ON THE NUMBER OF CERTAIN TYPES OF POLYHEDRA

BY

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1. Introduction. The discussions in the following pages are concerned with certain enumerations in the morphology of Eulerian polyhedra in 3-space. The "morphology" is actually the topology of the complexes consisting of the vertices, edges, and faces of polyhedra, with the restriction that these elements are linear. A class of polyhedra isomorphic to each other with respect to incidences is called a *type*. We impose also on this isomorphism the condition that it preserves the orientation. We shall call such a class a "type in the strict sense" in contradistinction to the usage in most classical papers in which the preservation of orientation was not required (Kirkman, Brückner). In Brückner's book [1] there is to be found the explicit statement "Verschiedene Typen ergeben sich nur, wenn . . ., denn weiterhin treten die Spiegelbilder der bisherigen Vielflache auf". That means that mirror-symmetric polyhedra belong in this sorting to the same type (in our terminology employed here "type in the wider sense"). Steinitz [9] in most parts of his book shares Brückner's point of view. However, on p. 86 he speaks of "direct isomorphy", which he defines as isomorphy under preservation of orientation. He formulates there his famous theorem on convex polyhedra, which is actually a homotopy theorem, stating that two convex polyhedra of the same type in the strict sense are homotopically equivalent, again with the "morphological" restriction that the complexes which constitute the continuous transition from one convex polyhedron to a directly isomorphic one remain always convex polyhedra of the same type in the strict sense.

It is clear that the number of types in the strict sense is at least as great as the number of types in the wider sense.

The number of types of polyhedra of a given number F of faces, whether the types are counted in the wider or the strict sense, is a problem mentioned by Euler, Steiner, Kirkman [4], Eberhardt [3], Brückner [1], and Steinitz [8]. Usually attention is only paid to trihedral polyhedra, i.e. those whose every vertex belongs to 3 faces and 3 edges. These polyhedra are considered as "general", whereas those with vertices of higher incidence are looked upon in such discussions as degenerate. We shall in this article also deal only with *trihedral polyhedra* and shall no longer mention this restriction.

For the types in the wider sense the enumeration has been carried out up to $F = 11$ by Brückner and recently, with the help of an electronic computer, by D. W. Grace, a student of G. Pólya. The number $\psi(F)$ of types (in the wider sense) increases rapidly with F, and no general formula has been found for it.

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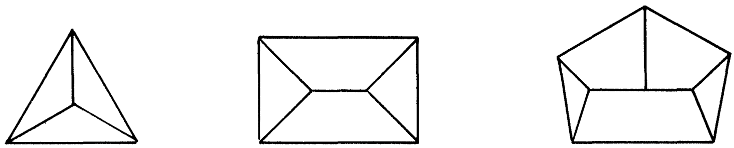


F=4 F=5 F=6

FIGURE 1

However, a special class of trihedral polyhedra has been singled out by Kirkman [4] namely those of F faces which possess a face of $F - 1$ sides, called the "base". Such polyhedra I shall call "based polyhedra"[1], and for these I shall derive explicit formulas for the number of types in the strict sense and in the wider sense. Examples of based polyhedra, seen in projection from above into the plane of the base, are given in Figure 1. Kirkman, in his paper [4] gives a recursive method to determine for each given F the number of types in the wider sense. However this method becomes soon unmanageable for increasing F. He does not arrive at an explicit formula nor an asymptotic expression for the number of types as a function of F. For the number $\chi_0(F)$ of types in the wider sense G. Pólya has proved that it satisfies the inequality

$$(1.1)\qquad \frac{1}{2(F-1)(F-2)}\binom{2F-6}{F-3} < \chi_0(F) \leqq \frac{6}{2(F-1)(F-2)}\binom{2F-6}{F-3}.$$

(Personal communication, in a letter dated 25th July 1964.)

Pólya's proof will be published in a dissertation of his student D. W. Grace. Pólya conjectured moreover that $\chi_0(F)$ is asymptotic to the left member of this inequality. I shall prove this conjecture and more, namely the following two theorems, which I enunciate with $F = n + 1$, n being the number of sides of the base of the based polyhedron.

THEOREM 1. *If $n \geqq 3$, the number of types in the strict sense of based polyhedra of $F = n + 1$ faces is*

$$(1.2)\qquad \begin{aligned} \psi_0(n+1) = g(n) = G(n) &+ \tfrac{1}{2}(n/2+1)G(n/2+1) \\ &+ \tfrac{2}{3}(n/3+1)G(n/3+1), \end{aligned}$$

where

$$(1.3)\qquad \begin{aligned} G(n) &= \frac{(2n-4)!}{n!\,(n-2)!} \quad \textit{for } n \textit{ an integer} \geqq 2 \\ &= 0 \qquad\qquad\quad \textit{otherwise.} \end{aligned}$$

[1] Following a suggestion of M. Kac. G. Pólya in a correspondence, which was the start of these investigations, calls them "roofless".

THEOREM 2. *If $n \geqq 3$, the number of types in the wider sense of based polyhedra of $F = n + 1$ faces is*

$$\chi_0(n+1) = k(n) = \tfrac{1}{2}G(n) + \tfrac{1}{4}(n/2+1)G(n/2+1) + \tfrac{1}{3}(n/3+1)G(n/3+1) + \tfrac{1}{2}([n/2]+1)G([n/2]+1). \tag{1.4}$$

We shall devote Section I to Theorem 1 and Section II to Theorem 2. The short Section III is concerned with the special types of based polyhedra with only 2 triangles, which have been discussed and enumerated by O. Hermes.

I wish to thank Professor George Pólya for mentioning this problem to me and letting me know his inequality (1.1) together with his conjectures.

I. Number of types in the strict sense

2. Description of based polyhedra. In order to construct the based polyhedra for small n and to understand their generation for all n we have to analyze their structure. We can and shall always refer to the projection of the polyhedron into the plane of its base. The polyhedron is viewed so that it lies *above* the base. Such a projection forms a graph in the plane.

There are 3 sorts of edges:

(1) those of the base, n in number;
(2) those issuing from the base, one from each vertex, also n in number;
(3) the remaining ones, forming a complex which we call the "ridge".

Since we have only trihedral polyhedra with V vertices and E edges, so that

$$3V = 2E$$

and $F = n + 1$, we find from Euler's formula

$$E = 3n - 3. \tag{2.1}$$

The ridge consists thus of

$$r = E - 2n = n - 3 \tag{2.2}$$

edges. The whole graph contains

FIGURE 2

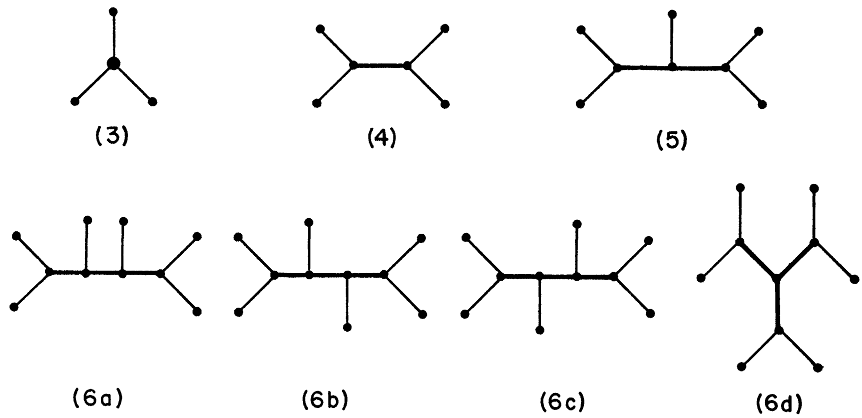


FIGURE 3

$$V = 2n - 2 \tag{2.3}$$

vertices. On the ridge are therefore $n - 2$ vertices.

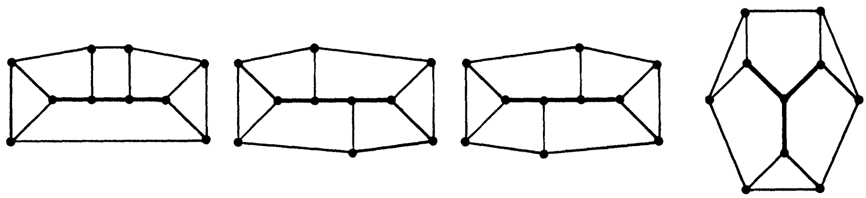
Taking the n edges of the base away from the graph we obtain a graph T_n of $2n - 3$ edges and $2n - 2$ vertices. This operation does not destroy connectedness of the graph. Since the number of vertices exceeds the number of edges by 1 we see that T_n is a *tree*. It contains only nodes (vertices) of order 1 (the endpoints, i.e. vertices of the base) and of order 3 (the branchpoints or inner nodes). Such a tree we shall call a *simple tree*. The tree T_n has n endpoints and $n - 2$ branchpoints.

The ridge R_n is obtained by taking away from T_n all the ending edges (endsegments). No such amputation disturbs the connectivity. The ridge is therefore also a tree, and indeed we found its number of edges $r = n - 3$ by 1 smaller than its number of vertices $V - n = n - 2$.

The trees T_n and R_n have to be considered as imbedded in the oriented plane, and not only in a purely combinatorial manner as complexes of segments and nodes put together by incidence relations.[2] We construct the based polyhedra starting from the ridges. Examples of ridges R_n are given in Figure 2, from which we derive the trees T_n of Figure 3. Connecting now the endpoints in the order in which they appear in the oriented plane, we obtain the full projections of the based polyhedra; see Figure 1 for $n = 3, 4, 5$ and Figure 4 for $n = 6$. These are all possible based polyhedra to $F = 7$ or $n = 6$. We notice that $n = 3$ and (6d) have a rotational symmetry (R.S.) of order 3; $n = 4$ and (6b) and (6c) have a rotational symmetry of order 2. Moreover, (6c) is the mirror image of (6b).

Since each type of a based polyhedron stands in a one-to-one correspondence

[2] The Figures (6a), (6b), (6c) would represent the same tree combinatorially, but here they are distinguished as imbedded in the plane of the drawing.



(6a) (6b) (6c) (6d)

FIGURE 4

to a type of (simple) tree in the plane, we may from now devote our attention solely to these trees.

3. Rotational symmetry. There can be no other R.S.'s than those of order 2 and 3. Of course, it is necessary in these cases that $2|n$ and $3|n$ respectively.

That there are no other R.S.'s becomes clear if we look at more stringent necessary conditions in the cases of R.S. of orders 2 and 3. For R.S. of order 2 we remove pairs of segments which correspond to each other under the rotation. We first remove the sides of the base polygon in pairs, which shows that n is even. Then we continue with the inner segments, i.e. those of T_n. But the number of segments of T_n is $2n-3$, thus odd. One segment must remain unmatched, its central segment, and this has the R.S. of order 2.

Similarly, R.S. of order 3 first necessitates n divisible by 3. Then T_n has $2n-3$ segments, a number also divisible by 3. We continue to remove corresponding segments in triples. There remains in the end a branchpoint, the central node, which with its adjacent segments has the R.S. of order 3.

Since T_n is a simple tree no other outcome is possible for a rotational symmetry.

We notice further that each *endpoint* of the ridge R_n produces a triangle of the polyhedron. Since any R_n must have at least 2 endpoints we observe: A based polyhedron has at least 2 triangles. And conversely: if a based polyhedron has only two triangles then its ridge has only two ends. But that means that the ridge is a *chain* of $r = n - 3$ segments linked in succession to each other.

We remark finally that the indicated construction of based polyhedra leads to configurations which fulfill Steinitz's condition for K-polyhedra, which can all be realized as convex polyhedra in Euclidean 3-space, [9, p. 192, pp. 227 ff].

4. Distinctions according to rotational symmetry. Before continuing the investigation I wish to introduce an abbreviated notation. There is no need to distinguish between a based polyhedron and the type which it represents. We shall speak of a based polyhedron T_n or T_n'' etc. as well as of the type T_n, T_n'', always understanding here the *type in the strict sense.* The set of all types T_n we shall write (T_n) and the cardinal number of this set as $|T_n|$.

We shall also speak of trees T_n, etc., corresponding to the type T_n. With $F = n + 1$ we put the number of types T_n

$$\psi_0(n + 1) = g(n)$$

and break this number down into the sum

$$g(n) = g_1(n) + g_2(n) + g_3(n), \tag{4.1}$$

where $g_1(n)$ is the number of types of polyhedra T'_n with base of n sides and without rotational symmetry (or R.S. of order 1), (see cases (5), (6a) in Figure 3), $g_2(n)$ the number of the types T''_n of R.S. of order 2 (cases (4), (6b), (6c)) and $g_3(n)$ that of types T'''_n of R.S. of order 3 (cases (3), (6d)). It is clear that

$$\begin{aligned} g_2(n) &= 0 \quad \text{for} \quad 2 \nmid n \\ g_3(n) &= 0 \quad \text{for} \quad 3 \nmid n, \end{aligned} \tag{4.2}$$

and *only* in these cases, as we shall see.

Instead of taking up $g(n)$ directly we shall consider the modified number

$$G(n) = g_1(n) + \tfrac{1}{2}g_2(n) + \tfrac{1}{3}g_3(n). \tag{4.3}$$

The first values are

$$G(3) = \tfrac{1}{3}, \qquad G(4) = \tfrac{1}{2}, \qquad G(5) = 1, \tag{4.4}$$

as our examples in **2** show.

We can express $g_2(n)$ and $g_3(n)$ conversely by $G(n)$.

Indeed in a T''_n (or R.S. of order 2) we cut its central segment into two segments and split thus T''_n into two identical trees T_l with $l = n/2 + 1$ endpoints each. Now this T_l can be, according to its own rotational symmetry, of the sort T'_l, T''_l, T'''_l. We can conversely use all T_l to construct all possible T''_n, choosing one of the l endpoints of T_l for attachment to an identical T_l (after rotation through 180°). We have only to observe that, whereas for a T'_l we have l choices of endpoints, for a T''_l we have only $l/2$ choices in order to avoid repetition, and for a T'''_l only $l/3$ choices. This gives

$$\begin{aligned} g_2(n) &= lg_1(l) + (l/2)g_2(l) + (l/3)g_3(l) \\ &= lG(l) \end{aligned}$$

and thus

$$g_2(n) = (n/2 + 1)G(n/2 + 1), \qquad 2 | n,\ n \geqq 4. \tag{4.5}$$

Splitting a T'''_n (of R.S. of order 3) at its central node into 3 identical T_k, $k = n/3 + 1$, we obtain similarly, distinguishing again the possibilities T'_k, T''_k, T'''_k of T_k,

$$g_3 = (n/3 + 1)G(n/3 + 1), \qquad 3 | n,\ n \geqq 4. \tag{4.6}$$

Our constructions show also that indeed (4.2) are the only cases for vanishing $g_2(n)$, $g_3(n)$ with $n \geqq 4$. From (4.1) and (4.3) we infer

$$g(n) = G(n) + \tfrac{1}{2}g_2(n) + \tfrac{2}{3}g_3(n)$$

and in view of (4.5), (4.6)

$$g(n) = G(n) + \tfrac{1}{2}(n/2 + 1)G(n/2 + 1) \qquad n \geqq 4,$$
$$(4.7) \qquad\qquad + \tfrac{2}{3}(n/3 + 1)G(n/3 + 1),$$

where

$$G(x) = 0$$

for x not integer. It suffices therefore to find the function $G(x)$.

5. The recursion formula. We take now $n > 4$. Then the ridge R_n (which is a tree as we know) contains at least one segment.

We pick now a segment of R_n and split it into two segments, one going to a tree T_λ, the other to a tree T_μ. Since T_λ and T_μ each contain a new endpoint we have

$$\lambda + \mu = n + 2;$$

moreover

$$\lambda \geqq 3, \qquad \mu \geqq 3,$$

since the segment was chosen in $R_n \subset T_n$, and to each edge of R_n there are at least 2 further edges of T_n attached.

We have now to treat the T_n', T_n'', T_n''' separately. If we split a T_n' in any of its $r = n - 3$ ridge segments we obtain $2(n - 3)$ of *ordered* pairs of subtrees (T_λ, T_μ). These ordered pairs are all different since the identity of T_λ and T_μ (after suitable rotation) would mean a symmetry as it appears in T_n'' only. For a T_n''' we obtain only $2\cdot(n - 3)/3$ different ordered pairs of subtrees (T_λ, T_μ). For a T_n'' we have $2\cdot(n - 4)/2$ different ordered pairs (T_λ, T_μ) and from the dissection of the central segment *one* pair (T_l, T_l) with $l = n/2 + 1$. If we count now all T_λ, T_μ with $\lambda + \mu = n + 2$, λ, $\mu \geqq 3$, each T_n' occurs $2(n - 3)$ times, each T_n'' occurs $2\cdot(n - 4)/2 + 1 = n - 3$ times, and each T_n''' occurs $2\cdot(n - 3)/3$ times, so that altogether there are

$$2(n - 3)(g_1(n) + \tfrac{1}{2}g_2(n) + \tfrac{1}{3}g_3(n)) = 2(n - 3)G(n)$$

ordered pairs (T_λ, T_μ).

On the other hand, each pair T_λ, T_μ gives rise to several T_n, namely according to the endpoints which we choose for attachments. We have here to distinguish the cases T_λ', T_λ'', T_λ''' and also T_μ', T_μ'', T_μ'''. Each of the λ endsegments of a T_λ' can be joined to each of those of a T_μ'. Whereas, e.g. only $\lambda/3$ endsegments of a T_λ''' have to be joined to $\mu/2$ of a T_μ'' and so on. This gives

$$\sum_{\substack{\lambda+\mu=n+2\\ \lambda+\mu\geqq 3}} \left(\lambda g_1(\lambda) + \frac{\lambda}{2}\, g_3(\lambda) + \frac{\lambda}{3}\, g_3(\lambda)\right)\left(\mu g_1(\mu) + \frac{\mu}{2}\, g_2(\mu) + \frac{\mu}{3}\, g_3(\mu)\right)$$
$$= \sum_{\substack{\lambda+\mu=n+2\\ \lambda,\mu\geqq 3}} \lambda G(\lambda)\mu G(\mu)$$

recombinations of T_λ, T_μ into T_n. These recombinations (or conversely splittings) we have just counted starting from the T_n. We obtain thus the relation

$$2(n-3)G(n) = \sum\nolimits_{\lambda+\mu=n+2;\lambda,\mu\geqq 3} \lambda G(\lambda)\mu G(\mu), \qquad n \geqq 4. \tag{5.1}$$

Let us put

$$nG(n) = h(n-2), \tag{5.2}$$

defined for $n \geqq 3$. We had in particular $G(3) = \frac{1}{3}$ and have thus

$$h(1) = 1. \tag{5.3}$$

With $m = n - 2$ the formula (5.1) goes over into

$$h(m) = \frac{m+2}{2(m-1)} \sum_{\substack{\rho+\sigma=m\\ \rho,\sigma\geqq 1}} h(\rho)h(\sigma), \qquad m \geqq 2. \tag{5.4}$$

Equation (5.3) is a recursion formula which gives the first few next values

$$h(2) = 2, \qquad h(3) = 5$$

in agreement with (4.4) and (5.2).

6. Solution of the recursion formula. We have now to find the solution of (5.4) with the initial condition (5.3). For this purpose we prove the

LEMMA. *The recursion formula*

$$\omega(m) = \sum\nolimits_{\rho+\sigma=m;\,\rho,\sigma\geqq 1} \omega(\rho)\omega(\sigma), \qquad m \geqq 2, \tag{6.1}$$

with $\omega(1) = 1$ *is solved by*

$$\omega(m) = \frac{1}{m}\binom{2m-2}{m-1}. \tag{6.2}$$

Proof. We consider the generating function

$$\Psi(x) = \sum\nolimits_{m=1}^{\infty} \omega(m)x^m. \tag{6.3}$$

Then we have

$$\Psi(x) = x + \sum\nolimits_{m=2}^{\infty} x^m \sum\nolimits_{\rho+\sigma=m} \omega(\rho)\omega(\sigma) = x + \Psi^2(x)$$

so that

$$\Psi(x) = \tfrac{1}{2}(1 \pm \sqrt{1-4x}).$$

Since $\Psi(0) = 0$ only the minus sign can be valid:

$$\Psi(x) = \tfrac{1}{2}(1 - \sqrt{1-4x}) = -\tfrac{1}{2}\sum_{m=1} \binom{\frac{1}{2}}{m}(-4x)^m.$$

This can be rewritten as

$$\Psi(x) = \sum_{m=1}^{\infty} \frac{1}{m}\binom{2m-2}{m-1} x^m,$$

so that (6.2) is established through comparison with (6.3), and the Lemma is proved.

We have thus established the formula

$$(6.4)\qquad \frac{1}{m}\binom{2m-2}{m-1} = \sum_{\substack{\rho+\sigma=m\\ \rho,\sigma\geqq 1}} \frac{1}{\rho}\binom{2\rho-2}{\rho-1}\frac{1}{\sigma}\binom{2\sigma-2}{\sigma-1}, \qquad m > 1.$$

Replacing here m by $m + 2$ we obtain

$$(6.5)\quad \frac{1}{m+2}\binom{2m+2}{m+1} = \sum_{\substack{\alpha+\beta=m\\ \alpha,\beta\geqq 1}} \frac{1}{\alpha+1}\binom{2\alpha}{\alpha}\frac{1}{\beta+1}\binom{2\beta}{\beta} + \frac{2}{m+1}\binom{2m}{m}.$$

Now a simple computation shows that

$$\frac{1}{m+2}\binom{2m+2}{m+1} - \frac{2}{m+1}\binom{2m}{m} = \frac{2(m-1)}{(m+1)(m+2)}\binom{2m}{m},$$

so that (6.5) can be rewritten as

$$(6.6)\qquad \frac{1}{m+1}\binom{2m}{m} = \frac{m+2}{2(m-1)} \sum_{\substack{\alpha+\beta=m\\ \alpha,\beta\geqq 1}} \frac{1}{\alpha+1}\binom{2\alpha}{\beta}\frac{1}{\beta+1}\binom{2\beta}{\beta}.$$

We realize further that for $m = 1$

$$\frac{1}{m+1}\binom{2m}{m} = \frac{1}{2}\binom{2}{1} = 1.$$

If we compare this initial value and (6.6) with (5.3) and (5.4) we see that we have proved

$$(6.7)\qquad h(m) = \frac{1}{m+1}\binom{2m}{m}.$$

This implies after (5.2)

THEOREM 3. *If* $n \geqq 3$,

$$(6.8)\qquad G(n) = \frac{1}{n\,(n-1)}\binom{2n-4}{n-4}.^{3}$$

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[3] The function $H(n) = nG(n)$ deserves some comments. It was already noticed by Euler (see e.g. [6, p. 102]) that it represents the number of different dissections of a convex polygon with n labelled vertices into triangles by diagonals. This has a connection with our problem, since the enumeration of based polyhedra is the dual to the counting of *topologically different* dissections of an unlabelled convex polygon of n sides into triangles by diagonals. This duality has already been noticed by Kirkman [5] and Steinitz [8]. On the other hand, Cayley [2, p. 114] investigated the number of rooted planar trees with m endpoints and a rootpoint, and interior nodes of order 3 only. He found $H(m + 1)$ as this number, which he, however, wrote "in the remarkably simple form"

$$\frac{1\cdot 3\cdot 5\cdot\ \cdots\ \cdot(2m-3)}{1\cdot 2\cdot 3\cdot\ \cdots\ \cdot m}\,2^{m-1}.$$

In view of (4.7) we have thus proved Theorem 1 for $n \geqq 4$. The case $n = 3$ of Theorem 1 is an easy verification. The formulas (1.2), (4.7) yield for some low values of F the following results:

F	4	5	6	7	8	9	10	11	12	13	14	15	16
$\psi_0(F)$	1	1	1	4	6	19	49	150	442	1424	4522	14924	49536

Remark. From its definition (4.3) it follows that the denominator of $G(n)$ is a divisor of 6. This can be verified by simple number theoretical arguments, also directly from the expression (6.8).

7. Another proof of Theorem 3. The determination of $G(n)$ can be achieved without the use of the Lemma by solving a differential equation for the generating function of $h(m)$.[4]

We define

$$\Phi(x) = \textstyle\sum_{m=1}^{\infty} h(m)x^m. \tag{7.1}$$

Writing the recursion formula (5.4) in the form

$$2mh(m) - 2h(m) = m\textstyle\sum_{\rho+\sigma=m;\, \rho,\sigma\geqq 1} h(\rho)h(\sigma) + 2\textstyle\sum_{\rho+\sigma=m;\, \rho,\sigma\geqq 1} h(\rho)h(\sigma)$$

valid for $m \geqq 2$, we obtain

$$2x\frac{d}{dx}(\Phi(x) - x) - 2(\Phi(x) - x) = x\frac{d}{dx}(\Phi(x)^2) + 2\Phi(x)^2$$

or

$$x\Phi' - \Phi = x\Phi\Phi' + \Phi^2,$$

or

$$\frac{1}{\Phi} - \frac{2}{1+\Phi}\,\Phi = \frac{1}{x}.$$

Integration on both sides yields

$$\log\Phi - 2\log(1+\Phi) = \log x + C$$

and thus

$$\Phi/(1+\Phi)^2 = Kx.$$

Now (7.1) shows that

$$\Phi(x)/x \to h(1) = 1 \quad \text{as} \quad x \to 0.$$

Cayley also observes that $H(m+1)$ moreover expresses the number of ways in which a product $A_1 \cdot A_2 \cdot \cdots \cdot A_m$ which does not obey the associative law can be understood through insertion of parentheses. Pólya, in his fundamental paper [6, footnote, p. 198] indicates an approach to the generating function of $H(n)$ through a functional equation. For comparison with our notation it should be noticed that he counts all nodes, except the rootpoint; his n is our $2n - 3$.

[4] I reproduce here a simplified version of my original proof, which I owe to a remark by Paul T. Bateman.

We conclude therefore $K = 1$ and have the quadratic equation for Φ

$$x(1 + \Phi)^2 = \Phi,$$

which has the solution

$$\Phi(x) = -1 + \frac{1}{2x} \pm \frac{1}{2x}\sqrt{1 - 4x}.$$

Here only the minus sign is acceptable since Φ has to be regular at $x = 0$. We obtain therefore

$$\Phi(x) = \frac{1}{2x}\left(-2x + 1 - \sum_{l=0}^{\infty}\binom{\frac{1}{2}}{l}\right)(-4x)^l$$

$$= \sum_{l=2}^{\infty}\frac{(2l-2)!}{l!\,(l-1)!}x^{l-1} = \sum_{m=1}^{\infty}\frac{1}{m+1}\binom{2m}{m}x^m.$$

From the comparison of this result with (7.1) we infer again the result (6.7).

8. Asymptotic estimates. Stirling's formula applied to (6.8) in the case of n integer yields immediately

$$G(n) \sim \frac{1}{16\sqrt{\pi}}n^{-5/2}4^n$$

and thus

$$(n/2 + 1)G(n/2 + 1) = O(n^{-3/2}4^{n/2}) = O(G(n)^{1/2}).$$

We have thus

THEOREM 4. *For $F = n + 1$ large*

(8.1) $$\psi_0(F) = \psi_0(n + 1) = g(n) = G(n) + O(G(n)^{1/2})$$

and less precisely

(8.2) $$\psi_0(F) \sim \frac{1}{64\sqrt{\pi}}F^{-5/2}4^F.$$

II. Number of types in the wider sense

9. Reduction of types in the wider sense to those in the strict sense. As explained in the Introduction, two types in the strict sense which are mirror symmetric are counted as the same type in the wider sense. It stands to reason that the majority of types (in the strict sense) is not mirror symmetric, and we can thus expect that the number $k(n)$ of types in the wider sense is about $\frac{1}{2}$ of $g(n)$ the number of types in the strict sense.

We introduce some notations. A type (in the strict sense) which is *not* mirror symmetric we shall denote by $\tilde{T}_n$. A type with mirror symmetry we shall denote by T_n^*. If $\tilde{g}(n)$ and $g^*(n)$ are the numbers of types (in the strict sense) of $\tilde{T}_n$ and T_n^* respectively, the number of types in the wider sense will be

$$k(n) = \tfrac{1}{2}\tilde{g}(n) + g^*(n).$$

Since evidently the number of all types in the strict sense is

$$g(n) = \tilde{g}(n) + g^*(n)$$

we obtain

$$k(n) = \tfrac{1}{2}(g(n) + g^*(n)). \tag{9.1}$$

The function $g(n)$ is given in Theorem 1. There remains thus only the determination of $g^*(n)$.

10. Let us have a mirror-symmetric based polyhedron with n-sided base. We discuss its type T_n^* by considering again the projection which is a tree in the base plane. We call this tree, representing T_n^*, also T_n^* for short.

The number of segments of T_n^* is odd, viz. $(2n - 3)$. We pair off the segments corresponding to each other under the mirror symmetry. An odd number of segments must remain unmatched, which means that they are involutory under mirror symmetry. This involution can either keep the endpoints of a segment fixed or can exchange the endpoints: in the first case the segment lies in the symmetry axis (of the base plane), in the second orthogonal to it.

Actually there is exactly one involutory segment. Firstly there cannot be several involutory elements on the symmetry axis. Since the whole graph is connected, and the nodes are threefold, there must appear also some nodes outside the axis, and these, of course, pairwise. The connectivity would then imply one or more cycles, so that T_n^* could not be a tree. Also several involutory segments among which there is one orthogonal to the symmetry axis are impossible for the same reason. (See Figure 5.)

Taking the symmetry axis "vertical" we say that T_n^* is an H_n if its involutory segment is orthogonal to the symmetry axis ("horizontal") and is a V_n if its involutory element lies in the symmetry axis (is "vertical").

A given T_n^* can, however, be an H_n as well as a V_n at the same time (see Fig. 6) after a rotation through 90°. These two symmetries, for group-theoretical

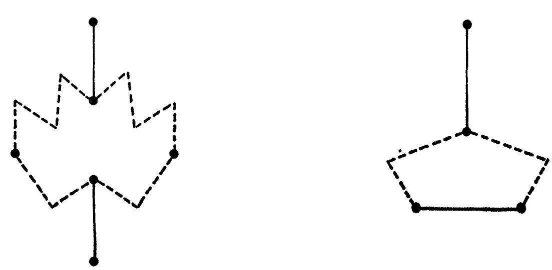

FIGURE 5

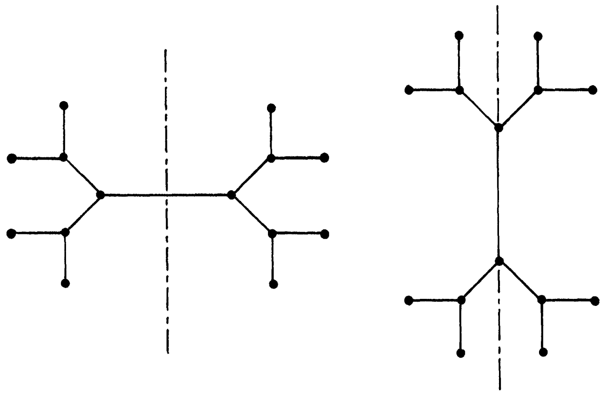

FIGURE 6

reasons, produce together a rotational symmetry of order 2, so that in such a case the T_n^* is also a T_n'', and conversely. We indicate this case by writing $T_n^{*''}$. In this case the number n of endpoints must clearly be divisible by 4.

With the pairing off of segments also the n endpoints are paired off. If n is odd there must be therefore an invariant endpoint under symmetry. This endpoint belongs to the involutory segment, and thus for n odd there exists only the class V_n.

11. Now let first $n = 2\nu$ be even. We shall establish among all the T_n^* a one-to-one correspondence between all the H_n and the V_n. We can change a T_n^* of class H_n into one (and only one) of class V_n by a process of "crossing-over". For this purpose we keep in a tree of class H_n the 4 nodes P_1, P_2, $\bar{P}_1$, $\bar{P}_2$ nearest to, but not on the involutory segment fixed. These 4 points are the endpoints of a subtree H_4. This subtree we replace by a subtree V_4 again with the endpoints P_1, $\bar{P}_1$, P_2, $\bar{P}_2$, as indicated in Figure 7. If and only if the tree subjected to the "crossing-over" process is of a type $T_n^{*''}$ then the two trees H_n and V_n related by the process will represent the same type (after rotation through 90°). Using the notation concerning the cardinal number of classes of types we can thus state

$$(11.1) \qquad \begin{aligned} g^*(n) &= |T^*(n)| \\ |H_{2\nu}| &= |V_{2\nu}|, \end{aligned}$$

$$(11.2) \qquad \begin{aligned} |T_{2\nu}| &= |H_{2\nu}| + |V_{2\nu}| - |T_{2\nu}^{*''}| \\ &= 2|H_{2\nu}| - |T_{2\nu}^{*''}|, \end{aligned}$$

since the class $(T_{2\nu}^{*''})$ is counted in $(H_{2\nu})$ as well as in $(V_{2\nu})$.

We take now $n = 2\nu + 1$ as odd. We have seen that a T_n^* for n odd must be of the sort V_n, with the involutory segment in the symmetry axis.

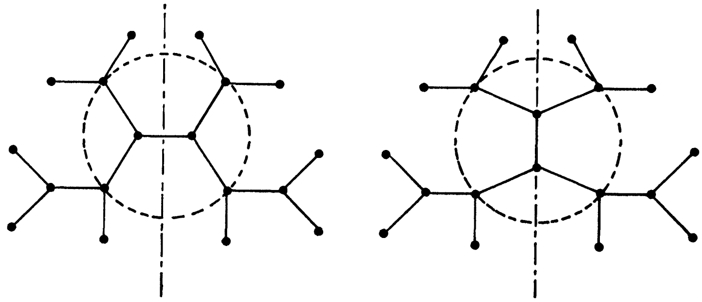

FIGURE 7

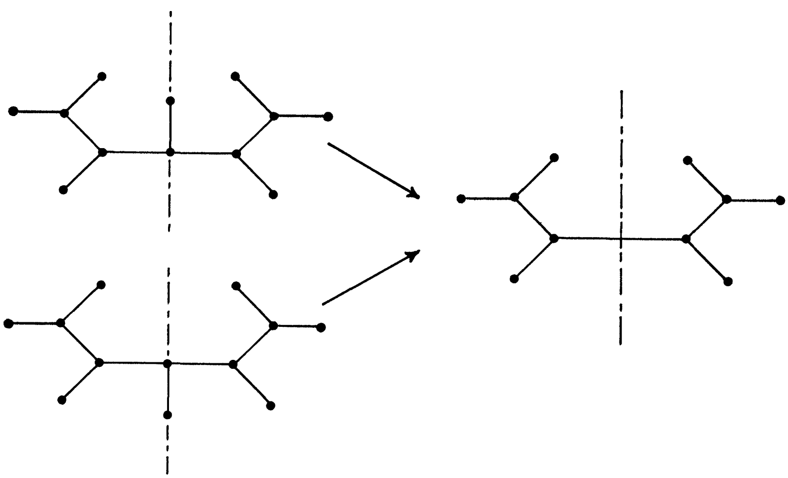

FIGURE 8

We take now this involutory segment away, with its two endpoints and retain a tree T^*_{n-1} of class H_{n-1}. (See Figure 8.) The same T^*_{n-1} is obtained whether the involutory element points "up" or "down". In general, therefore *two* trees T^*_n correspond to one T^*_{n-1}. However, if this T^*_{n-1} has also a second mirror symmetry, i.e. is of the class $(T^{*\prime\prime}_{n-1})$, then the two T^*_n just mentioned are identical, going over into each other by a rotation through 180°. The enumeration is thus

$$(11.3) \qquad | T_{2\nu+1} | = 2 | H_{2\nu} | - | T^{*\prime\prime}_{2\nu} |$$

since the $T^{*\prime\prime}_{n-1}$ are among the $H_{2\nu}$ and should be counted only once as we just stated.

Comparison of (11.2) and (11.3) leads to

THEOREM 5. *The number of mirror-symmetric types T^*_n is the same for $n = 2\nu$ and $n = 2\nu + 1$, that is,*

$$(11.4) \qquad g^*(2\nu) = g^*(2\nu + 1), \qquad \nu \geqq 2.$$

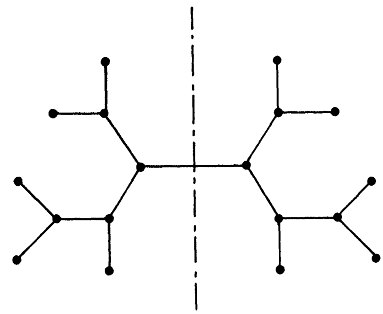

FIGURE 9

12. We are now going to build up all the T_n^* out of their symmetric halves. In view of the preceding theorem we need only discuss the case n even, $n = 2\nu$. Moreover we restrict our attention to T_n^* of class (H_n), since those of class (V_n) can be obtained through the "crossing-over" process.

If we have before us a $T_{2\nu}^*$ which is also an $H_{2\nu}$ then each half is a $T_{\nu+1}$ and the two halves have an endpoint in common on the symmetry axis. Conversely, we take a $T_{\nu+1}$, put one of its endpoints on the desired symmetry axis for $T_{2\nu}^*$, and join $T_{\nu+1}$ at the corresponding endpoint to $\bar{T}_{\nu+1}$, the mirror image of $T_{\nu+1}$, and then delete the point of junction in order to obtain a $T_{2\nu}^*$. (See Figure 9.) The number of different $T_{2\nu}^*$ obtained from one $T_{\nu+1}$ depends on the symmetry character of $T_{\nu+1}$. We have 6 classes of $T_{\nu+1}$ to consider.

Let us write, for short,

$$\nu + 1 = \mu.$$

A T_μ can either be a $\tilde{T}_\mu$ or T_μ^*, i.e. a tree without or with mirror symmetry. These trees can further be distinguished according to their rotational symmetry, so that we have to consider the six kinds $\tilde{T}'$, $\tilde{T}''$, $\tilde{T}'''$, $T^{*\prime}$, $T^{*\prime\prime}$, $T^{*\prime\prime\prime}$.

We firstly carry out the process of constructing a T_n^* out of a $\tilde{T}_\mu'$. Each of the μ ends of $\tilde{T}_\mu'$ can be joined to the corresponding end of $\bar{T}_\mu'$, the mirror image of $\tilde{T}_\mu'$. This will yield μ different T_n^*. However the $\bar{T}_\mu'$ (which is by definition different from $\tilde{T}_\mu'$) will also produce the same μ different T_n^*, which in our construction belong all to the class (H_n). We apply now the crossing-over process to each of the constructred H_n and obtain μ trees V_n so that the 2 trees $\tilde{T}_\mu'$, $\bar{T}_\mu'$ produce 2μ different T_n^*, $n = 2\nu = 2(\mu - 1)$.

Now let us take a $\tilde{T}_\mu''$. The rotational symmetry of order 2 in this case demands $2|\mu$. We can make use only of $\mu/2$ ends of $\tilde{T}_\mu''$ in order to obtain different T_n^*. Again $\bar{T}_\mu''$ will not produce new T_n^*, but only such which are obtained through rotation by 180° from the previous ones. The process of crossing-over will produce another set of T_n^*, of the sort V_n, so that we obtain altogether $2(\mu/2) = \mu$ different T_n^* from a pair $\tilde{T}_\mu^*$, $\bar{T}_\mu''$.

Similarly we obtain from a pair $\tilde{T}_\mu'''$, $\bar{T}_\mu'''$ by reflexion and crossing-over $2(\mu/3)$ different T_n^*.

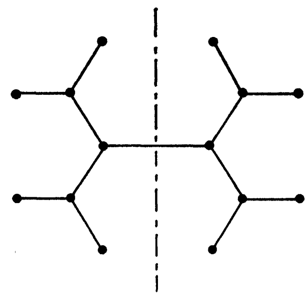

FIGURE 10

We have thus so far from all the $\tilde{T}_\mu$ constructed

$$\mu \cdot \{ | \tilde{T}'_\mu | + \tfrac{1}{2} | \tilde{T}''_\mu | + \tfrac{1}{3} | \tilde{T}'''_\mu | \}$$

different T^*_n .

13. We come now to the class (T^*_μ) and its subclasses $(T^{*\prime}_\mu)$, $(T^{*\prime\prime}_\mu)$, $(T^{*\prime\prime\prime}_\mu)$.

In dealing with a $T^{*\prime}_\mu$ we have to distinguish μ even and odd. If, firstly, μ is even, then the endsegments and endpoints of $T^{*\prime}_\mu$ are paired off through the mirror symmetry. We take two copies of $T^{*\prime}_\mu$ and attach the partners of a pair of endsegments to each other, deleting the node in common. In this way we obtain from the use of each of the $\mu/2$ pairs of symmetric endsegments a symmetric T^*_n with $n = 2\mu - 2$ of the sort H_n . Using then the process of crossing-over we get again $\mu/2$ other T^*_n , (of the sort V_n), together thus μ different T^*_n .

If, however, μ is odd, then we have $(\mu - 1)/2$ pairs of corresponding endsegments and one unmatched (involutory) endsegment. Through the process just described we construct $(\mu - 1)/2$ T^*_n of the sort H_n and then also $(\mu - 1)/2$ further T^*_n of the sort V_n . The single unmatched segment of $T^{*\prime}_\mu$ lies on its symmetry axis. If we connect it to the corresponding $T^{*\prime}_\mu$ in mirrored position (see Figure 10) we obtain a T^*_n which is of the sort H_n , but also a V_n , by rotation through 90°. Again we have $2(\mu - 1)/2 + 1 = \mu$ different T^*_n gained from one $T^{*\prime}_\mu$.

If we have a $T^{*\prime\prime}_\mu$ to start with, then we know that $4|\mu$. Again we pair off the endsegments by mirror symmetry. Because of R.S. of order 2 each such pair goes over into another one by rotation through 180°. We use therefore only $\mu/4$ pairs for establishing a T^*_n by connecting in mirror fashion one end of $T^{*\prime\prime}_\mu$ with the corresponding end of another copy of it. (See Figure 11.) In this way we produce $\mu/4$ different T^*_n , all of the sort H_n . By means of crossing-over we get just as many different types T^*_n of sort V_n , thus $\mu/2$ altogether.

A $T^{*\prime\prime\prime}_\mu$ requires $3|\mu$ because of rotational symmetry of order 3. If now μ is *odd*, then the mirror symmetry expressed by the asterisk * requires that there

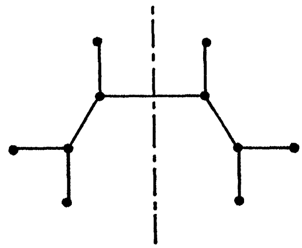

FIGURE 11

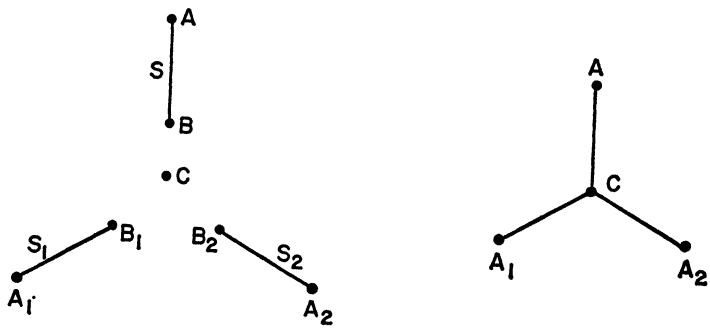

FIGURE 12

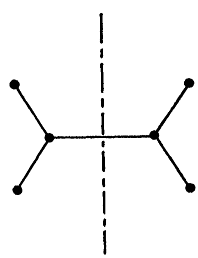

FIGURE 13

is an endsegment $s = AB$ in the symmetry axis with endpoints fixed under reflexion. Rotation around the central node C through 120° and 240° shows that then $T_\mu^{*'''}$ possesses 2 other such endsegments $s = A_1 B_1$ and $s_2 = A_2 B_2$. (See Figure 12.) Since they are part of a connected graph, there must be a connection between s and s_1. The connection by segments BC and CB_1 would not be possible since it would create nodes of even order, which is excluded. Any other connection between s and s_1 would through rotation imply a connection between s_1 and s_2 and the corresponding one between s_2 and s and would thus mean the existence of a cycle in the tree $T_\mu^{*'''}$ which is a contradiction. The only remaining possibility is that $B = B_1 = B_2 = C$ and that the whole $T_\mu^{*'''}$ is $T_3^{*'''}$, $\mu = 3$. This $T_3^{*'''}$ produces only *one* T_4^* through reflexion, viz., the one shown in Figure 13. For $\mu > 3$ we can have a $T_\mu^{*'''}$ only with $6|\mu$.

Through mirror symmetry we pair off the endsegments of $T_\mu^{*'''}$ into $\mu/2$ pairs. Through rotational symmetry each such pair goes over to 2 further equivalent pairs. We have thus $\mu/6$ pairs of endsegments which we use through the process of joining in reflexion to produce each a T_n^*. This gives $\mu/6$ different T^* of sort H_n. Crossing-over gives another set of $\mu/6$ different T_n^*, so that each $T_\mu^{*'''}$ produces $\mu/3$ types T_n^*.

If we collect the results of **12** and **13** we see that all $T_\mu = T_{\nu+1}$ produce the following number of T_n^*, $n = 2\nu$:

$$\begin{aligned} g^*(n) = \mu\{| T'_\mu | + \tfrac{1}{2} | T''_\mu | + \tfrac{1}{3} | T'''_\mu |\} \\ + \mu\{| T^{*\prime}_\mu | + \tfrac{1}{2} | T^{*\prime\prime}_\mu | + \tfrac{1}{3} | T^{*\prime\prime\prime}_\mu |\}. \end{aligned} \tag{13.1}$$

We have obtained all T_n^*, since the process is reversible and leads from a T_n^* to exactly one T_μ with one of its endsegments used for the process of reflexion and attachment.

Since the $\tilde{T}'_\mu, \cdots T^{*'''}_\mu$ are mutually exclusive we know that

$$g_1(\mu) = | \tilde{T}'_\mu | + | T^{*\prime}_\mu |$$

$$g_2(\mu) = | \tilde{T}''_\mu | + | T^{*\prime\prime}_\mu |$$

$$g_3(\mu) = | \tilde{T}'''_\mu | + | T^{*\prime\prime\prime}_\mu |.$$

Formula (13.1) implies thus, for $n = 2\nu$, $\mu = \nu + 1$:

$$\begin{aligned} g^*(2\nu) &= (\nu + 1)\{g_1(\nu + 1) + \tfrac{1}{2}g_2(\nu + 1) + \tfrac{1}{3}g_3(\nu + 1)\} \\ &= (\nu + 1)G(\nu + 1). \end{aligned}$$

In view of Theorem 5 we have thus proved the important

THEOREM 6. *If* $n \geqq 4$,

$$g^*(n) = ([n/2] + 1)G([n/2] + 1). \tag{13.2}$$

If we insert this result together with (1.2) in (9.1), we obtain for $n \geqq 4$ the statement of Theorem 2, which was our goal. For $n = 3$ the result of Theorem 2 is immediate. For a few low values of $F = n + 1$, formula (1.4) yields the following table:

F	4	5	6	7	8	9	10	11	12	13	14	15	16
$\chi_0(F)$	1	1	1	3	4	12	27	82	228	733	2282	7528	24834

for which the values up to $F = 11$ have already been found by Kirkman [4] and Brückner [1].[5] Theorem 2 implies Pólya's result (1.1) and proves also Pólya's conjecture

[5] Brückner [1a] gave also values up to $F = 16$, of which, however, those for 13, 15, and 16 are in error. In particular his value for $F = 16$ is too small by 522 and violates even Pólya's inequality (1.1).

$$\chi_0(F) \sim \frac{1}{2(F-1)(F-2)}\binom{2F-6}{F-3}.$$

III. On based polyhedra with only two triangles

14. The enumeration of certain very special types of based polyhedra, viz. of those with only two triangles has been carried out by O. Hermes (1894), see [1, p. 97] and [8, p. 55].

We can derive his formula easily from the fact that these polyhedra have a ridge consisting of a chain of $r = n - 3$ segments and $n - 2$ nodes, of which the two terminal ones are already occupied by the legs of the two triangles. We have thus only the choices to attach the missing $n - 4$ endsegments "up" or "down" at the $n - 4$ free nodes.

We first count the number $s(n)$ of different types in the strict sense. Let us take $n > 4$. We have 2^{n-4} possibilities to attach the endsegments. If n is odd all these choices are matched off in pairs through rotation by 180°, so that we obtain 2^{n-5} different polyhedra types (all belonging to T'_n).

If however n is even, there are among the 2^{n-4} graphs those in number $2^{(n-4)/2}$ which are not matched by another one through rotation by 180° but go over into themselves, belonging thus to T''_n. We have then

$$\tfrac{1}{2}(2^{n-4} - 2^{(n-4)/2}) + 2^{(n-4)/2} = 2^{n-5} + 2^{(n-6)/2}.$$

Since this last expression gives the correct value also when $n = 4$, we have proved

THEOREM 7. *If $n \geqq 4$, the number of types of based polyhedra with $F = n + 1$ faces and only 2 triangles, the types counted in the strict sense, is*

$$\sigma(F) = \sigma(n+1) = s(n) = \begin{cases} 2^{n-5} & n \text{ odd} \\ 2^{n-5} + 2^{(n-6)/2} & n \text{ even.} \end{cases} \tag{14.1}$$

Let us now go over to the enumeration in the classical sense of Kirkman and Steinitz, where orientation does not have to be preserved.

We proceed as in the general case treated in Section II. Let $\tilde{s}(n)$ be the number of types (in the strict sense) without mirror symmetry and $s^*(n)$ the number of those with mirror symmetry. Obviously we have

$$s(n) = \tilde{s}(n) + s^*(n).$$

Steinitz now counts the number of types in the wider sense:

$$t(n) = \tfrac{1}{2}\tilde{s}(n) + s^*(n) = \tfrac{1}{2}(s(n) + s^*(n)). \tag{14.2}$$

We have thus still to determine $s^*(n)$. This time we take $n > 5$ (in order to avoid the possibility that the set of choices might be void in the following arguments).

Let first n be even. In view of the mirror symmetry we have only to make the choices "up" or "down" only for one half of the available free nodes.

That gives $2^{(n-4)/2}$ possibilities. But if we interchange all choices "up" and "down" by the opposite ones we arrive at the same T_n^*, only turned through 180°. Therefore in this case

$$s^*(n) = 2^{(n-6)/2}, \qquad n \text{ even.}$$

If n is odd we choose the middle endsegment as "up". Then there are $(n - 5)/2$ endsegments to be chosen as "up" or "down", which gives $2^{(n-5)/2}$ possibilities. This time rotation cannot be used. We have thus here

$$s^*(n) = 2^{(n-5)/2}, \qquad n \text{ odd.}$$

If we insert now these results and the statement (14.1) into (14.2) we have completed the proof of the theorem of O. Hermes:

THEOREM 8. *If $n \geqq 5$, the number of types in the wider sense of based polyhedra with $F = n + 1$ faces and only 2 triangles is*

$$t(F) = t(n + 1) = t(n) = 2^{n-6} + 2^{[(n-6)/2]}. \tag{14.3}$$

Of course the case $n = 5$ can be verified directly.

The formula found in [1] and [8] is expressed in F and distinguishes F even and odd, but is equivalent to (14.3).

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