PERIODIC TRANSFORMATIONS OF 3-MANIFOLDS[1]

BY

P. A. SMITH

1. M will denote a triangulated 3-manifold, G a finite group, (G, M) an effective simplicial action, *orientation-preserving whenever M is orientable.* Concerning the action we assume that (1) for every $g \in G$, the fixedpoint set $F = F(g)$ is a subcomplex of M; (2) the natural cell structure of the orbit space $\mathfrak{M} = M/G$ and the projection $\phi : M \to \mathfrak{M}$ are simplicial and (3) ϕ maps each simplex homeomorphically and (4) if σ, σ' are oriented simplexes of M, then $\phi\sigma = \phi\sigma'$ implies $\sigma' = g\sigma$ for some $g \in G$.

From the piecewise linear point of view, these conditions are not restrictive. In fact if (G, M) is simplicial, there is an induced action (G, M_1), M_1 a simplicial subdivision of M, which satisfies (1). If (G, M) satisfies (1), it is a straightforward exercise to show that the induced action (G, M''), where M'' is the second barycentric subdivision, satisfies (1), (2), (3), (4).

We shall assume from here on that $G = Z_p$, $p \geq 2$ and $F = F(G)$ is a simple closed curve. From condition (1), F is a polygon, subcomplex of M.

Moise [1] proved

THEOREM 1. *If M is homeomorphic to a euclidean 3-sphere there exists a compact orientable polyhedral 2-manifold Y in M (i.e. piecewise linearly imbedded in M) such that[2] $\partial Y = F$ and such that the p images of $Y - F$ are disjoint.*

Moise showed further that if F is unknotted in the 3-sphere M, then (G, M) is equivalent to a rotation. It is sufficient to prove

THEOREM 2. *If M is homeomorphic to a euclidean 3-sphere and F is unknotted, there exists a manifold Y which has the properties stated in Theorem 1 and is a disc.*

The proof of Theorem 1 in [1] employs a number of special technical devices. We give here an alternative proof which seems shorter and more direct. The same proof in conjunction with Dehn's lemma gives Theorem 2. Theorem 1 will be proved essentially by producing a 2-manifold $\mathcal{C}$ in M/G such that $\partial\mathcal{C} = \mathfrak{F}(=\phi F)$. The required 2-manifold in M is the union of F and a component of $\phi^{-1}(\mathcal{C} - \mathcal{C} \cap \mathfrak{F})$.

If M is oriented and without boundary, and if the induced action $(G, M - F)$ is free, then $\mathfrak{M} = M/G$ *is an oriented manifold without boundary.* For let x be a vertex of M, $\kappa = \phi x$, $W_x = \mathrm{St}(x, M)$ (= star of x in M). Since $\phi \mid M - F$ is a local homeomorphism, one sees that if $x \in M - F$, ϕ maps W_x

Received January 17, 1964.

[1] This work has been supported by the Office of Naval Research.

[2] ∂X denotes the boundary of X in the sense of manifold theory. If X is oriented, so is ∂X.

isomorphically onto $\mathrm{St}(\kappa, \mathfrak{M})$. If $x \in F$, there is an induced orientation-preserving action $(G, \partial W_x)$. ∂W_x is a 2-sphere and the action leaves fixed two vertices say a, b and is free in $W_x - \{a, b\}$. Clearly $\phi(\partial W_x)$ is an orientable 2-manifold and a count of simplexes shows that its Euler characteristic is 2, hence it is a 2-sphere. Hence ϕW_x, which equals $\mathrm{St}(\kappa, \mathfrak{M})$ is the join of $\phi(\partial W_x)$ with κ, hence is a 3-ball around κ. It follows that $\mathfrak{M}$ is a manifold without boundary. Let M be oriented by a fundamental cycle z (infinite if the complex is infinite). We may write $z = \sum_g \sum_i g\sigma_i$ where $\sigma_1, \sigma_2, \cdots$ are oriented 3-simplexes such that no relation $g\sigma_i = \sigma_j$ $(i \neq j, g \in G)$ exists. Then $\phi(z) = p \sum \sigma_i$ is not zero. $\mathfrak{M}$ is oriented by $\phi(z)$.

2. LEMMA 1. *Let P be an oriented (simple) polygon, subcomplex of M and let i be the inclusion $P \to M$. If $i_* H_1(P) = 0$ there exists a compact oriented polyhedral 2-manifold Y in M such that $\partial Y = P$.*

A proof of this lemma under the additional assumption that $P \subset \partial M$ is given in [3, Lemma (5.2)]. The general case follows immediately. (As it happens, the assumption $P \subset \partial M$ holds in the situation where the lemma is to be applied.)

LEMMA 2. *Let (K, ψ) be a regular covering of a connected manifold $\mathcal{K}$* [4, p. 195] *and let $x \in K$, $\kappa = \psi(x)$.* Let $\Gamma = \pi_1(\mathcal{K}, \kappa)/\psi_* \pi_1(K, x)$ (the subgroup is normal by regularity). *There is a free action (Γ, K) such that $\psi gy = \psi y$* for *$g \in \Gamma$, $y \in K$. Let $\mathcal{Y}$ be an arcwise connected subset of $\mathcal{K}$ containing κ and let i be the inclusion $\mathcal{Y} \subset \mathcal{K}$. If*

$$(1) \qquad i_* \pi_1(\mathcal{Y}, \kappa) \subset \psi_* \pi_1(K, x)$$

there exists a set $Y \subset K$ such that the sets gY, $g \in \Gamma$, are disjoint, their union is $\psi^{-1}\mathcal{Y}$, and each is mapped homeomorphically onto $\mathcal{Y}$ by ψ.

(K, ψ) can be realized as the totality of equivalence classes of paths modulo $\psi_* \pi_1(K, x)$ emanating from κ [4, p. 189]; ψ maps each class onto the common terminal point of its members. The action of Γ is obvious. Referring to (1), one sees that those classes having representatives which lie in $\mathcal{Y}$ form a subset Y with the stated properties.

LEMMA 3. *Let (Γ, K) be a free action in which Γ is finite and K is a connected manifold. Let $\mathcal{K} = K/\Gamma$ and let ψ be the projection $K \to \mathcal{K}$. Let $x \in K$, $\kappa = \psi x$. Then $\mathcal{K}$ is a connected manifold, (K, ψ) a regular covering, and*

$$\Gamma \cong \pi_1(\mathcal{K}, \kappa)/\psi_* \pi_1(K, x).$$

(See [4, p. 195].)

3. *Toroidal neighborhoods of F.* Let M be orientable, without boundary. Denote successive barycentric subdivisions by M', M'', $\cdots$. Let $v_0, \cdots, v_{k-1}$ be the vertices of F' named in cyclic order; the indices are to be taken as elements of Z_k. Let $L_i = \mathrm{St}(v_i, M'')$ and let b_i be the barycenter of $v_i v_{i+1}$. Let $D_i = L_i \cap L_{i+1}$. D_i is a disc, union of those 2-simplexes σ of M'' such that $\sigma \cap F = b_i$. If the M''-stars of v_i, v_j $(i \neq j)$ intersect, so do the interiors of the corresponding M'-stars. Then v_j is a vertex of the M'-star of v_i, and $v_i v_j$ is then a 1-simplex of M', hence of F', and this implies that $i - j = \pm 1$. It follows that L_i meets L_j if and only if $i = j = 0, 1$, or -1. The discs D_i are therefore disjoint. Since each L_i is a 3-ball and M is orientable, $L = \bigcup L_i$ *is a solid torus, neighborhood of* F.

Let $T = \partial L$, $T_i = T \cap L_i$, $J_i = T \cap D_i$. T_i is an annulus and $\partial T_i = J_{i-1} \cup J_i$.

Let P be a *T-circuit*, that is a simple polygon in T which meets each J_i in a single point e_i. Since $k \geq 3$, each $P_i = P \cap T_i$ is a simple arc in T_i with ends e_{i-1}, e_i. Let $A(P) = \bigcup \Omega(P_i)$ where $\Omega(P_i)$ is the join of v_i and the simple arc $b_{i-1} e_{i-1} \cup P_i \cup e_i b_i$. $\Omega(P_i)$ is a polyhedral disc in L_i and

$$\partial\Omega(P_i) = b_{i-1} e_{i-1} \cup P_i \cup b_i e_i \cup b_{i-1} b_i .$$

Hence $A(P)$ *is an annulus in* L *and* $\partial A(P) = F \cup P$.

4. *Notation.* Let X be an oriented simple closed curve in some set W and let i be the inclusion $X \to W$. We denote by $h(X)$ the generator of $H_1(X)$ which corresponds to the orientation and by $h(X, W)$ the element $i_* h(X)$ of $H_1(W)$.

Let $J = J_0$ oriented (any J_i would do). Note that *if* P *is any oriented T-circuit* $h(J, T)$ *and* $h(P, T)$ *generate* $H_1(T)$.

(4.1) *If* M *is oriented without boundary and if* $H_1(M) = H_2(M) = 0$, *there exists an oriented T-circuit* P *such that* $h(P, M - F) = 0$.

Proof. In the exact homology sequence for $(M, M - F)$ the connecting homomorphism $\alpha : H_2(M, M - F) \to H_1(M - F)$ is bijective. Now $H_2(M, M - F) = Z$, in fact a generator is represented by a fundamental cycle for D_0 mod J. The image of this generator under α is $h(J, M - F)$. Hence $h(J, M - F)$ is a generator of $H_1(M - F) = Z$. Let P^* be any fixed oriented T-circuit. Then $h(P^*, M - F) = qh(J, M - F)$, q an integer. Consider the generators $h(P^*, T)$, $h(J, T)$ of $H_1(T)$. It is easy to see that there exists a T-circuit P such that $h(P, T) = h(P^*, T) - qh(J, T)$. Since $T \subset M - F$, this relation holds when T is replaced by $M - F$. Hence $h(P, M - F) = 0$.

5. Let M be orientable, without boundary and assume that the induced action $(G, M - F)$ is free. The projection $\phi : M \to \mathfrak{M}$ maps F isomorphically onto $\mathfrak{F} = \phi F$. Let L, T, L_i, D_i, J_i be as in §3 and let $\mathfrak{L}$, $\mathfrak{J}$, $\cdots$ be the corresponding subcomplexes of $\mathfrak{M}$. Evidently $\phi^{-1}\,\mathrm{St}(\mathfrak{v}_i, \mathfrak{M}) = \mathrm{St}(v_i, M)$; hence $\phi^{-1}\mathfrak{L} = L$; hence L is invariant under the action.

(5.1) *Let* M *be orientable, without boundary, and assume that* $(G, M - F)$ *is* *free. If P is an oriented T-circuit, there exists an oriented* $\mathfrak{J}$*-circuit* $\mathcal{P}$ *such that* $\phi_* h(P, T) = h(\mathcal{P}, \mathfrak{J})$.

Proof. Write $P = \bigcup P_i$ as in §2. Then ϕP_i is a polygonal arc, not necessarily simple, which joins $\mathcal{J}_{i-1}$ to $\mathcal{J}_i$ and, except for its endpoints, lies in the interior of the annulus $\mathfrak{J}_i$. By a homotopy in $\mathfrak{J}_i$ with fixed endpoints, ϕP_i is homotopic to a simple polygonal arc which, except for its endpoints lies in Int $\mathfrak{J}_i$. Thus ϕP is homotopic in $\mathfrak{J}$ to an oriented $\mathfrak{J}$-circuit $\mathcal{P}$ and so $\phi_* h(P, T) = h(\mathcal{P}, \mathfrak{J})$.

Notation. From here on we shall write $X_\flat$ for $X - X \cap F$, and $\mathcal{X}_\flat$ for $\mathcal{X} - \mathcal{X} \cap \mathfrak{F}$.

(5.2) *Let M be orientable, without boundary and assume that* $(G, M_\flat)$ *is free.* Let P, $\mathcal{P}$ *be oriented T- and* $\mathfrak{J}$*-circuits such that* $\phi_* h(P, T) = h(\mathcal{P}, \mathfrak{J})$ *and let* $\mathcal{A} = A(\mathcal{P})$. *There exists a polyhedral annulus A in M such that* (1) ϕ *maps A homeomorphically onto* $\mathcal{A}$; (2) *the sets* $gA_\flat$, $g \in G$, *are disjoint*; (3) $\phi^{-1}\mathcal{A} = \bigcup gA$; (4) $\partial A = F \cup B$ *where B is a polygon in M.*

Proof. First we show that there exists in $L_\flat$ a set Y which is mapped homeomorphically onto $\mathcal{A}_\flat$ by ϕ and is such that the images of Y are disjoint and their union is $\phi^{-1}\mathcal{A}_\flat$. This will be a consequence of lemmas 2 and 3 with $\Gamma = G$, $K = L_\flat$, $\psi = \phi \mid L_\flat$, $\mathcal{Y} = \mathcal{A}_\flat$, provided we show that

$$i_* \pi_1(\mathcal{A}_\flat, \chi) \subset \phi_* \pi_1(L_\flat, x), \qquad x \in L_\flat,$$

where $\chi = \phi x$ and $i : \mathcal{A}_\flat \to \mathcal{L}_\flat$ is the inclusion. Since T is a strong deformation retract of $L_\flat$,

$$\pi_1(T, x) = \pi_1(L_\flat, x) = Z \times Z.$$

Thus $\pi_1(\mathcal{A}_\flat, \chi)$ and $\pi_1(L_\flat, x)$ are abelian and it is sufficient therefore to show that $i_* H_1(\mathcal{A}_\flat) \subset \phi_* H_1(L_\flat)$. Since $\mathcal{P}$ is a strong deformation retract of $\mathcal{A}_\flat$, $H_1(\mathcal{A}_\flat)$ is generated by $h(\mathcal{P}, \mathcal{A}_\flat)$. Hence $i_* \mathcal{H}_1(\mathcal{A}_\flat)$ is generated by

$$i_* h(\mathcal{P}, \mathcal{A}_\flat) = h(\mathcal{P}, \mathcal{L}_\flat) = \phi_* h(P, L_\flat) \subset \phi_*(H_1(L_\flat))$$

and so the inclusion in question follows. Now let $A = Y \cup F$ so that $A_\flat = Y$. ϕ maps A onto $\mathcal{A}$ and the map $\phi_1 = \phi \mid A$ of A onto $\mathcal{A}$ is bijective. We assert that ϕ_1^{-1} is continuous. It is sufficient to prove continuity at an arbitrary point $\mathfrak{f}$ of $\mathfrak{F}$. Let $\phi_1^{-1} \mathfrak{f} = f$ and let U be an open neighborhood of f. It is sufficient to show that there exists an open neighborhood $\mathcal{U}$ of $\mathfrak{f}$ such that $\phi_1^{-1}(\mathcal{U} \cap \mathcal{A}) \subset U$. Let $V = \bigcap_g gU$ $(g \in G)$. V is an open neighborhood of f and is invariant, hence a union of orbits so that $\phi^{-1}\phi V = V$. Let $\mathcal{U} = \phi V$. Since ϕ is an open map, $\mathcal{U}$ is an open neighborhood of $\mathfrak{f}$. We have

$$\phi_1^{-1}(\mathcal{A} \cap \mathcal{U}) \subset \phi^{-1}(\mathcal{A} \cap \mathcal{U}) \subset \phi^{-1}\mathcal{U} = V \subset U.$$

Since the domain $\mathcal{A}$ of ϕ_1^{-1} is compact, ϕ_1^{-1} is a homeomorphism. Hence A is an annulus. It is readily seen that A is polyhedral since ϕ is simplicial.

6. *Proof of Theorem* 1. Assume that the hypotheses of Theorem 1 are satisfied. The induced action $(G, M_\flat)$ is free [5, remark on p. 708]. Choose P, $\mathcal{P}$ so that $h(P, M_\flat) = 0$, $\phi_* h(P, T) = h(\mathcal{P}, \mathfrak{I})$ ((4.1) and (5.1)). The second relation holds with $\mathfrak{I}$ replaced by $\mathfrak{M}_\flat$ and the first then implies $h(\mathcal{P}, \mathfrak{M}_\flat) = 0$. Now the manifold $\mathfrak{M} - \text{Int}\,\mathcal{L}$ is a strong deformation retract of $M_\flat$, hence $h(\mathcal{P}, \mathfrak{M} - \text{Int}\,\mathcal{L}) = 0$. From the definition of $\mathcal{L}$ (§3), $\mathcal{P}$ is a subcomplex of $(\mathfrak{M} - \text{Int}\,\mathcal{L})''$. By Lemma 1 there exists in $\mathfrak{M} - \text{Int}\,\mathcal{L}$ a compact oriented polyhedral manifold $\mathcal{W}$ with $\partial\mathcal{W} = \mathcal{P}$. Let $\mathcal{A} = A(\mathcal{P})$ (§3) and let $\mathcal{A}$ and $\mathfrak{F}$ be oriented so that $\partial\mathcal{A} = \mathfrak{F} \cup (-\mathcal{P})$ (which implies $\partial\mathcal{A}_\flat = -\mathcal{P}$). Then $\mathcal{C} = \mathcal{W} \cup \mathcal{A}$ is an oriented 2-manifold with boundary $\mathfrak{F}$, and $\mathcal{C}_\flat$ is an oriented (noncompact) manifold without boundary. Now $\phi \mid M_\flat$ is a local homeomorphism and hence $\phi^{-1}\mathcal{A}_\flat$, $\phi^{-1}\mathcal{W}$, $\phi^{-1}\mathcal{C}_\flat$, are oriented manifolds and $\phi^{-1}\mathcal{C}_\flat = \phi^{-1}\mathcal{A}_\flat \cup \phi^{-1}\mathcal{W}$. Since $\phi^{-1}\mathcal{C}_\flat$ is without boundary (because $\mathcal{C}_\flat$ is), we have

$$\partial\phi^{-1}\mathcal{A}_\flat = -\partial\phi^{-1}\mathcal{W}. \tag{2}$$

If we refer to (4.2) and keep in mind that $\phi g\sigma = \phi\sigma$ $(g \in G)$ for every oriented simplex σ of M we see that there exists an oriented annulus A such that $\phi^{-1}\mathcal{A}_\flat = \bigcup gA_\flat$ (disjoint union) and such that ϕ maps A homeomorphically onto $\mathcal{A}$ with preservation of orientation. We have $\partial A = F \cup (-B)$, $\partial A_\flat = -B$, where B is an oriented polygon in $M_\flat$ such that $\phi B = \mathcal{P}$ and F is oriented so that $\phi F = \mathfrak{F}$. From (2) we have

$$\partial\phi^{-1}\mathcal{W} = \bigcup gB. \tag{3}$$

Let W be a component of $\phi^{-1}\mathcal{W}$. Then $\partial W = \bigcup g'B$ where g' ranges over a subset G' of G. Let $Y = (\bigcup g'A) \cup W$. From (3) and the relations $\partial g'A = F \cup (-g'B)$, we have, formally at least, $\partial Y = kF$ where k is the number of elements in G'. Thus Y is, so to speak, an oriented 2-manifold with oriented boundary kF. This simply means that Y is homeomorphic to the complex obtained from a compact oriented surface with k boundary curves by identifying the boundaries with orientations matching. Suppose that $k > 1$. A simple cell decomposition shows that $H_2(Y, Z_k) = Z_k$, $H_2(Y, Z_j) = 0$ if $(j, k) = 1$. By the Alexander duality theorem $M - Y$ has two components which, again by duality, implies that $H_2(Y, Z_j) = Z_j$ for every $j > 1$, which is impossible. We conclude that $k = 1$, so $Y = W \cup g'A$ for some $g' \in G$, and $\partial Y = F$. The images of $Y_\flat$ are disjoint. For if $Y_\flat$ meets $gY_\flat$, $g \neq 1$, then W meets gW since $A \cap gA = \emptyset$. Since W is a component of the invariant set $\phi^{-1}\mathcal{W}$ so is gW; hence $W = gW$. Hence $B = gB$ which is impossible since $B \subset A$. This concludes the proof.

7. *Proof of Theorem* 2. Assume that the hypotheses of Theorem 2 hold. Orient F. Since F is unknotted, there exists an oriented polyhedral disc Δ in M such that $\partial\Delta = F$. Then $\phi\Delta$ is a singular disc in $\mathfrak{M}$ with boundary $\mathfrak{F}$. We shall modify Δ to obtain Δ_1 say, such that no singularity of $\phi\Delta_1$ lies on $\mathfrak{F}$. By

Dehn's lemma [2] there exists a polyhedral disc Ω in $\mathfrak{M}$ with boundary $\mathfrak{F}$. By Lemmas 2 and 3 with $\Gamma = G, K = M_\flat, \mathfrak{K} = \mathfrak{M}_\flat, \psi = \phi \mid M_\flat, Y = \Omega_\flat$ there is a set Y in $M_\flat$ with disjoint images and mapped homeomorphically onto $\Omega_\flat$ by ϕ. Then $Y \cup F$ is the required disc in M (see proof of (5.2)).

To obtain Δ_1, decompose $\Delta : \Delta = \Delta^* \cup E$ where Δ^* is an oriented disc, E an oriented annulus, $\partial\Delta^* = B^*$ say, $\partial E = (-B^*) \cup F$. Let this be done in such a way that $E \subset L$.

Let A, B, $\mathfrak{A}$, $\mathfrak{P}$ be as in §6 and recall that $\partial A = (-B) \cup F$ and that ϕ maps A, B homeomorphically onto $\mathfrak{A}$, $\mathfrak{P}$. Evidently ϕ maps $B \cap L_i$ (§3) homeomorphically onto $\mathfrak{P} \cap \mathfrak{L}_i$ from which we see that B is a T-circuit. Hence $h(B, T)$, $h(J, T)$ generate $H_1(T) = Z \times Z$ and since T is a strong deformation retract of $L_\flat$, $h(B, L_\flat)$ and $h(J, L_\flat)$ generate $H(L_\flat) \cong H(T)$. Hence $h(B^*, L_\flat) = ah(B, L_\flat) + bh(J, L_\flat)$ say. This holds with $L_\flat$ replaced by $M_\flat$ and since $h(J, M_\flat) \neq 0$ (see proof of (4.1)) and $h(B^*, M_\flat) = 0$, we have $b = 0$ so that $h(B^*, L_\flat) = ah(B, L_\flat)$. This holds with $L_\flat$ replaced by L and since $\partial A = (-B) \cup F$ and $\partial E = (-B^*) \cup F$, we see that $h(B^*, L) = h(F, L) = h(B, L)$; hence $a = 1$. Hence $h(B^*, L_\flat) = h(B, L_\flat)$. Since the fundamental group of $L_\flat$ is abelian, there is a singular annulus A_1 in $L_\flat$ with boundary $(-B^*) \cup B$. Then $\Delta_1 = A \cup A_1 \cup \Delta^*$ is a singular disc with boundary $(-B^*) \cup B$. Then $\Delta_1 = A \cup A_1 \cup \Delta^*$ is a singular disc with boundary F. Since A is non-singular and $A_1 \cup \Delta^* \subset M_\flat$, no singularity of Δ_1 is in F. Since ϕ maps A homeomorphically, no singularity of $\phi\Delta_1$ is in $\mathfrak{F}$.

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Columbia University
New York, New York