$(A_k\psi)(t) = \int_0^t \psi(s)\dot{a}_k(s)\, ds, \quad t \in [0,T], \quad k = 0, 1, 2, \ldots, m$$

for all $\psi \in W_0$, where $a_0(t) := t$ and the integral is understood as a Bochner integral. View L, L_r as operators acting on the spaces W_j in the natural way

$$(Lv)(t) = Lv(t), \quad (L_r v)(t) = L_r v(t), \quad t \in [0,T],$$

and embed φ, f, f_r into W_j as constant functions of $t \in [0,T]$, i.e.,

$$\varphi(t) = \varphi, \quad f(t) = f, \quad f_r(t) = f_r \quad \text{for all} \;\; t \in [0,T].$$

It is seen that equations (4.1)–(4.2) and (4.8) take the form of equations (1.1) and (1.2), respectively.

To verify that Assumption 2.1 and 2.3 are satisfied we suppose that

$$\nu \geq \mu + 2l. \tag{4.10}$$

Then Assumption 2.1 is obviously satisfied with a constant depending only on T, K, ν, d, m, p.

To check Assumption 2.3 first suppose that $\mu \geq 2$. Then by Theorem 3.1 of [2], for any $g \in W_0$ equation (2.7) has a unique solution $u \in W_0$. We call this solution $\mathcal{R}_k g$ and in this way construct the operators $\mathcal{R}_k$. The fact that they satisfy Assumption 2.3 with a constant depending only on T, K, ν, d, m, p easily follows again from Theorem 3.1 of [2].

If $\mu = 0$ or 1 we need to say more. Since $\nu \geq 2$ (for that matter $\nu \geq 4$ by (4.10)), by Theorem 3.1 of [2] for any $g \in W_l$ equation (2.7) has a unique solution $u \in W_l$ and

$$\sup_{t\in[0,T]} \|u(t)\|^p_{\mu,p} \leq N \sup_{t\in[0,T]} \|g(t)\|^p_{\mu,p}, \tag{4.11}$$

where N depends only on T, K, ν, p, d. Hence, on W_l we have the operators $\mathcal{R}_k$. Owing to (4.11) and the denseness of W_l in W_0, we can extend $\mathcal{R}_k$ to a bounded operator acting in W_i for all $i = 0, 1, \ldots, l$. That it also enjoys property (ii) follows from the fact that by definition (2.7) holds for $g \in W_l$ and $A_0 L\mathcal{R}$ is a bounded operator from W_1 to W_0. To check property (iii) we use one more assertion of Theorem 3.1 of [2], which in our setting says that, under the conditions in (iii), similarly to (4.11) we have

$$\sup_{t\in[0,T]} \|u(t)\|^p_{\mu,p} \leq N \sum_{r=1}^{m} \sup_{t\in[0,T]} \|g_r(t)\|^p_{\mu,p}.$$

By taking $\bar{g}_r \in W_l$ and using this estimate we get that

$$\sup_{t\in[0,T]} \|u(t) - \bar{u}(t)\|^p_{\mu,p} \leq N \sum_{r=1}^{m} \sup_{t\in[0,T]} \|g_r(t) - \bar{g}_r(t)\|^p_{\mu,p},$$

where $\bar{u} := \sum_r \mathcal{R}_r \bar{g}_r$. Since W_l is dense in W_0 and $\mathcal{R}_r$ are continuous in W_0 we see that the property (iii) of Assumption 2.3 holds as well.

Next we introduce the operators $b^{\pm}_{\alpha}$, B_α and constants c_α in the same way as in Remark 2.10 allowing k to run through $0, 1, \ldots, m$ and taking $\bar{\Theta}_\alpha$ to be the identity operators. It is almost obvious that for these objects Assumptions 2.5-2.7 are satisfied. Thus, Theorems 2.14, 2.15, and 2.18 are applicable.

Observe that by Lemma 2.13 the constants c_α are independent of τ and $b_\alpha(i\tau) = 0$ for all integers $i \geq 0$. Therefore, by taking in Theorems 2.14 and 2.15 functionals $\langle w^*, u\rangle$ to be restrictions of $u(\cdot) \in W_l$ at the times in T_τ and also taking into account (4.9) we immediately arrive at the following two results.

THEOREM 4.3. *Let Assumptions 4.1 and 4.2 hold with ν satisfying*

$$\nu \geq 4 + \mu + 4k. \tag{4.12}$$

Then for all $\tau \in (0, 1]$ and $t \in T_\tau$, $x \in \mathbb{R}^d$, the following representation holds:

$$\begin{aligned} v_\tau(t, x) = {} & v(t, x) + \tau v^{(1)}(t, x) \\ & + \tau^2 v^{(2)}(t, x) + \cdots + \tau^k v^{(k)}(t, x) + R^{(k)}_\tau(t, x), \end{aligned} \tag{4.13}$$

where the functions $v^{(1)}$,…,$v^{(k)}$, and $R^{(k)}_\tau$, defined on $[0, T]$, are W^μ_p-valued and weakly continuous. Furthermore, $v^{(j)}$, $j = 1, 2, \ldots, k$, are independent of

τ, and

$$\max_{t\in T_\tau} \|R_\tau^{(k)}(t)\|_{\mu,p} \le N\tau^{k+1}$$

for all $\tau \in (0,1]$, where N depends only on k, μ, d, m, K, p, T.

THEOREM 4.4. *Let Assumptions 4.1 and 4.2 hold with ν satisfying* (4.12). *Then for some constants $\lambda_0, \lambda_1, \dots, \lambda_k$, depending only on k we have*

$$\max_{t\in T_\tau} \left\| \sum_{j=0}^{k} \lambda_j v_{\tau_j}(t) - v(t) \right\|_{\mu,p} \le N\tau^{k+1},$$

where v_{τ_j} denotes the splitting-up approximation on the grid T_{τ_j} with step size $\tau_j := 2^{-j}\tau$. Here N is a constant, depending only on $k, d, m, K,\ \mu, p, T$.

REMARK 4.5. Assume that $v^{(1)} = v^{(2)} = \cdots = v^{(s)} = 0$ in expansion (4.13) for some integer $1 \le s \le k$. In this case we need only take $k+1-s$ terms, $v_\tau, v_{\tau_1}, \dots, v_{\tau_{k-s}}$, in the linear combination to achieve accuracy of order $k+1$. Namely, we define now

$$\bar v_\tau = \sum_{j=0}^{k-s} \lambda_j v_{\tau_j}(t), \quad t \in T_\tau,$$

with

$$(\lambda_0, \lambda_1, \dots, \lambda_{k-s}) := (1, 0, \dots, 0)V^{-1}, \tag{4.14}$$

where V is now a $(k+1-q) \times (k+1-q)$ Vandermonde matrix with entries $V_{i1} := 1$, $V_{i,j} := 2^{-(i-1)(j+s-1)}$ for $i = 1, 2, \dots, k+1-s$ and $j = 2, \dots, k+1-s$. Then Theorem 4.4 remains valid with $\bar v_\tau$ in place of $\sum_{j=0}^{k} \lambda_j v_{\tau_j}$. One can get this from Theorem 4.3 by a simple calculation in the same way as Theorem 2.15 is obtained from Theorem 2.14. For example, if $v^{(1)} = 0$, then

$$\bar v(t) := -\frac{1}{3} v_\tau(t) + \frac{4}{3} v_{\tau_1}(t), \quad t \in T_\tau$$

is an approximation of accuracy τ^3 for the solution v.

Strang's splitting

$$v_\tau(t) : = \mathbb{S}^{t/\tau}(\tau)\varphi, \quad t \in T_\tau \tag{4.15}$$

where

$$\mathbb{S}(\tau) = \mathbb{P}^{(1)}_{\tau/2}\mathbb{P}^{(2)}_{\tau/2} \dots \mathbb{P}^{(m)}_{\tau/2}\mathbb{P}^{(m)}_{\tau/2} \dots \mathbb{P}^{(2)}_{\tau/2}\mathbb{P}^{(1)}_{\tau/2}, \tag{4.16}$$

is known to be of accuracy τ^2. We will see how to obtain this from our results and below describe a whole class of splitting-up approximations, containing Strang's splitting, for which $v^{(1)} = 0$ in expansion (2.34).

Clearly, we get Strang's splitting if we consider the splitting-up method defined by (4.6) with the operators

$$\frac{1}{2}L_1, \frac{1}{2}L_2, \dots, \frac{1}{2}L_m, \frac{1}{2}L_m, \dots, \frac{1}{2}L_2, \frac{1}{2}L_1$$

and free terms

$$\frac{1}{2}f_1, \frac{1}{2}f_2, \dots, \frac{1}{2}f_m, \frac{1}{2}f_m, \dots, \frac{1}{2}f_2, \frac{1}{2}f_1$$

in place of $L_1, \dots, L_m$ and $f_1, \dots, f_m$ in (4.4).

To generalize this scheme, we fix the operators $L_1, \dots, L_m$ and free terms $f_1, \dots, f_m$, satisfying Assumptions 4.1 and 4.2. Let $\xi \geq m$ be an integer, $s_1, \dots, s_\xi \in (0, 1]$ and $k_1, \dots, k_\xi \in \{1, 2, \dots, m\}$ such that

$$\sum_{i=1}^{\xi} s_i \delta_{rk_i} = 1 \quad \text{for all} \quad r = 1, 2, \dots, m. \tag{4.17}$$

Consider the splitting-up approximation (4.15) with

$$\mathbb{S}(\tau) = \mathbb{P}^{(k_\xi)}_{s_\xi \tau} \dots \mathbb{P}^{(k_2)}_{s_2 \tau} \mathbb{P}^{(k_1)}_{s_1 \tau}. \tag{4.18}$$

We say that $\mathbb{S}(\tau)$ is a *symmetric product* if the sequences $k_1, \dots, k_\xi$ and $s_1, \dots, s_\xi$ remain the same when we reverse them. In accordance with the product (4.18) we define now the functions a_r, $r = 1, 2, \dots, m$, by

$$a_r(t) = \tau \kappa_r(jt/\tau), \quad t \geq 0, \tag{4.19}$$

where κ_r is an absolutely continuous function, such that $\kappa_r(0) = 0$, $\dot{\kappa}_r(t)$ is periodic with period ξ, and

$$\dot{\kappa}_r(t) = \sum_{i=1}^{\xi} s_i \delta_{rk_i} \mathbf{1}_{[i-1,i)} \quad \text{for} \quad t \in [0, \xi).$$

Then it is again easy to see that

$$v_\tau(t) = w(t) \qquad \text{for} \quad t \in T_\tau,$$

where now v_τ is defined by (4.15) and (4.18) and w is the solution of the Cauchy problem (4.8) with the functions a_r defined above. Set, as before, $a_0(t) = t$, and define the numbers c_α and the operators $b^{\pm}_\alpha$ and B_α as before. There is nothing to add to what was said before Theorems 4.3 and 4.4 about validity of these theorems for v_τ defined by (4.15) and (4.18) and generally about applicability of Theorems 2.14, 2.15, and 2.18.

Here is a specification of $v^{(1)}$.

THEOREM 4.6. *Under the conditions of Theorem 4.3 define v_τ by* (4.15) *and* (4.18). *Then in* (4.13)

$$v^{(1)} = \frac{1}{2}\sum_{i,j=1}^{m}\int_0^\xi (\kappa_i(t)\,d\kappa_i(t) - \kappa_j(t)\,d\kappa_i(t))\,\mathcal{R}v_{ij},$$

where $\mathcal{R}v_{ij}$ is the solution of (4.1) *with $f = v_{ij} = L_iL_jv + L_if_j$ and 0 initial condition. Thus $v^{(1)}$ vanishes if*

$$\int_0^\xi (\kappa_i(t)\,d\kappa_j(t) - \kappa_j(t)\,d\kappa_i(t)) = 0 \quad \textit{for all} \;\; i,j = 1,2,\ldots,m, \tag{4.20}$$

which is equivalent to

$$\int_0^\xi \kappa_i(t)\,d\kappa_j(t) = \frac{1}{2} \quad \textit{for all} \;\; 1 \le i < j \le m. \tag{4.21}$$

In particular, $v^{(1)} = 0$ if (4.18) *is a symmetric product, which is the case of, say, Strang's approximation* (4.15)–(4.16).

Proof. By Theorem 2.18 expansion (4.13) holds with

$$v^{(1)} = \sum_{i,j=1}^{m}(c_{ij} - c_{j0})\mathcal{R}v_{ij},$$

so $v^{(1)} = 0$ if $c_{ij} - c_{j0} = 0$. Notice that for all $i,j = 0,1,2,\ldots,m$

$$c_{ij} = \int_0^1 (a_i(t) - a_0(t))\;da_j(t) = \int_0^\xi (\kappa_i(t) - \kappa_0(t))\,d\kappa_j(t),$$

where $\kappa_0(t) := t/\xi$. Therefore

$$\begin{aligned}2(c_{ij} - c_{j0}) &= 2\int_0^\xi (\kappa_i(t) - \kappa_0(t))\,d\kappa_j(t) - 2\int_0^\xi (\kappa_j(t) - \kappa_0(t))\,d\kappa_0(t)\\ &= 2\int_0^\xi \kappa_i(t)\,d\kappa_j(t) - 2\kappa_0(\xi)\kappa_j(\xi) + \kappa_0^2(\xi)\\ &= 2\int_0^\xi \kappa_i(t)\,d\kappa_j(t) - 1,\end{aligned}$$

and we have

$$2(c_{ij} - c_{j0}) = \int_0^\xi \kappa_i(t)\,d\kappa_j(t) - \int_0^\xi \kappa_j(t)\,d\kappa_i(t)$$

by taking into account

$$1 = \kappa_i(\xi)\kappa_j(\xi) = \int_0^\xi \kappa_i(t)\,d\kappa_j(t) + \int_0^\xi \kappa_j(t)\,d\kappa_i(t).$$

In particular, $c_{ij} - c_{j0} = -(c_{ji} - c_{i0})$, so $c_{ij} - c_{j0} = 0$ implies $c_{ji} - c_{i0} = 0$. Hence conditions (4.20), (4.21) and their equivalence follow immediately. If $\mathbb{S}(\tau)$ is a symmetric product, then obviously

$$\dot{\kappa}_i(\xi - t) = \dot{\kappa}_i(t) \quad \text{for all} \quad t \in (0, \xi] \setminus \{1, \dots, \xi\},$$

and $\kappa(t) + \kappa_i(\xi - t) = 1$ for all $t \in [0, \xi]$ and $i = 1, 2, \dots, \xi$. Hence

$$\begin{aligned}\int_0^\xi \kappa_i(t)\dot{\kappa}_j(t)\,dt &= \int_0^\xi \kappa_i(\xi - s)\dot{\kappa}_j(\xi - s)\,ds \\ &= \int_0^\xi (1 - \kappa_i(s))\dot{\kappa}_j(s)\,ds = 1 - \int_0^\xi \kappa_i(s)\dot{\kappa}_j(s)\,ds,\end{aligned}$$

which immediately implies equation (4.21). The theorem is proved. □

REMARK 4.7. Clearly, every symmetric product is a product of type (4.16) with respect to a new set of operators L'_i and free terms f'_i, obtained from L_r and f_r by $L'_1 := 2s_1 L_{k_1}$, $f'_1 := 2s_1 f_{k_1}, \dots$.

REMARK 4.8. There are infinitely many non-symmetric products which still satisfy (4.21) and consequently define splitting-up approximations with accuracy of order τ^2. For example, when $m = 2$, every product of the form

$$\mathbb{P}(\tau) = \mathbb{P}^{(2)}_{(1-b)\tau}\mathbb{P}^{(1)}_{(1-a)\tau}\mathbb{P}^{(2)}_{b\tau}\mathbb{P}^{(1)}_{a\tau} \tag{4.22}$$

with $a \neq 1$, and $b = \frac{1}{2(1-a)}$, satisfies (4.21). If $a = 1/2$, then (4.22) is Strang's product with $m = 2$. For $a \neq 1/2$ these products are not symmetric.

Indeed, for κ_1, κ_2 characterizing (4.22) we have

$$\dot{\kappa}_1(t) = a1_{[0,1)}(t) + (1 - a)1_{[2,3)}(t), \quad \dot{\kappa}_2(t) = b1_{[1,2)}(t) + (1 - b)1_{[3,4)}(t),$$

for $t \in (0, 4)$, and

$$\int_0^4 \kappa_1(t)\dot{\kappa}_2(t)\,dt = ab + 1 - b = 1 - b(1 - a) = \frac{1}{2},$$

i.e., condition (4.21) holds. If $a \neq 1/2$, then clearly (4.22) is not symmetric. If $a = 1/2$, then $b = 1$, and (4.22) is Strang's symmetric product with $m = 2$.

5. An application to systems of parabolic PDEs and hyperbolic PDEs

As in Section 4 we consider the problem (4.1)–(4.2) with an operator L given by (4.3) but this time instead of unknown real-valued functions v we consider $\mathbb{R}^q$-valued functions, where q is a fixed number. Accordingly, we assume that a^i, a, a^i_r, a_r are $q \times q$-matrix valued functions with entries $a^{i,\alpha\beta}$, $a^{0\alpha\beta}$,$a^{i,\alpha\beta}_r$, $a^{0\alpha\beta}_r$, respectively, and f, f_r and φ are $\mathbb{R}^q$-valued. Yet, a^{ij} and a^{ij}_r are assumed to be real-valued as in Section 4. We set $p = 2$ and impose the same assumptions as in Section 4 with the obvious interpretation of the norms $\|\cdot\|_{\nu,2}$ for vector-valued functions. We also need the following:

ASSUMPTION 5.1. For each $x, \lambda \in \mathbb{R}^d$, $r = 1, \ldots, m$, and $\alpha, \beta = 1, \ldots, q$ we have

$$\left| \sum_{i=1}^{d} \bar{a}_r^{i,\alpha\beta}(x)\lambda^i \right| \le K \left(\sum_{i,j=1}^{d} a_r^{ij}(x)\lambda^i\lambda^j \right)^{1/2}, \tag{5.1}$$

where $\bar{a}_r^{i,\alpha\beta} = a_r^{i,\alpha\beta} - a_r^{i,\beta\alpha}$.

Observe that Assumption 5.1 is obviously satisfied if

(a) the matrices (a_r^{ij}) are uniformly nondegenerate, so that the systems (4.4) are uniformly parabolic, or

(b) $a_r^{ij} \equiv 0$ and the matrices a_r^i are symmetric, so that the systems (4.4) are first-order symmetric hyperbolic.

It turns out that under Assumptions 4.1, 4.2, and 5.1 all the results of Section 4 are true in the present case. To prove this it suffices to check that the counterpart of Theorem 3.1 of [2] holds for systems. This is a standard albeit somewhat tedious task. The main tool is energy estimates in L_2 of the solution and of its derivatives. One proves these estimates following the proof of Theorem 3.1 of [2] with only one additional observation that can be found, for instance, in Section 7.3 of [1]. Namely, while estimating the L_2-norm of v one has to estimate from above

$$\int_{\mathbb{R}^d} v^\alpha a^{i,\alpha\beta} D_i v^\beta \, dx.$$

Here

$$2v^\alpha a^{i,\alpha\beta} D_i v^\beta = a^{i,\alpha\beta} D_i \left(v^\alpha v^\beta\right) + \bar{a}^{i,\alpha\beta} v^\alpha D_i v^\beta.$$

The integral of the first term on the right is

$$-\int_{\mathbb{R}^d} v^\alpha v^\beta D_i a^{i,\alpha\beta} \, dx \le N\|v\|_{0,2}^2,$$

whereas by Assumption 5.1 and Hölder's inequality the integral of the second term is less than

$$N\|v\|_{0,2} \sum_{\alpha,\beta} \left(\int_{\mathbb{R}^d} \left| \sum_i \bar{a}^{i,\alpha\beta} D_i v^\beta \right|^2 dx \right)^{1/2}$$
$$\le NK\|v\|_{0,2} \left(\int_{\mathbb{R}^d} \sum_{i,j,\beta} a^{ij} D_i v^\beta D_j v^\beta \, dx \right)^{1/2}.$$

We estimate this further by using Young's inequality: $ab \le \varepsilon^{-1}a^2 + \varepsilon b^2$. We note that the appearance of $\|v\|_{0,2}^2$ with large coefficient causes no harm due to Gronwall's inequality, and the term

$$\int_{\mathbb{R}^d} \sum_{i,j,\alpha} a^{ij} D_i v^\alpha D_j v^\alpha \, dx$$

with negative sign appears when we integrate by parts

$$2 \int_{\mathbb{R}^d} \sum_{i,j,\alpha} v^\alpha a^{ij} D_{ij} v^\alpha \, dx,$$

that is, the first term in the formula for $\partial \|v\|^2 / \partial t$.

We hope that after these somewhat sketchy explanations the reader will be able to fill in the necessary details and see that, indeed, all the results of Section 4 are true in the present case.

As an excuse we can say that the main aim of this article is far from proving existence theorem and a priori estimates. Also it is worth noting that certainly one can consider more general degenerate parabolic systems when, say, a^{ij} are matrices.

We want to comment further on the case of hyperbolic symmetric systems when $a^{ij} \equiv 0$. Such systems are extensively treated in the literature from the splitting-up point of view. In that case the direction of time plays no role and it make sense to consider $\mathbb{P}_t^{(r)}$ for negative t. Then in (4.17) one can admit $s_1, \dots, s_\xi \in \mathbb{R}$ rather than $\in (0,1]$ and assert that Theorems 4.3, 4.4, and 4.6 still hold for v_τ defined by (4.15) and (4.18).

Note that, by using the Baker-Campbell-Hausdorff formula, in [17] a splitting-up method is constructed for any even order of accuracy. In particular it is proved that the product

$$\bar{\mathbb{S}}(\tau) = \mathbb{S}(a\tau)\mathbb{S}(b\tau)\mathbb{S}(a\tau),$$

with

$$a = \frac{1}{2 - 2^{1/3}}, \quad b = -\frac{2^{1/3}}{2 - 2^{1/3}}$$

and with Strang's product $\mathbb{S}(\tau)$ with $m = 2$, defines a splitting-up method of fourth order of accuracy. This certainly can be obtained from Theorem 2.18 by computing the coefficients. Then from our results we get that the linear combination

$$-\frac{1}{7}\mathbb{S}^{t/\tau}(\tau)\varphi + \frac{8}{7}\mathbb{S}^{2t/\tau}(\tau/2)\varphi, \quad t \in T_\tau,$$

is an approximation of fifth order of accuracy.

6. An application to ODEs

We consider the ordinary differential equation

$$\dot{x}_t = b_1(x_t) + \dots + b_m(x_t) =: b(x_t), \quad t \geq 0, \tag{6.1}$$

in $\mathbb{R}^d$ with sufficiently smooth and bounded vector fields $b_1, \dots, b_m$ on $\mathbb{R}^d$. We want to investigate the splitting-up method for solving this equation on the basis of solving the equations

$$\dot{x}_t = b_k(x_t) \tag{6.2}$$

for each particular $k = 1, 2, \dots, m$.

Let us denote by $\mathbb{P}_t$ and $\mathbb{P}_t^{(k)}$ the mappings $x \to x_t$, where x_t denotes the solution of (6.1) and (6.2), respectively, with starting point x. Taking a parameter $\tau > 0$, we want to approximate $\mathbb{P}_t$ by means of the products

$$\mathbb{S}(\tau) := \mathbb{P}^{(k_\xi)}_{s_\xi \tau} \cdot \dots \cdot \mathbb{P}^{(k_1)}_{s_1 \tau}, \quad k_1, \dots, k_\xi \in \{1, \dots, m\}, \tag{6.3}$$

at the points t of the grid (1.5), where $\xi \geq m$ is a fixed integer and $s_1, s_2, \dots, s_\xi$ are some real numbers such that (4.17) holds.

It is well-known that for every $x \in \mathbb{R}^d$

$$\max_{t \in T_\tau} |\mathbb{P}_t x - \mathbb{S}^{t/\tau}(\tau) x| \leq N\tau, \tag{6.4}$$

for all $\tau > 0$, where N is a constant which does not depend on τ. It is also known that if (6.3) is a symmetric product, then this estimate holds with τ^2 in place of τ on the right-hand side. We say that the product (6.3) is a method of order k if (6.4) holds for every x with τ^k in place of τ on the right-hand side.

Though for any given $k \geq 1$ the existence of methods of order k is known in the literature (see, e.g., [8], [12], [13], [15], [17]), it is useful to investigate if one can further accelerate any given method by mixing the approximations corresponding to different step sizes. In practice one computes the approximations using the same method with many different step sizes τ anyway, and it takes very little additional computation to mix them.

To formulate our results, let $W_i = C([0,T], C_0^i(\mathbb{R}^d))$ denote the space of bounded continuous functions $u(t,x)$ on $[0,T] \times \mathbb{R}^d$ with values in $\mathbb{R}^d$, such that their derivatives in x up to order i are also bounded and continuous, and

$$\lim_{|x| \to \infty} \sup_{t \in [0,T]} |u(t,x)| = 0.$$

Recall that the functions $\kappa_1, \dots, \kappa_m$, associated with (6.3) are defined after (4.19).

THEOREM 6.1. *Let $k \geq 0$ and l be integers such that $l \geq 2k+2$. Assume that the derivatives of the vector fields $b_1, \dots, b_m$ up to order l are bounded and continuous functions. Then, for $\tau \in (0,1]$, $t \in T_\tau$, $x \in \mathbb{R}^d$, we have*

$$\mathbb{S}^{t/\tau}(\tau) x = x_t(x) + \sum_{j=1}^{k} \tau^j h_j(t,x) + R_\tau^{(k)}(t,x), \tag{6.5}$$

where $h_1, h_2, \dots, h_k \in W_0$ are some functions independent of τ and $R_\tau^{(k)} \in W_0$ is such that for any compact set $\mathbb{K} \subset \mathbb{R}^d$ there exists a constant N independent of τ such that

$$\sup_{t \in T_\tau, x \in \mathbb{K}} |R_\tau^{(k)}(t,x)| \leq N\tau^{k+1}.$$

Furthermore, if $k \geq 1$, then for the function h_1 we have

$$h_1 = \sum_{i,j=1}^{m} (c_{ij} - c_{j0}) h_{ij}, \quad c_{ij} = \int_0^\xi (\kappa_i(t) - \kappa_0(t))\, d\kappa_j(t) \tag{6.6}$$

for some $h_{ij} \in W_0$ for $i, j = 1, 2, \dots m$.

Our approach to proving this theorem is based on the observation that the solutions of equation (6.1) are characteristics of the partial differential equation

$$D_t u(t,x) = Lu(t,x),$$

where

$$Lu(t,x) = b^i(x) u_{x^i}(t,x) = \sum_{k=1}^{m} L_k u(t,x), \quad L_k u(t,x) = b_k^i(x) u_{x^i}(t,x).$$

That Theorem 6.1 can be deduced from Theorem 2.18 is shown in [5]. The same approach is applicable to equations on smooth manifolds, one replaces $\mathbb{P}_t x$ in (6.5) with $\varphi(\mathbb{P}_t x)$, and time dependent systems when one just adds one additional coordinate t.

From Theorem 6.1 we easily obtain the following result about accelerating any given splitting-up method after defining

$$(\lambda_0, \lambda_1, \dots, \lambda_{k-q+1}) = (1, 0, \dots, 0) V^{-1},$$

where V is a $(k-q+2) \times (k-q+2)$-matrix with entries $V_{i1} = 1$ and $V_{i,j} = 2^{-(i-1)(q+j-2)}$ for $i = 1, 2, \dots, k-q+2$, $j = 2, \dots, k-q+2$.

THEOREM 6.2. *Let the conditions of Theorem 6.1 hold. Let the product (6.3) be a method of order $q \geq 1$. Then for every compact set $\mathbb{K} \subset \mathbb{R}^d$ there exists a constant N, such that*

$$\max_{t \in T_\tau} \sup_{x \in \mathbb{K}} \left| \mathbb{P}_t x - \sum_{j=0}^{k-q+1} \lambda_j \mathbb{S}_{2^{-j}\tau}^{2^j t/\tau} x \right| \leq N\tau^{k+1}$$

for all $\tau \in (0, 1]$.

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