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VECTOR-VALUED ANALYTIC FUNCTIONS OF BOUNDED MEAN OSCILLATION AND GEOMETRY OF BANACH SPACES

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Introduction

When dealing with vector-valued functions, sometimes is rather difficult to give non trivial examples, meaning examples which do not come from tensoring scalar-valued functions and vectors in the Banach space, belonging to certain classes. This is the situation for vector valued *BMO*. One of the objectives of this paper is to look for methods to produce such examples.

Our main tool will be the vector-valued extension of the following result on multipliers, proved in [MP], which says that the space of multipliers between H^1 and *BMOA* can be identified with the space of Bloch functions $\mathcal{B}$, i.e. $(H^1, BMOA) = \mathcal{B}$ (see Section 3 for notation), which, in particular gives $g * f \in BMOA$ whenever $f \in H^1$ and $g \in \mathcal{B}$.

Given two Banach spaces X, Y it is rather natural to define the convolution of an analytic function with values in the space of operators $\mathcal{L}(X, Y)$, say $F(z) = \sum_{n=0}^{\infty} T_n z^n$, and a function with values in X, say $f(z) = \sum_{n=0}^{\infty} x_n z^n$, as the function given by $F * g(z) = \sum_{n=0}^{\infty} T_n(x_n) z^n$.

It is not difficult to see that the natural extension of the multipliers' result to the vector-valued setting does not hold for general Banach spaces. To be able to get a proof of such a result we shall be using the analogue of certain inequalities, due to Hardy and Littlewood [HL3], in the vector valued setting, namely

$$\left(\int_0^1 (1-r) M_1^2(f', r)\,dr\right)^{\frac{1}{2}} \leq C\|f\|_{H^1}$$

and its dual formulation

$$\|f\|_* \leq C\left(\int_0^1 (1-r) M_\infty^2(f', r)\,dr\right)^{\frac{1}{2}}.$$

This leads us to consider spaces where these inequalities hold when the absolute value is replaced by the norm in the Banach space, which turn out to be very closely related to notions as (Rademacher) cotype 2 and type 2.

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The paper is divided into five sections. We start with a section of preliminary character to recall the notions on geometry of Banach spaces to be used throughout the paper and which also contains the basic properties of vector-valued analytic functions of bounded mean oscillation and functions in the vector valued Bloch space.

Section 2 is devoted to give some sufficients conditions on the derivative of the function or on the Taylor coefficients of the function to assure that the function belongs to $BMOA(X)$. It is shown that one has $M_p(f',r) = O((1-r)^{-\frac{1}{p'}})$ for some $1 \le p < \infty$ or $||x_n|| = O(\frac{1}{n})$ (in the case of B-convex spaces) implies that $f(z) = \sum_{n=0}^{\infty} x_n z^n \in BMOA(X)$.

Section 3 deals with multipliers between spaces of vector valued functions defined on different Banach spaces X and Y. This is done by looking at functions with values in the space of operators $\mathcal{L}(X,Y)$ and considering the natural convolution mentioned above. We also introduce two new notions based on the vector-valued formulations of the Hardy-Littlewood inequalities previously pointed out, called the (HL)-property and the $(HL)^*$-property respectively. It is shown that under the assumptions of (HL)-property on X and $(HL)^*$-property on Y one has $(H^1(X), BMOA(Y)) = \mathcal{B}(\mathcal{L}(X,Y))$.

Section 4 is devoted to the study of these properties. It is shown that the spaces having the so-called (HL) and $(HL)^*$ must satisfy the Paley property (see definition in Section 1) and have type 2 respectively. It is also shown that the natural duality between them holds for UMD spaces. We investigate Lebesgue spaces L^p and Schatten classes σ_p having such properties. The tools to deal with Schatten classes are the use of certain factorization and interpolation results holding for functions in Hardy spaces with values in them.

Finally Section 5 is devoted to present several applications of different nature of the previous results.

Throughout the paper all spaces are assumed to be complex Banach spaces, D stands for the unit disc and $\mathbb{T}$ for its boundary. Given $1 \le p < \infty$, we shall denote by $L^p(X)$ the space of X-valued Bochner p-integrable functions on the circle $\mathbb{T}$ and write $\|f\|_{p,X} = (\int_0^{2\pi} \|f(e^{it})\|^p \frac{dt}{2\pi})^{\frac{1}{p}}$ and $M_{p,X}(F,r) = \|F_r\|_{p,X} = (\int_0^{2\pi} \|F(re^{it})\|^p \frac{dt}{2\pi})^{\frac{1}{p}}$ whenever F is any X-valued analytic function on D. We shall write $L^1(D,X)$ for the space of X-valued Bochner integrable functions on D with respect to the area measure $dA(z)$, and $H^p(X)$ (resp. $H_0^p(X)$) for the vector-valued Hardy spaces, i.e., the space of functions in $L^p(X)$ whose negative (resp. non positive) Fourier coefficients vanish.

Of course Hardy spaces $H^p(X)$ (resp. $H_0^p(X)$) can be regarded as spaces of analytic functions on the disc. Actually they coincide with the closure of the X-valued polynomials, denoted by $\mathcal{P}(X)$ (resp. those which vanish at $z=0$, denoted by $\mathcal{P}_0(X)$,) under the norm given by $\sup_{0<r<1} M_{p,X}(f,r)$.

The reader should be aware that the analytic functions we are considering have boundary values a.e. on $\mathbb{T}$, but this in general does not hold (such a fact actually corresponds to the so called ARNP introduced in [BuD]).

Finally let us point out a notation to be used in the sequel. Whenever a scalar-valued function ϕ is given we write $\phi_z(w) = \phi(zw)$ and look at $z \to \phi_z$ as a vector-valued function. As usual p' is the conjugate exponent of p when $1 \leq p \leq \infty$, i.e. $\frac{1}{p}+\frac{1}{p'} = 1$ and C will stand for a constant that may vary from line to line.

1. Preliminaries on geometry of Banach spaces and vector valued functions

The connection between certain properties in geometry of Banach spaces and vector-valued Hardy spaces is well known. Some of them, like ARNP [BuD] and Paley [BP], were actually introduced to have certain theorems on Hardy spaces hold in the vector-valued setting; others, like UMD [Bu2], B-convexity [MPi] or Fourier type [Pee], were connected to this theory through the boundedness of classical operators like Hilbert transform, Paley projection or Fourier transform for vector-valued functions. In this section we shall recall those to be used in the sequel and give some references to get more information about them.

One of the more relevant properties in the vector-valued Fourier analysis is the so called UMD property. It was introduced in the setting of vector valued martingales, but was shown (see [Bu1], [Bo1]) to be equivalent to the boundedness of the Hilbert transform on $L^p(X)$ for any $1 < p < \infty$. Because of this it is a natural assumption when dealing with vector-valued Hardy spaces.

We shall say that a complex Banach space X is a UMD space if the Riesz projection $\mathcal{R}$, defined by $\mathcal{R}(f) = \sum_{n\geq 0} \hat{f}(n)e^{int}$, is bounded from $L^2(X)$ into $H^2(X)$.

One of the basic facts on this property that we shall use is that the vector valued version of the Fefferman's $H^1 - BMO$-duality theorem holds for UMD spaces (see, for instance, [Bo3], [B2], [RRT]). The reader is referred to the surveys [RF], [Bu2] for information on the UMD property.

Another useful property for our purposes will be the notion of Fourier-type introduced by Peetre [Pee] which corresponds to spaces where the vector-valued analogue of Hausdorff-Young's inequalies holds.

Let us recall that for $1 \leq p \leq 2$, a Banach space X is said to have Fourier type p if there exists a constant $C > 0$ such that

$$\left(\sum_{n=-\infty}^{\infty} \|\hat{f}(n)\|^{p'}\right)^{\frac{1}{p'}} \leq C\|f\|_{L^p(X)}.$$

It is not hard to see that X has Fourier type p if and only if X^* has Fourier type p. Typical examples are L^r for $p \leq r \leq p'$ or those obtained by interpolation between any Banach space and a Hilbert space. The reader is referred to [Pee], [GKT], [K] for some equivalent formulations, connections with interpolation and several examples in the contex of function spaces.

Let us now recall two fundamental notions in geometry of Banach spaces associated to Khintchine's inequalities. Although they are defined in terms of the Rademacher functions, to be denoted r_n, we shall replace them by lacunary sequences $e^{i2^n t}$, which gives an equivalent definition [MPi], [Pi1].

Let $1 \leq p \leq 2 \leq q \leq \infty$. A Banach space has cotype q (resp. type p) if there exists a constant $C > 0$ such that for all $N \in \mathbb{N}$ and for all $x_0, x_1, x_2, \dots x_N \in X$ one has

$$\left(\sum_{k=0}^{N} \|x_k\|^q\right)^{\frac{1}{q}} \leq \left\|\sum_{k=0}^{N} x_k e^{2^k it}\right\|_{1,X}$$

and respectively

$$\left\|\sum_{k=0}^{N} x_k e^{2^k it}\right\|_{1,X} \leq C\left(\sum_{k=0}^{N} \|x_k\|^p\right)^{\frac{1}{p}}.$$

A Banach space is called B-convex if it has (Rademacher)-type > 1.

The reader is referred to [LT] and [Pi2] for some applications of such notions to the Banach space theory.

Let us now state two fundamental theorems, to be used in the sequel, due to J. Bourgain and S. Kwapien respectively.

THEOREM A. [Bo4], [Bo5] *Let X be a complex Banach space. X has Fourier type > 1 if and only if X is B-convex.*

THEOREM B. [Kw] *Let X be a complex Banach space. X is isomorphic to a Hilbert space if and only if X has type 2 and cotype 2.*

Let us finally recall another property, stronger than cotype 2, to be used later on that was introduced in [BP] and depends upon the vector-valued analogue of Paley's inequality [Pa] for Hardy spaces. A complex Banach space X is said to be a Paley space if

$$\left(\sum_{k=0}^{\infty} \|x_{2^k}\|^2\right)^{\frac{1}{2}} \leq C\|f\|_{1,X}$$

for any $f(z) = \sum_{n=0}^{\infty} x_n z^n \in H^1(X)$.

Let us now consider the vector-valued version of analytic *BMO* and the space of Bloch functions $\mathcal{B}$. The reader is referred to [GR], [G] and [Z] for scalar-valued theory on *BMO* and to [ACP] and [Z] for results on scalar-valued Bloch functions.

DEFINITION 1.1. *Given a complex Banach space X, we shall denote by $BMOA(X)$ the space functions $f \in L^1(X)$ with $\hat{f}(n) = 0$ for $n < 0$ such that*

$$\|f\|_{*,X} = \sup_I \frac{1}{|I|} \int_I \|f(e^{it}) - f_I\| \frac{dt}{2\pi} < \infty,$$

where the supremum is taken over all intervals $I \in [0, 2\pi)$, $|I|$ stands for the normalized Lebesgue measure of I and $f_I = \frac{1}{|I|}\int_I f(e^{it})\frac{dt}{2\pi}$.

The norm in the space is given by

$$\|f\|_{BMO(X)} = \left\| \int_0^{2\pi} f(e^{it}) \frac{dt}{2\pi} \right\| + \|f\|_{*,X}.$$

From John-Nirenberg's lemma [G], [GF], which holds in the vector-valued setting, one can actually replace the L^1 norm in the definition by any other L^p norm; that is, for any $1 \leq p < \infty$,

$$\|f\|_{*,X} \approx \sup_I \left(\frac{1}{|I|} \int_I \|f(e^{it}) - f_I\|^p \frac{dt}{2\pi} \right)^{\frac{1}{p}}.$$

Let us point out certain results on the duality to be used later on. Although most of the results on the duality $H^1 - BMO$ for vector-valued functions (see [B2], [Bo3], [RRT]) are given for the space H^1 defined in terms of atoms, it is easy to deduce from the known results the following facts:

For any Banach space X, $BMOA(X^*)$ continuously embeds into $(H^1(X))^*$. Actually if $f \in BMOA(X^*)$ and $g \in \mathcal{P}(X)$ then

$$\left| \int_0^{2\pi} < f(e^{it}), g(e^{-it}) > \frac{dt}{2\pi} \right| \leq \|f\|_{BMOA(X^*)} \|g\|_{1,X}.$$

If X is a UMD space then we actually have the validity of Fefferman's duality result

$$(H^1(X))^* = BMOA(X^*).$$

In the particular case of Hilbert spaces we also have that following formulation in terms of Carleson measures holding (see [G] for a proof which can be reproduced in the Hilbert-valued case).

If $f(z) = \sum_{k=0}^{\infty} x_k z^k$ with $x_k \in H$ then

$$\|f\|_{*,X} \approx \sup_{|z|<1} \left(\int_D (1 - |w|) \|f'(w)\|^2 P_z(\bar{w}) dA(w) \right)^{\frac{1}{2}} \tag{1.1}$$

where P_z is the Poisson Kernel $P_z(w) = \frac{1-|z|^2}{|1-zw|^2}$.

Let us now recall some results on vector-valued Bloch functions.

DEFINITION 1.2. *Given a complex Banach space E we shall use the notation $\mathcal{B}(E)$ for the space of E-valued analytic functions on D, say $f(z) = \sum_{n=0}^{\infty} x_n z^n$, such that*

$$\sup_{|z|<1} (1 - |z|) \|f'(z)\| < \infty.$$

We endow the space with the norm

$$\|f\|_{\mathcal{B}(E)} = \max\{\|f(0)\|, \sup_{|z|<1} (1 - |z|) \|f'(z)\|\}.$$

The next result, although known, is included for the sake of completeness.

PROPOSITION 1.1 (see [ACP], [AS]). Let E be a Banach space and $x_n \in E$.

(i) If $\sup_{\|x^*\|\le 1}\sup_{n\ge 0}\sum_{k=2^n}^{2^{n+1}} |\langle x^*, x_k\rangle| < \infty$ then $\sum_{n=0}^{\infty} x_n z^n \in \mathcal{B}(E)$.

(ii) $\| \sum_{n=0}^{\infty} x_n z^{2^n} \|_{\mathcal{B}(E)} \approx \sup_{n\ge 0} \|x_n\|$.

Proof. (i) Note that for each $\|x^*\| \le 1$,

$$\begin{aligned}
\left\| \sum_{n=1}^{\infty} n\langle x^*, x_n\rangle z^{n-1} \right\| &\le \sum_{n=0}^{\infty}\sum_{k=2^n}^{2^{n+1}} k|\langle x^*, x_k\rangle||z|^{k-1} \\
&\le \left(\sup_{n\ge 0}\sum_{k=2^n}^{2^{n+1}} |\langle x^*, x_k\rangle|\right)\left(\sum_{n=0}^{\infty} 2^{n+1}|z|^{2^n-1}\right) \\
&\le \frac{C}{1-|z|}.
\end{aligned}$$

Hence $\sum_{n=0}^{\infty}\langle x^*, x_n\rangle z^n \in \mathcal{B}$ uniformly in $\|x^*\| \le 1$.
(ii) Take $f(z) = \sum_{n=0}^{\infty} x_n z^{2^n}$. From (i) we have $\| \sum_{n=1}^{\infty} x_n z^{2^n} \|_{\mathcal{B}(E)} \le C \sup_{n\ge 0} \|x_n\|$. The other estimate follows by taking $r = 1 - \frac{1}{2^n}$ in the inequality

$$2^n \|x_n\| r^{2^n-1} \le \sup_{|z|=r} \|f'(z)\| \le \frac{C}{1-r}. \qquad \square$$

PROPOSITION 1.2. *Let X be a Banach space, let K_z denote the Bergman Kernel $K_z(w) = \frac{1}{(1-zw)^2}$ and $T \in \mathcal{L}(L^1(D), X)$. Then $f(z) = T(K_z)$ is a X-valued Bloch function.*

Proof. Observe that $f(z) = \sum_{n=0}^{\infty}(n+1)T(u_n)z^n$ for $u_n(w) = w^n$.
Therefore $f'(z) = \sum_{n=1}^{\infty} n(n+1)T(u_n)z^{n-1} = T(\frac{-2w}{(1-wz)^3})$ and then we have

$$\begin{aligned}
\|f'(z)\| &\le \|T\| \int_0^1 \int_0^{2\pi} \frac{2r}{|1-rze^{it}|^3}\frac{dt}{2\pi}dr \\
&\le C\|T\| \int_0^1 \frac{2r}{(1-r|z|)^2}dr \le C\frac{1}{1-|z|}. \qquad \square
\end{aligned}$$

2. Elementary properties and examples on $BMOA(X)$

Let us start by pointing out some procedures to obtain non trivial examples of vector valued *BMOA* functions. We shall give some simple necessary conditions following [CP] and [BSS].

For such a purpose we shall need some well known lemmas.

LEMMA A. *Let* $0 < p \le q \le \infty$ *and let* g *be an* X*-valued analytic function. Then*

$$M_{q,X}(g, r^2) \le C(1-r)^{\frac{1}{q}-\frac{1}{p}} M_{p,X}(g, r) \qquad (\textit{see } [D, \textit{page } 84]). \tag{2.1}$$

Let $\gamma > 1$; *then*

$$\int_0^{2\pi} \frac{d\theta}{|1 - ze^{i\theta}|^{\gamma}} = O((1-|z|)^{1-\gamma}) \quad (\textit{see } [D, \textit{page } 65]). \tag{2.2}$$

Let $\gamma < \beta$; *then*

$$\int_0^1 \frac{(1-r)^{\gamma-1}}{(1-rs)^{\beta}} dr = O\left((1-s)^{\gamma-\beta}\right) \quad (\textit{see } [SW, \textit{Lemma } 6]). \tag{2.3}$$

The next result has an straightforward generalization to the vector-valued setting.

LEMMA B (Hardy-Littlewood [D, Theorem 5.4]). Let $f\colon D \to X$ be analytic, $i \le p \le \infty$ and $0 < \alpha < 1$.

If $M_{p,X}(f', r) = O(\frac{1}{(1-r)^{1-\alpha}})(r \to 1)$ then

$$\left(\int_0^{2\pi} \|f(e^{it}) - f(e^{i(t+h)})\|^p dt\right)^{\frac{1}{p}} = O(|h|^{\alpha}), (h \to 0).$$

THEOREM 2.1. *Let* f *be a* X*-valued analytic function. If there exists* $0 < p < \infty$ *such that*

$$M_{p,X}(f', r) = O\left((1-r)^{-\frac{1}{p'}}\right)$$

then $f \in BMOA(X)$.

Proof. Note that (2.1) implies that if there exists $0 < p_0 < \infty$ such that $M_{p_0,X}(f', r) = O((1-r)^{-\frac{1}{p_0'}})$ then the same property holds for any $p \ge p_0$. Therefore it suffices to prove the result for $2 < p < \infty$.

In such a case to see that $f \in BMOA(X)$ we can use Lemma B (for $\alpha = \frac{1}{p}$) and the argument in [BSS, Theorem 2.5] that we include for sake of completeness.

Note that Lemma B implies $\int_{-\pi}^{\pi} \|f(e^{i(t-s)}) - f(e^{it})\|^p \frac{dt}{2\pi} \le C|s| \; \forall \delta$.

Assume $I = [-\delta, \delta]$ for some $0 < \delta < \frac{\pi}{2}$ (the general case follows by using translation invariance of the space).

$$\begin{aligned}
\frac{1}{|I|}\int_I \|f(e^{it}) - f_I\|^p \frac{dt}{2\pi} &= \frac{1}{2\delta}\int_{-\delta}^{\delta} \left\| \frac{1}{2\delta}\int_{-\delta}^{\delta} (f(e^{it}) - f(e^{is}))\frac{ds}{2\pi}\right\|^p \frac{dt}{2\pi} \\
&\le \frac{1}{2\delta}\int_{-\delta}^{\delta} \frac{1}{2\delta}\left(\int_{-\delta}^{\delta} \|f(e^{it}) - f(e^{is})\|^p \frac{ds}{2\pi}\right)\frac{dt}{2\pi}
\end{aligned}$$

$$= \frac{1}{4\delta^2}\int_{-\delta}^{\delta}\left(\int_{t-\delta}^{t+\delta}\|f(e^{it})-f(e^{i(t-s)})\|^p\frac{ds}{2\pi}\right)\frac{dt}{2\pi}$$
$$\leq \frac{1}{4\delta^2}\int_{-\pi}^{\pi}\left(\int_{-2\delta}^{2\delta}\|f(e^{it})-f(e^{i(t-s)})\|^p\frac{ds}{2\pi}\right)\frac{dt}{2\pi}$$
$$= \frac{1}{4\delta^2}\int_{-2\delta}^{2\delta}\left(\int_{-\pi}^{\pi}\|f(e^{i(t-s)})-f(e^{it})\|^p\frac{dt}{2\pi}\right)\frac{ds}{2\pi}$$
$$\leq C\frac{1}{4\delta^2}\int_{-2\delta}^{2\delta}|s|\frac{ds}{2\pi}\leq C. \qquad \square$$

Let us now go a bit further and find conditions on the sequence of Taylor coefficients x_n which guarantees that the corresponding analytic function belongs to $BMOA(X)$. Some conditions can easily be achieved for spaces of Fourier type p.

COROLLARY 2.1. *Let $1 < p \leq 2$, let X be Banach space with Fourier type p and (x_n) a sequence in X such that*

$$\sum_{n=1}^{N}\|nx_n\|^p = O(N).$$

Then $f(z)=\sum_{n=1}^{\infty}x_nz^n \in BMOA(X)$.

Proof. Let us first observe that the assumption implies

$$\sup_{n\in\mathbb{N}} 2^{n(p-1)}\sum_{k=2^n}^{2^{n+1}}\|x_k\|^p < \infty.$$

Let us now show that $M_{p',X}(f',r) = O((1-r)^{-\frac{1}{p}})$ and then the result will follow from Theorem 2.1.

It is not difficult to see, using duality, that Fourier type p can be also formultated as

$$\|f\|_{p',X} \leq C\left(\sum_{n\in\mathbb{Z}}\|\hat{f}(n)\|^p\right)^{\frac{1}{p}}.$$

Therefore, from the Fourier type p condition, it follows

$$M_{p',X}(f',r) \leq C\left(\sum_{n=1}^{\infty}n^p\|x_n\|^pr^{p(n-1)}\right)^{\frac{1}{p}}$$

$$\leq C\left(\sum_{n=0}^{\infty}\left(\sum_{k=2^n}^{2^{n+1}}\|x_k\|^p\right)2^{pn}r^{p2^n}\right)^{\frac{1}{p}}$$

$$\leq C\left(\sup_{n\in\mathbb{N}}2^{n(p-1)}\sum_{k=2^n}^{2^{n+1}}\|x_k\|^p\right)^{\frac{1}{p}}\left(\sum_{n=0}^{\infty}2^n r^{p2^n}\right)^{\frac{1}{p}} \leq \frac{C}{(1-r)^{\frac{1}{p}}}. \quad \square$$

Example 2.1. Consider $f_1(z) = (\frac{z^n}{n})_{n\in\mathbb{N}}$ and $f_2(z) = \sum_{n=1}^{\infty}\frac{r_n}{n}z^n$ (r_n are the Rademacher functions).

Observe that

$$\|f_1(z)\|_{l^1} = \|f_2(z)\|_{L^\infty([0,1])} = \sum_{n=1}^{\infty}\frac{|z|^n}{n} = \log\frac{1}{1-|z|}.$$

Hence $\|\frac{e_n}{n}\|_{l^1} = \|\frac{r_n}{n}\|_{L^\infty([0,1])} = \frac{1}{n}$ but $f_1 \notin BMOA(l^1)$ and $f_2 \notin BMOA\,(L^\infty([0,1]))$ (because $f_i \notin H^1(X_i)$ for $X_1 = l^1, X_2 = L^\infty([0,1])$).

This shows that, in general, the simple condition $\|x_n\| = O(\frac{1}{n})$ does not imply that $f \in BMOA(X)$.

COROLLARY 2.2. *Let X be Banach space. The following are equivalent*

(*i*) *If $\|x_n\| = O(\frac{1}{n})$ then $\sum_{n=1}^{\infty}x_n z^n \in BMOA(X)$.*
(*ii*) *X is B-convex.*

Proof.

(i) $\Rightarrow$ (ii). Assume X is not a B-convex space. Then it contains l^1 uniformly which allows to built a function as f_1 in Example 2.1, leading to a contradiction with (i).

(ii) $\Rightarrow$ (i).We can invoke Theorem A to find $1 < p \leq 2$ such that X has Fourier type p. Now apply Corollary 2.1 for such a p. $\square$

Remark 2.1. The reader should be aware that Corollary 2.3 is nothing but the dual formulation of a result inlcuded in [BP] due to Pisier, which says that X is B-convex if and only if the functions in $H^1_{at}(X)$ satisfy Hardy inequality.

3. Vector valued multipliers from $H^1(X)$ into $BMOA(Y)$

Let us denote by $(H^1, BMOA)$ the space of convolution multipliers between H^1 and $BMOA$, that is the set of functions $F(z) = \sum_{n=0}^{\infty}\lambda_n z^n$ such that there exists a constant $C > 0$ for which

$$\left\|\sum_{n=0}^{\infty}\lambda_n\alpha_n z^n\right\|_{BMOA} \leq C\left\|\sum_{n=0}^{\infty}\alpha_n z^n\right\|_{H^1}.$$

the following scalar-valued result was proved in [MP]

$$(*) \qquad (H^1, BMOA) = \mathcal{B}$$

where $\mathcal{B}$ stands for the space of Bloch functions.

We shall be interested in this section in the vector valued formulation of this result. First of all we need to give sense to the notion of convolution multiplier acting between two different Banach spaces. We present here two possible interpretations.

Let us recall that given Banach spaces X, Y we denote by $X\hat{\otimes}Y$ the completion of $X \otimes Y$ endowed with the projective tensor norm, i.e. for $u \in X \otimes Y$

$$\|u\|_{X\hat{\otimes}Y} = inf \left\{ \sum_{i=1}^{n} \|x_i\| \|y_i\| \right\}$$

where the infimum goes over all possible representations of $u = \sum_{i=1}^n x_i \otimes y_i$, $x_i \in X$, $y_i \in Y$.

DEFINITION 3.1. *Given $f(z) = \sum_{n=0}^\infty x_n z^n \in H^1(X)$ and $g(z) = \sum_{n=0}^\infty y_n z^n \in H^1(Y)$ we shall define the $X\hat{\otimes}Y$-valued analytic function*

$$f\hat{*}g(z) = \int_0^{2\pi} f(ze^{-it}) \otimes g(e^{it}) \frac{dt}{2\pi} = \sum_{n=1}^{\infty} x_n \otimes y_n z^n. \tag{3.1}$$

It is rather simple to observe that $f\hat{*}g(z) \in H^1(X\hat{\otimes}Y)$.

DEFINITION 3.2. *Let X, Y be complex Banach spaces and let $F(z) = \sum_{n=0}^\infty T_n z^n$ be a $\mathcal{L}(X, Y)$-valued analytic function and $f(z) = \sum_{n=0}^\infty x_n z^n \in H^1(X)$. We define the Y-valued function*

$$F * f(z) = \sum_{n=0}^{\infty} T_n(x_n) z^n = \int_0^{2\pi} F(ze^{it}) \left(f(e^{-it})\right) \frac{dt}{2\pi}. \tag{3.2}$$

We shall denote by $\left(H^1(X), BMOA(Y)\right)$ the set of analytic functions $F\colon D \to \mathcal{L}(X, Y)$ such that $F * f \in BMOA(Y)$ for any $f \in H^1(X)$.
This becomes a closed subspace of $\mathcal{L}\left(H^1(X), BMOA(Y)\right)$.

Let us notice first that we have the following obvious extension.

LEMMA 3.1. *Let X, Y be two complex Banach spaces. Then*

$$(H^1(X), BMOA(Y)) \subset \mathcal{B}(\mathcal{L}(X, Y)).$$

Proof. Given $F \in (H^1(X), BMOA(Y))$ and $x \in X$, $y^* \in Y^*$ then $\langle F(z)(x), y^*\rangle \in \left(H^1, BMOA\right)$. Hence, from the scalar-valued case $(*)$,

$$\|\langle F(z)(x), y^*\rangle\|_{\mathcal{B}} \le \|F\|_{(H^1(X), BMOA(Y))} \|x\| \|y^*\|.$$

This clearly shows $F \in \mathcal{B}(\mathcal{L}(X, Y))$. $\square$

Nevertheless let us first point out that there is no hope for the analogue of $(*)$ to hold for general pairs of Banach spaces as the following remark shows.

Remark 3.1. Let us assume $\mathcal{B}(\mathcal{L}(X,X)) \subset \big(H^1(X), BMOA(X)\big)$ then taking $T_n = I$, the identity operator, part (ii) in Proposition 1.1 shows that $F(z) = \sum_{n=0}^{\infty} T_n z^{2^n} \in \mathcal{B}(\mathcal{L}(X,X))$ and then one should have

$$\left\| \sum_{n=0}^{\infty} x_{2^n} z^{2^n} \right\|_{1,X} \leq \|F * f\|_{*,X} \leq C\|f\|_{1,X}.$$

This cannot be true as soon as we take X being a cotype 2 space but not a Paley space (for instance $X = \frac{L^1}{H_0^1}$, see [BP]). In fact it will be shown later that actually under such an assumption X has to be isomorphic to a Hilbert space.

DEFINITION 3.3. *Let X, Y be complex Banach spaces. The pair (X, Y) is said to have the $(H^1, BMOA)$-property if*

$$\big(H^1(X), BMOA(Y)\big) = \mathcal{B}(\mathcal{L}(X,Y)).$$

Let us now present various properties holding for pairs having the $(H^1, BMOA)$-property.

THEOREM 3.1. *Let (X, Y^*) have the $(H^1, BMOA)$-property.*
If $f \in H^1(X)$ and $g \in H^1(Y)$ then $(f \hat{*} g)' \in L^1(D, X\hat{\otimes}Y)$.

Proof. Let us recall that $\big(X\hat{\otimes}Y\big)^* = \mathcal{L}(X, Y^*)$ under the pairing

$$\left(T, \sum_{k=1}^{n} x_k \otimes y_k\right) = \sum_{k=1}^{n} \langle T(x_k), y_k\rangle,$$

where $\langle , \rangle$ stands for the pairing on (Y, Y^*).

On the other hand for any Banach space E one has $L^1(D,E) = L^1(D)\hat{\otimes}E$ what gives $\big(L^1(D,E)\big)^* = \mathcal{L}\left(L^1(D), E^*\right)$ under the pairing given by

$$\left[T, \sum_{k=1}^{n} e_k\phi_k\right] = \sum_{k=1}^{n} \langle\langle T(\phi_k), e_k\rangle\rangle$$

for any $e_k \in E$ and $\phi_k \in L^1(D)$ where $\langle\langle , \rangle\rangle$ stands for the pairing on (E, E^*).

Assume now $f(z) = \sum_{n=0}^{m} x_n z^n$ and $g(z) = \sum_{n=0}^{m} y_n z^n$. Hence $(f\hat{*}g)'(z) = \sum_{n=0}^{m} n x_n \otimes y_n z^{n-1}$.

According to the previous dualities, and denoting $u_n(w) = w^n$, we can write

$$\|(f\hat{*}g)'\|_{L^1(D,X\hat{\otimes}Y)} = \sup\left\{\left|\sum_{n=1}^{m} n(T(u_{n-1}), x_n \otimes y_n)\right|\right\}$$

where the supremum is taken over $T \in \mathcal{L}\left(L^1(D), \mathcal{L}(X, Y^*)\right)$ with $\|T\| = 1$.

Note that for each $T \in \mathcal{L}\left(L^1(D), \mathcal{L}(X, Y^*)\right)$ with $T_n = T(u_{n-1}) \in \mathcal{L}(X, Y^*)$ and $\|T\| = 1$ we have

$$\sum_{n=1}^{m} n(T(u_{n-1}), x_n \otimes y_n) = \sum_{n=1}^{m} n\langle T_n(x_n), y_n\rangle.$$

On the other hand, with $F(z) = \sum_{n=1}^{\infty} nT_n z^n$, observe that $F(z) = zT(K_z)$ and therefore, from Proposition 1.2, it is a $\mathcal{L}(X, Y^*)$-valued Bloch function with $\|T(K_z)\|_{\mathcal{B}(\mathcal{L}(X,Y^*))} \leq \|T\|$.

Now notice that

$$\begin{aligned}\left|\sum_{n=1}^{m} n\langle T_n(x_n), y_n\rangle\right| &= \left|\int_0^{2\pi}\int_0^{2\pi}\langle F(re^{i(t-s)})(f(e^{it})), g(e^{-is})\rangle\frac{dt}{2\pi}\frac{ds}{2\pi}\right| \\ &\leq \|F * f\|_{BMOA(Y^*)}\|g\|_{1,Y} \\ &\leq C\|F\|_{\mathcal{B}(\mathcal{L}(X,Y^*))}\|f\|_{1,X}\|g\|_{1,Y}.\end{aligned}$$

□

COROLLARY 3.1. *Let (X, Y^*) have the $(H^1, BMOA)$-property. If $f(z) = \sum_{n=0}^{\infty} x_n z^n \in H^1(X)$ and $g(z) = \sum_{n=0}^{\infty} y_n z^n \in H^1(Y)$ then*

$$\sum_{n=0}^{\infty} \|x_{2^n}\|\|y_{2^n}\| \leq C\|f\|_{1,X}\|g\|_{1,Y}.$$

Proof. Let $h(z) = (f\hat{*}g)'(z) \in L^1(D, X\hat{\otimes}Y)$. Obviously one has

$$n\|x_n\|\|y_n\|r^{n-1} \leq M_{1,X\hat{\otimes}Y}(h, r) \qquad (n \in \mathbb{N}).$$

Therefore

$$\begin{aligned}\int_0^1 M_{1,X\hat{\otimes}Y}(h, r)dr &\geq \sum_{k=0}^{\infty}\int_{1-2^{-k}}^{1-2^{-(k+1)}} 2^k\|x_{2^k}\|\|y_{2^k}\|r^{(2^k-1)}dr \\ &\geq C\sum_{k=0}^{\infty}\|x_{2^k}\|\|y_{2^k}\|.\end{aligned}$$

□

COROLLARY 3.2. *If $(\mathbb{C}, Y^*)$ have the $(H^1, BMOA)$-property then Y is a Paley space.*

Proof. Apply Corollary 3.1 to $f(z) = \sum_{n=0}^{\infty} \alpha_n z^{2^n} \in H^1$ and $g \in H^1(Y)$ and recall that $\|f\|_1 \approx (\sum_{n=0}^{\infty} |\alpha_n|^2)^{\frac{1}{2}}$.

LEMMA 3.2. *If (X, Y) has the $(H^1, BMOA)$-property then $(X, \mathbb{C})$ and $(\mathbb{C}, Y)$ also have the $(H^1, BMOA)$-property.*

Proof. (i) Let $F(z) = \sum_{n=0}^{N} x_n^* z^n \in \mathcal{B}(X^*)$ and $f(z) = \sum_{n=0}^{\infty} x_n z^n \in H^1(X)$. Fix $y \in Y$ with $\|y\| = 1$ and consider $\hat{F}(z) = \sum_{n=0}^{N} T_n z^n$ where T_n are the operators in $\mathcal{L}(X, Y)$ defined by $T_n(x) = \langle x_n^*, x\rangle y$. It is elementary to show that $\hat{F} \in \mathcal{B}(\mathcal{L}(X, Y))$ and $\|\hat{F}\|_{\mathcal{B}(\mathcal{L}(X,Y))} = \|\hat{F}\|_{\mathcal{B}(X^*)}$. Therefore

$$\begin{aligned}\left\|\sum_{k=0}^{\infty}\langle x_k^*, x_k\rangle z^k\right\|_{BMOA} &= \left\|\sum_{k=0}^{\infty} T_k(x_k) z^k\right\|_{BMOA(Y)}\\ &\le C\|\hat{F}\|_{\mathcal{B}(\mathcal{L}(X,Y))}\left\|\sum_{k=0}^{\infty} x_k z^k\right\|_{1,X}.\end{aligned}$$

(ii) Let $F(z) = \sum_{n=0}^{\infty} y_n z^n \in \mathcal{B}(Y)$ and $\phi(z) = \sum_{n=0}^{\infty} \alpha_n z^n \in H^1$. Fix $x_0 \in X$ and $x_0^* \in X^*$ with $\|x_0\| = 1$ and $\langle x_0^*, x_0\rangle = 1$. Define $\hat{F}(z) = \sum_{n=0}^{\infty} T_n z^n$ where T_n are defined by $T_n(x) = \langle x_0^*, x\rangle y_n$. It is elementary to show that $\hat{F} \in \mathcal{B}(\mathcal{L}(X, Y))$ and $\|\hat{F}\|_{\mathcal{B}(\mathcal{L}(X,Y))} = \|\hat{F}\|_{\mathcal{B}(Y)}$. Observe that

$$\sum_{n=1}^{\infty} \alpha_n y_n z^n = \sum_{n=1}^{\infty} T_n(\alpha_n x_0) z^n = \hat{F} * f$$

where $f(z) = \phi(z)x_0$. Then we have

$$\begin{aligned}\left\|\sum_{n=1}^{\infty} y_n \alpha_n z\right\|_{BMO(Y)} &\le C\|\hat{F}\|_{\mathcal{B}(\mathcal{L}(X,Y))}\left\|\sum_{n=0}^{\infty} \alpha_n x_0 z^n\right\|_{1,X}\\ &\le C\|F\|_{\mathcal{B}(Y)}\left\|\sum_{n=0}^{\infty} \alpha_n z^n\right\|_{H^1}.\end{aligned}$$

□

PROPOSITION 3.1. *Let X, Y be two complex Banach spaces.*
(i) If $(X, \mathbb{C})$ has the $(H^1, BMOA)$-property then X is a Paley space.
(ii) If $(\mathbb{C}, Y)$ has the $(H^1, BMOA)$-property then Y has type 2.

Proof. (i) Let $f(z) = \sum_{n=0}^{\infty} x_n z^n \in H^1(X)$ and choose $x_n^* \in X^*$ with $\|x_n^*\| = 1$ and $\langle x_n^*, x_{2^n}\rangle = \|x_{2^n}\|$. Let us recall that Khintchine's inequalities hold for *BMO* functions ([G]); i.e.,

$$\left(\sum_{k=0}^{\infty} |\alpha_k|^2\right)^{\frac{1}{2}} \approx \left\|\sum_{k=0}^{\infty} \alpha_k z^{2^k}\right\|_{BMOA}.$$

Then, using the previous fact,

$$\left(\sum_{k=1}^{\infty} \|x_{2^k}\|^2\right)^{\frac{1}{2}} = \left(\sum_{k=1}^{\infty} |\langle x_k^*, x_{2^k}\rangle|^2\right)^{\frac{1}{2}} \approx \left\|\sum_{k=1}^{\infty} \langle x_k^*, x_{2^k}\rangle z^{2^k}\right\|_{BMOA}.$$

Let us observe that, from (ii) in Proposition 1.1, $F(z) = \sum_{n=1}^{\infty} x_n^* z^{2^n}$ belongs to $\mathcal{B}(X^*)$ and therefore

$$\left(\sum_{k=1}^{\infty} \|x_{2^k}\|^2\right)^{\frac{1}{2}} \leq \left\| \sum_{k=1}^{\infty} \langle x_k^*, x_{2^k}\rangle z^{2^k} \right\|_{BMOA}$$
$$\leq C\|F\|_{\mathcal{B}(X^*))} \left\| \sum_{k=1}^{\infty} x_k z^k \right\|_{1,X} \leq C\|f\|_{1,X}$$

This shows that X is a Paley space.

(ii) Now given $y_0, y_1, y_2, \ldots y_N \in X$ with $y_j \neq 0$ we define $F(z) = \sum_{n=0}^{N} \frac{y_n}{\|y_n\|} z^{2^n}$. From Proposition 1.1 again we have $F \in \mathcal{B}(Y)$ and $\|F\|_{\mathcal{B}(Y)} \leq C$.

Observe that

$$\sum_{k=0}^{N} y_k z^{2^k} = \sum_{k=0}^{N} \|y_k\| \frac{y_k}{\|y_k\|} z^{2^k} = F * \phi$$

where $\phi(z) = \sum_{k=0}^{N} \|y_k\| z^{2^k}$; then we have

$$\left\| \sum_{k=0}^{N} y_k z^{2^k} \right\|_{1,Y} \leq \left\| \sum_{k=0}^{N} y_k z^{2^k} \right\|_{BMO(Y)}$$
$$\leq C\|F\|_{\mathcal{B}(Y)} \left\| \sum_{k=0}^{N} \right\| y_k\| z^{2^k} \|_1$$
$$\leq C\left(\sum_{k=0}^{N} \|y_k\|^2\right)^{\frac{1}{2}}.$$

This shows that X has type 2. □

We shall now introduce two new properties which are motivated by the inequality due to Hardy and Littlewood mentioned in the introduction and its dual formulation and will be connected with the $(H^1, BMOA)$-property.

Let us recall the notation $\mathcal{P}(X)$ and $\mathcal{P}_0(X)$ for the X-valued polynomials and those which vanish at $z = 0$ respectively.

DEFINITION 3.4. *A complex Banach space X is said to have the $(HL)^*$-property if there exists a constant $C > 0$ such that*

$$\|f\|_{*,X} \leq C\left(\int_0^1 (1-r) \sup_{|z|=r} \|f'(z)\|^2 dr\right)^{\frac{1}{2}} \tag{3.3}$$

for any $f \in \mathcal{P}(X)$.

PROPOSITION 3.2. *Let H be a Hilbert space. Then H has the $(HL)^*$ property.*

Proof. Using (1.1) we have

$$\|f\|_{*,H} \approx \sup_{|z|<1} \int_D \frac{(1-|z|^2)(1-|w|)\|\|f'(w)\|_H^2}{|1-\bar{w}z|^2} dA(w).$$

Now for any $z \in D$ one has

$$\begin{aligned}\int_D & \frac{(1-|z|^2)(1-|w|)\|\|f'(w)\|_H^2}{|1-\bar{w}z|^2} dA(w) \\ &\leq \int_0^1 (1-r) \sup_{|w|=r} \|f'(w)\|_H^2 \left(\int_0^{2\pi} \frac{1-r^2|z|^2}{|1-re^{-it}z|^2} \frac{dt}{2\pi} \right) dr \\ &= \int_0^1 (1-r) \sup_{|w|=r} \|f'(w)\|_H^2 dr.\end{aligned}$$

Therefore

$$\|f\|_{*,H} \leq C \left(\int_0^1 (1-r) \sup_{|w|=r} \|f'(w)\|_H^2 dr \right)^{\frac{1}{2}} < \infty. \qquad \square$$

The next example shows that $X = l^1$ fails to have the $(HL)^*$ property.

Example 3.1. Let $X = l^1$ and $f(z) = \left(\frac{1}{n\log(n+1)} z^n\right)_{n=0}^{\infty}$. Then

$$\int_0^1 (1-r) \sup_{|z|=r} \|f'(z)\|_{l^1}^2 dr < \infty$$

but $f \notin H^1(l^1)$.

Indeed, since $\|f(z)\|_{l^1} = \sum_{n=1}^{\infty} \frac{1}{n\log(n+1)} |z|^n$ then

$$\lim_{r\to 1} M_{1,l^1}(f,r) = \sum_{n=1}^{\infty} \frac{1}{n\log(n)} = \infty,$$

which gives $f \notin H^1(l^1)$.

On the other hand, (see [L, page 93–96]),

$$\|f'(z)\|_{l^1} = \sum_{n=1}^{\infty} \frac{1}{\log(n+1)} |z|^n \approx \frac{|z|}{(1-|z|)(\log \frac{1}{1-|z|})}.$$

Therefore

$$\int_0^1 (1-r) \sup_{|z|=r} \|f'(z)\|_{l^1}^2 dr \leq C \int_0^1 \frac{dr}{(1-r)(\log \frac{1}{1-r})^2} < \infty. \qquad \square$$

DEFINITION 3.5. *A complex Banach space X is said to have the (HL)-property if there exists a constant $C > 0$ such that*

$$\left(\int_0^1 (1-r)M_{1,X}^2(f',r)dr\right)^{\frac{1}{2}} \leq C\|f\|_{1,X} \tag{3.4}$$

for any $f \in \mathcal{P}_0(X)$.

Remark 3.2. Observe that

$$\int_0^1 (1-r)M_{1,X}^2(f',r)dr = \sum_{k=0}^{\infty}\int_{r_k}^{r_{k+1}} (1-r)M_{1,X}^2(f',r)dr,$$

for $r_k = 1 - 2^{-k}$ and then, since $M_{1,X}(f,r)$ is increasing the inequalities (3.3) and (3.4) can be replaced by

$$\|f\|_{*,X} \leq C\left(\sum_{k=0}^{\infty} 2^{-2k} \sup_{|z|=r_k} \|f'(z)\|^2\right)^{\frac{1}{2}} \tag{3.4}$$

and

$$\left(\sum_{k=0}^{\infty} 2^{-2k} M_{1,X}^2(f',r_k)\right)^{\frac{1}{2}} \leq C\|f\|_{1,X}. \tag{3.5}$$

Therefore inequality (3.6) says that X has the (HL)-property if and only if the operator $f \to \left(2^{-k}f'(r_k e^{it})\right)_k$ is bounded from $H_0^1(X)$ into $l^2\left(L^1(X)\right)$.

Example 3.2. Let $X = c_0$; then X fails to have (HL)-property.

Indeed, take $f_N(z) = \sum_{n=1}^N e_n z^n$ and then clearly $\sup_{N\in\mathbb{N}} \|f_N\|_{1,c_0} = 1$.
On the other hand $M_{1,c_0}(f_N', r_k) \geq C2^k$ for $N \geq 2^k$. Therefore

$$\left(\sum_{k=0}^{\infty} 2^{-2k} M_{1,c_o}^2(f_N',r_k)\right)^{\frac{1}{2}} \geq CN^{\frac{1}{2}}. \qquad \square$$

THEOREM 3.2. *Let X, Y be Banach spaces.*
If X has the (HL)-property and Y has the $(HL)^*$-property then (X, Y) has the $(H^1, BMOA)$-property.

Proof. From Lemma 3.1. we only have to prove

$$\mathcal{B}\left(\mathcal{L}(X,Y)\right) \subset \left(H^1(X), BMOA(Y)\right).$$

Let $F(z)=\sum_{n=0}^{\infty}T_nz^n \in \mathcal{B}(\mathcal{L}(X,Y))$ and $f(z)=\sum_{n=0}^{\infty}x_nz^n \in H^1(X)$.

Now observe that

$$\begin{aligned} z(F*f)'(z^2) &= \sum_{n=1}^{\infty} nT_n(x_n)z^{2n-1} \\ &= \int_0^{2\pi} F'(ze^{it})\left(f(ze^{-it})\right)e^{it}\frac{dt}{2\pi} \\ &= 2\int_0^1\int_0^{2\pi}\left(\sum_{n=1}^{\infty} nT_nz^{n-1}s^{n-1}e^{i(n-1)t}\right) \\ &\quad \times \left(\sum_{n=1}^{\infty} nx_ns^{n-1}e^{-i(n-1)t}\right)\frac{dt}{2\pi}sds \\ &= 2\int_0^1\int_0^{2\pi} F'(zse^{it})\left(f'(se^{-it})\right)se^{it}\frac{dt}{2\pi}ds. \end{aligned}$$

Therefore, since $F \in \mathcal{B}(\mathcal{L}(X,Y))$, we have

$$\begin{aligned} \|z(F*f)'(z^2)\| &\le C\|F\|_{\mathcal{B}(\mathcal{L}(X,Y))}\int_0^1 \frac{M_{1,X}(f_1,s|z|)}{(1-s|z|)}ds \\ &\le C\|F\|_{\mathcal{B}(\mathcal{L}(X,Y))}\left(\int_0^1\frac{ds}{(1-s|z|)^2}\right)^{\frac{1}{2}}\left(\int_0^{|z|}M_{1,X}^2(f',s)ds\right)^{\frac{1}{2}} \\ &\le \frac{C\|F\|_{\mathcal{B}(\mathcal{L}(X,Y))}}{(1-|z|)^{\frac{1}{2}}}\left(\int_0^{|z|}M_{1,X}^2(f',s)ds\right)^{\frac{1}{2}}. \end{aligned}$$

Hence

$$\sup_{|z|=r}\|z(F*f)'(z^2)\| \le \frac{C}{(1-r)^{\frac{1}{2}}}\left(\int_0^r M_{1,X}^2(f',s)ds\right)^{\frac{1}{2}}.$$

Now, using the $(HL)^*$-property on Y and the (HL)-property on X, we can estimate

$$\begin{aligned} \|F*f\|_{*,Y}^2 &\le \int_0^1(1-r^2)\sup_{|z|=r^2}\|(F*f)'(z)\|^2rdr \\ &\le C\int_0^1\left(\int_0^r M_{1,X}^2(f',s)ds\right)dr \\ &= C\int_0^1(1-s)M_{1,X}^2(f',s)ds \le C\|f\|_{1,X}. \end{aligned}$$

Clearly $\|\int_0^{2\pi}F*f(e^{it})\frac{dt}{2\pi}\| = \|T_0(x_0)\| \le C\|f\|_{1,X}$. This combined with the previous estimate finishes the proof. □

4. Lebesgue spaces and Schatten classes with *(HL)*-property

In this section we study these new properties and investigate the Lebesgue spaces and the Schatten classes having the (HL)-property and the $(HL)^*$-property.

Let us start with some general facts and their relations with the notions of type and cotype.

PROPOSITION 4.1. *(i) If X has the (HL)-property then X is a Paley space.*
(ii) If X has the $(HL)^*$-property then X has type 2.

Proof. Combine Theorem 3.2 together with Proposition 3.1. □

Let us now establish the duality existing between both notions.

THEOREM 4.1 (DUALITY). *(i) If X^* has the $(HL)^*$-property then X has the (HL)-property.*
(ii) Let X be an UMD space. Then X^* has the $(HL)^*$-property if and only if X has the (HL)-property.

Proof. Let $f(z) = \sum_{n=1}^{\infty} x_n z^n \in H_0^1(X)$ with $\|f\|_{1,X} = 1$. Using the embedding

$$l^2(L^1(X)) \subseteq \left(l^2(C(X^*))\right)^*,$$

and setting $r_k = 1 - 2^{-k}$, we have

$$\left(\sum_{k=0}^{\infty} 2^{-2k} M_{1,X}^2(f', r_k)\right)^{\frac{1}{2}} = \sup \left|\sum_{k=0}^{\infty} \int_0^{2\pi} \langle 2^{-k} f'(r_k e^{it}), g_k(e^{-it})\rangle \frac{dt}{2\pi}\right|,$$

where the supremum is taken over the set of sequences $(g_k)_{k\in\mathbb{N}} \subset C(X^*)$ such that $\sum_{k=0}^{\infty} \|g_k\|_{\infty,X^*}^2 = 1$.

Letting

$$G_k(z) = \int_0^{2\pi} \frac{g_k(e^{it})}{(1 - ze^{-it})} \frac{dt}{2\pi}$$

we have, for $|z| = r$,

$$\|G_k''(r_k z)\|_{X^*} \leq \|g_k\|_{\infty,X^*} \int_0^{2\pi} \frac{1}{|1 - zr_k e^{-it}|^3} \frac{dt}{2\pi} \leq C \frac{1}{(1 - r_k r)^2} \|g_k\|_{\infty,X^*}.$$

Therefore for any sequence (g_k) with $\sum_{k=0}^{\infty} \|g_k\|_{\infty,X^*}^p = 1$,

$$\left|\sum_{k=0}^{\infty} \int_0^{2\pi} \langle 2^{-k} f'(r_k e^{it}), g_k(e^{-it})\rangle \frac{dt}{2\pi}\right|$$

$$= \left| \int_0^{2\pi} \langle f(e^{it}), \sum_{k=0}^{\infty} 2^{-k} G_k'(r_k e^{-it}) \rangle \frac{dt}{2\pi} \right|$$
$$\leq \|f\|_{1,X} \left\| \sum_{k=0}^{\infty} 2^{-k} G_k'(r_k e^{it}) \right\|_{*,X^*}$$
$$\leq C \left(\int_0^1 (1-r) \sup_{|z|=r} \left\| \sum_{k=0}^{\infty} 2^{-k} G_k''(r_k z) \right\|_{X^*}^2 dr \right)^{\frac{1}{2}}$$
$$\leq C \left(\int_0^1 (1-r) \left(\sum_{k=0}^{\infty} 2^{-k} \frac{\|g_k\|_{\infty,X^*}}{(1-r_k r)^2} \right)^2 dr \right)^{\frac{1}{2}} = I$$

Using Hölder's inequality and the facts

$$\sum_{k=0}^{\infty} 2^{-k} \frac{1}{(1-r_k r)^2} \approx \int_0^1 \frac{ds}{(1-rs)^2},$$

$$\int_0^1 \frac{ds}{(1-rs)^2} = \frac{1}{1-r},$$

we can write

$$I \leq C \left(\int_0^1 (1-r) \left(\sum_{k=0}^{\infty} 2^{-k} \frac{\|g_k\|_{\infty,X^*}^2}{(1-r_k r)^2} \right) \left(\sum_{k=0}^{\infty} 2^{-k} \frac{1}{(1-r_k r)^2} \right) dr \right)^{\frac{1}{2}}$$
$$\leq C \left(\int_0^1 (1-r) \left(\sum_{k=0}^{\infty} 2^{-k} \frac{\|g_k\|_{\infty,X^*}^2}{(1-r_k r)^2} \right) \left(\int_0^1 \frac{ds}{(1-rs)^2} \right) dr \right)^{\frac{1}{2}}$$
$$\leq C \left(\sum_{k=0}^{\infty} 2^{-k} \|g_k\|_{\infty,X^*}^2 \int_0^1 \frac{dr}{(1-r_k r)^2} \right)^{\frac{1}{p}} \leq C.$$

(ii) From part (i) we only have to show that if X is a UMD space having the (HL)-property implies X^* has the $(HL)^*$-property .

Given an X^*-valued polynomial, say $f(z) = \sum_{n=0}^{m} x_n^* z^n$, and using the duality $(H^1(X))^* = BMOA(X^*)$, we have

$$\|f\|_{*,X^*} = \sup \left\{ \int_0^{2\pi} \langle f(e^{it}), g(e^{-it}) \rangle \frac{dt}{2\pi} : g \in H_0^1(X), \|g\|_{1,X} = 1 \right\}.$$

Now let us observe that for $g(z) = \sum_{n=1}^{\infty} x_n z^n$

$$\int_0^{2\pi} \langle f(e^{it}), g(e^{-it}) \rangle \frac{dt}{2\pi} = \sum_{n=1}^{m} \langle x_n^*, x_n \rangle$$

$$= 2\int_0^1 (1-r^2)$$

$$\int_0^{2\pi} \left\langle \sum_{n=1}^{m} n x_n^* r^{n-1} e^{-i(n-1)t}, \sum_{n=1}^{\infty}(n+1)x_n r^n e^{i(n-1)t} \right\rangle \frac{dt}{2\pi} dr$$

$$= 2\int_0^1 (1-r^2) \int_0^{2\pi} \langle f'(re^{it}), g_1'(re^{-it})\rangle e^{it} \frac{dt}{2\pi} dr$$

where $g_1(z) = zg(z)$. Hence

$$\left| \int_0^{2\pi} \langle f(e^{it}), g(e^{-it})\rangle \frac{dt}{2\pi} \right|$$

$$\leq \int_0^1 (1-r) M_{1,X}(g_1', r) \sup_{|z|=r} \|f'(z)\|_{X^*} dr$$

$$\leq \left(\int_0^1 (1-r) M_{1,X}^2 (g_1', r) dr \right)^{\frac{1}{2}} \left(\int_0^1 (1-r) \sup_{|z|=r} \|f'(z)\|_{X^*}^2 dr \right)^{\frac{1}{2}}$$

$$\leq C\|g_1\|_{1,X} \left(\int_0^1 (1-r) \sup_{|z|=r} \|f'(z)\|_{X^*}^2 dr \right)^{\frac{1}{2}}$$

$$\leq C\|g\|_{1,X} \left(\int_0^1 (1-r) \sup_{|z|=r} \|f'(z)\|_{X^*}^2 dr \right)^{\frac{1}{2}}. \qquad \square$$

PROPOSITION 4.2. *Hilbert spaces have the $(HL)^*$-property and the (HL)-property.*

Proof. The $(HL)^*$-property was proved in Proposition 3.2. Now apply Theorem 4.1 to get the (HL)-property. □

COROLLARY 4.1 [B2]. *X is isomorphic to a Hilbert space if and only if (X, X) has the the (H^1, BMO)-property.*

PROPOSITION 4.3. Let (Ω, Σ, μ) be a measure space. If X has the (HL)-property then $L^1(\mu, X)$ has the (HL)-property.

Proof. It follows from the vector-valued Minkowsky's inequality that

$$\left(\sum_{k=0}^{\infty} \|f_k\|_{L^1(\mu)}^2 \right)^{\frac{1}{2}} \leq C \left\| \left(\sum_{k=0}^{\infty} |f_k(.)|^2 \right)^{\frac{1}{2}} \right\|_{L^1(\mu)}, \tag{4.1}$$

for any sequence $(f_k) \in L^1(\mu)$.

Now, given an $L^1(\mu, X)$-valued analytic polynomial, say $F(z) = \sum_{n=0}^m x_n z^n$, for a.a. $\omega \in \Omega$ the X-valued polynomial $F(\omega)(z) = \sum_{n=0}^m x_n(\omega) z^n$ satisfies

$$\left(\sum_{k=0}^{\infty} 2^{-2k} M_{1,X}^2(F'(\omega), r_k)\right)^{\frac{1}{2}} \leq C \int_0^{2\pi} \|F(\omega)(e^{it})\|_X \frac{dt}{2\pi} \qquad \omega \in \Omega$$

Now integrating over Ω,

$$\left\| \left(\sum_{k=0}^{\infty} 2^{-2k} M_{1,X}^2(F'(\omega), r_k)\right)^{\frac{1}{2}} \right\|_{L^1(\mu)} \leq C \|F\|_{1,L^1(\mu,X)}.$$

On the other hand, from (4.1),

$$\begin{aligned}\left(\sum_{k=0}^{\infty} 2^{-2k} M_{1,L^1(\mu,X)}^2(F', r_k)\right)^{\frac{1}{2}} &= \left(\sum_{k=0}^{\infty} \|2^{-k} M_{1,X}(F'(\omega), r_k)\|_{L^1(\mu)}^2\right)^{\frac{1}{2}} \\ &\leq \left\| \left(\sum_{k=0}^{\infty} 2^{-2k} \|F(.,r_k)\|_X^2\right)^{\frac{1}{2}} \right\|_{L^1(\mu)} \\ &\leq \|F\|_{1,L^1(\mu,X)}.\end{aligned}$$

□

PROPOSITION 4.4. *Let (Ω, Σ, μ) be a measure space.*
(i) *$L^p(\mu)$ has the (HL)-property if and only if $1 \leq p \leq 2$.*
(ii) *$L^p(\mu)$ has the $(HL)^*$-property if and only if $2 \leq p < \infty$.*

Proof. (i) From Proposition 4.1 the (HL)-property implies cotype 2 and then $1 \leq p \leq 2$.

On the other hand $L^1(\mu)$ has the (HL)-property according to Proposition 4.3.

The case $1 < p \leq 2$ follows from the fact that L^p is isometrically isomorphic to a subspace of L^1 (see [R]).

(ii) Follows from (i) and Theorem 4.1. □

Now let us investigate the $(HL)^*$-property and the (HL)-property for the Schatten classes. Given $1 \leq p < \infty$ we shall denote by σ_p the Banach space of compact operators on l^2 such that

$$\|A\|_p = \left(\mathrm{tr}(A^*A)^{\frac{p}{2}}\right)^{\frac{1}{p}} < \infty.$$

It is well known that σ_1 coincides with the space of nuclear operators on l^2 and σ_2 with the space of Hilbert-Schmidt operators on l^2. The reader is referred to [GK] for general properties on σ_p and to [TJ] for results on (Rademacher) type and cotype on these classes. The key point in dealing with them is the use of factorization of analytic functions with values on theses classes. The reader is referred to [BP], [L-PP], [Pi3] for the use of factorization in related questions. Let us establish the result to be used later.

LEMMA D (NONCOMMUTATIVE FACTORIZATION [S]). *Let $f \in H^1(\sigma_1)$. Then there exist two functions $h_1, h_2 \in H^2(\sigma_2)$ such that*

$$f(e^{it}) = h_1(e^{it})h_2(e^{it}) \text{ and } \|f\|_{1,\sigma_1} = \|h_1\|^2_{2,\sigma_2} = \|h_2\|^2_{2,\sigma_2}.$$

THEOREM 4.2. *σ_1 has the (HL)-property.*

Proof. Given $f \in H^1(\sigma_1)$ take $h_1, h_2 \in H^2(\sigma_2)$ such that

$$f(e^{it}) = h_1(e^{it})h_2(e^{it}), \qquad \|h_1\|^2_{2,\sigma_2} = \|h_2\|^2_{2,\sigma_2} = \|f\|_{1,\sigma_1}.$$

Note that for $i, j \in \{1, 2\}, i \neq j$,

$$\begin{aligned}\int_0^{2\pi} \|h_i'(re^{it})h_j(re^{it})\|_{\sigma_1}\frac{dt}{2\pi} &\leq \int_0^{2\pi} \|h_i'(re^{it})\|_{\sigma_2}\|h_j(re^{it})\|_{\sigma_2}\frac{dt}{2\pi} \\ &\leq \left(\int_0^{2\pi} \|h_i'(re^{it})\|^2_{\sigma_2}\frac{dt}{2\pi}\right)^{\frac{1}{2}} \\ &\quad \times \left(\int_0^{2\pi} \|h_j(re^{it})\|^2_{\sigma_2}\frac{dt}{2\pi}\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore

$$M_{1,\sigma_1}(f', r) \leq M_{2,\sigma_2}(h_1', r)M_{2,\sigma_2}(h_2, r) + M_{2,\sigma_2}(h_1, r)M_{2,\sigma_2}(h_2', r).$$

This gives

$$\left(\int_0^{2\pi} (1-r)M^2_{1,\sigma_1}(f', r)dr\right)^{\frac{1}{2}} \leq \|f\|^{\frac{1}{2}}_{1,\sigma_1} \sum_{i=1}^{2} \left(\int_0^{2\pi} (1-r)M^2_{2,\sigma_2}(h_i', r)dr\right)^{\frac{1}{2}}.$$

Since σ_2 is a Hilbert space we have, using Plancherel,

$$\int_0^{2\pi} (1-r)M^2_{2,\sigma_2}(h_i', r)dr = \sum_{n=1}^{\infty} \|\hat{h}_i(n)\|^2_{\sigma_2} n^2 \int_0^{2\pi} (1-r)r^{2n-2}dr \leq C\|h_i\|^2_{\sigma_2}$$

This shows

$$\left(\int_0^{2\pi} (1-r)M^2_{1,\sigma_1}(f', r)dr\right)^{\frac{1}{2}} \leq C\|f\|_{1,\sigma_1}. \qquad \square$$

To cover other values of p we shall use some of the recent advances on interpolation of vector-valued Hardy spaces. It is known (see [BX]) that interpolation spaces by complex or real method, $(H^{p_1}(X_1), H^{p_2}(X_2))_\theta$ or $(H^{p_1}(X_1), H^{p_2}(X_2))_{\theta,p}$ do not coincide, in general, with $H^{p_\theta}(X_\theta)$ or $H^{p_\theta}(X_{\theta,p})$, but nevertheless there are some positive results that still can be used to find the (HL)-property of certain spaces.

For some particular spaces, like L^p in the commutative and non-commutative versions, the expected result remains true (see [X1], [X2], [BX], [Pi4]):

If $0 < \theta < 1$ and $\frac{1}{p} = 1 - \frac{\theta}{2}$ then

$$\big(H^1(L^1(\mu)), H^1(L^2(\mu))\big)_\theta = H^1(L^p(\mu)). \tag{4.2}$$

$$(H^1(\sigma_1), H^1(\sigma_2))_\theta = H^1(\sigma_p). \tag{4.3}$$

$$(H^1(L^1(\mu)), H^1(L^2(\mu)))_{\theta,1} = H^1(L^{p,1}(\mu)). \tag{4.4}$$

where $L^{p,1}(\mu)$ stands for the corresponding Lorentz space.

PROPOSITION 4.5. *Let X_i $(i = 1, 2)$ be spaces having the (HL)-property and assume*

$$(H^1(X_1), H^1(X_2))_\theta = H^1((X_1, X_2)_\theta).$$

Then $(X_1, X_2)_\theta$ has the (HL)-property.

Proof. Since

$$T(f) = (2^{-k} f'(r_k e^{it}))_k$$

defines a bounded operator $T\colon H_0^1(X_i) \to l^2(L^1(\mathbb{T}, X_i))$ for $i = 1, 2$, the assumption together with the well-known result of interpolation

$$\big(l^2(L^1(X_1)), l^2(L^1(X_2))\big)_\theta = l^2(L^1((X_1, X_2)_\theta))$$

shows that T is also bounded from $H_0^1((X_1, X_2)_\theta)$ into $l^2(L^1((X_1, X_2)_\theta))$ which shows that $(X_1, X_2)_\theta$ has the (HL)-property. □

Combining the results (4.3), (4.2) and the previous proposition we easily obtain the following corollary.

PROPOSITION 4.6. *Let $1 \le p < \infty$. Then*:
(i) σ_p has the (HL)-property if and only if $1 \le p \le 2$.
(ii) σ_p has the $(HL)^*$-property if and only if $2 \le p < \infty$.
(iii) $L^{p,1}(\mu)$ has the (HL)-property for $1 \le p < 2$.

Remark 4.1. Some of the previous ideas appeared already in [BP]. Proposition 4.6 gives an alternative proof of the Paley property of σ_p for $1 \le p \le 2$ and then the cotype 2 condition (see [TJ]). Another approach was also obtained in [L-PP].

5. Applications

Let us start this section with some new examples of vector-valued *BMOA* functions. Observe that Theorem 3.2 actually provides a procedure to find functions in $BMOA(X)$ for spaces with the $(HL)^*$-property.

PROPOSITION 5.1. *Let* $0 < \alpha \leq \frac{1}{2}$ *and* $p = \frac{1}{\alpha}$. *Define*

$$I_\alpha(\phi)(z) = \int_0^{2\pi} \frac{\phi(e^{-it})}{(1 - ze^{it})^\alpha} \frac{dt}{2\pi}.$$

Then the operator given by $\phi \to f_\alpha(z) = I_\alpha(\phi)_z$ *is bounded from* H^1 *to* $BMOA(L^p)$.

Proof. *Take* $g(z) = \frac{1}{(1-z)^\alpha}$ *and* $G(z) = g_z$. *First observe that* G *is an* H^p*-valued Bloch function. Indeed*

$$\|G'(z)\|_p \leq M_p(g, |z|) = \left(\int_0^{2\pi} \frac{1}{(1 - |z|e^{-it})^{p\alpha+p}} \frac{dt}{2\pi}\right)^{\frac{1}{p}} \leq \frac{C}{1 - |z|}.$$

Now invoke Theorem 3.2. □

PROPOSITION 5.2. *Let* $(\mathbb{C}, X)$ *have the* $(H^1, BMOA)$*-property and let* $T \in \mathcal{L}(L^1(D), X)$. *Then* $f(z) = T(\phi_z') \in BMOA(X)$ *for any* $\phi \in H^1$.

Proof. Recall that by Proposition 1.2, $g(z) = T(K_z) \in \mathcal{B}(X)$ where K_z is the Bergman Kernel $K_z(w) = \frac{1}{(1-zw)^2}$.

Now for any $\phi \in H^1$,

$$g * \phi(z) = \int_0^{2\pi} T(K_{ze^{it}}\phi(e^{-it}))\frac{dt}{2\pi} = T\left(\int_0^{2\pi} K_{ze^{it}}\phi(e^{-it})\frac{dt}{2\pi}\right) = T(\phi_z'),$$

so $f(z) \in BMOA(X)$ by Theorem 3.2. □

Now let us give a couple of applications to sequences of scalar-valued functions.

Note that if (f_n) is a sequence of functions in H^1 such that $\sum_{n\in\mathbb{N}} \|f_n\|_1 < \infty$ and let (g_n) is a sequence of Bloch functions such that $\sup_{n\in\mathbb{N}} \|g_n\|_{\mathcal{B}} < \infty$ then $\sum_{n\in\mathbb{N}} f_n * g_n$ is absolutely convergent in *BMOA*. This shows that if $f = (f_n) \in H^1(l^1)$ and $g = (g_n) \in \mathcal{B}(l^\infty)$ then $f * g \in BMOA$. We now produce an extension of this result to other values of p different from 1.

PROPOSITION 5.3. *Let* $1 < p \leq 2$. *Let* (f_n) *be a sequence of functions in* H^1 *such that* $(\sum_{n\in\mathbb{N}} |f_n(e^{it})|^p)^{\frac{1}{p}} \in L^1$, *and* (g_n) *be a sequence of Bloch functions such that* $(\sum_{n\in\mathbb{N}} |g_n'(z)|^{p'})^{\frac{1}{p'}} = O(\frac{1}{1-|z|})$. *Then* $\sum_{n\in\mathbb{N}} f_n * g_n$ *converges in BMOA.*

Proof. Note that $f = (f_n) \in H^1(l^p)$ and $g = (g_n) \in \mathcal{B}(l^{p'})$. Since l^p has (HL)-property then we can apply Theorem 3.2 to $(l^p, \mathbb{C})$ to get $f * g = \sum_{n\in\mathbb{N}} f_n * g_n \in BMOA$. □

PROPOSITION 5.4. *Let $\phi \in H^1$ and let (g_n) be a sequence of Bloch functions such that $(\sum_{n\in\mathbb{N}} |g_n'(z)|^2)^{\frac{1}{2}} = O(\frac{1}{1-|z|})$. Then $d\mu(z) = (1-|z|)\sum_{n\in\mathbb{N}} |(g_n*\phi)'(z)|^2 dA(z)$ is a Carleson measure on D.*

Proof. It folows from (1.1) that $d\mu(z) = (1-|z|)\sum_{n\in\mathbb{N}} |(g_n*\phi)'(z)|^2 dA(z)$ is a Carleson measure on D if and only if $(g_n * \phi)_{n\in\mathbb{N}} \in BMOA(l^2)$.

This now follows again from Theorem 3.2 applied to $(\mathbb{C}, l^2)$. □

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