ZASSENHAUS' LEMMA ON SECTORIAL NORM-DISTANCES[1]

BY

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1. Introduction

A norm-distance on the Euclidean space, E_n, is a function, say F, from $E_n \times E_n$ to the reals having the properties that for any points P and Q in E_n:

(i) $F(P, Q) \geq 0$.
(ii) $F(P, Q) = F(Q, P)$.
(iii) $F(P + \bar{a}, Q + \bar{a}) = F(P, Q)$ where $\bar{a}$ is any vector in R_n and $P + \bar{a}$, $Q + \bar{a}$ denote respectively the points to which P and Q are translated by $\bar{a}$.
(iv) $F(P, X) + F(X, Q) = F(P, Q)$ where X is any point of the segment PQ.

The translation invariance expressed in (iii) implies that $F(P + \bar{a}, P)$ is independent of P so that $F(P + \bar{a}, P) = f(\bar{a})$ defines a non-negative real-valued function, f, on R_n. $f(\bar{a})$ is called the norm-length of the vector $\bar{a}$. (See for example Cassels [1, Chapter IV].)

In view of (iv), f has the property that

$$f(t\bar{a}) = |t| f(\bar{a})$$

for any real t.

The gauge body of F at P, P a point of E_n, is the set,

$$B(P, F) = \{X \mid X \text{ in } E_n, F(P, X) \leq 1\}.$$

It is a star set having P as center of symmetry. If P is an interior point then $B(P, F)$ is called a star body.

In E_2 a packing with respect to F, in the sense of Minkowski-Hlawaka, consists of a finite set of points, E, which is admisible with respect to F (i.e. $F(P, Q) \geq 1$ for any two points P and Q of E) and a Jordan polygon, Π, the vertices of which belong to E and which contains the remaining points of E, if any, in its interior. Such a pair, (Π, E), will also be called an F-distribution.

The term "sectorial norm-distance" has been introduced by Zassenhaus [2] to describe a norm-distance, F, which has the following special property: The complement of $B(0, F)$ consists of a finite and, because of (ii), an even number of disjoint open convex sets $K_1, \cdots, K_{2r}$; each K_i is contained in a sector (i.e. a cone with vertex 0) S_i of E_n $(i = 1, \cdots, 2r)$; int $S_i \cap$ int $S_j = \emptyset$ if $i \neq j$; $\bigcup S_i = E_n$.

A vector $\bar{a}$ is said to belong to the sector S_i if $0 + \bar{a}$ is in S_i.

A sectorial norm-distance is non-degenerate if and only if $r > 1$. That $r > 1$ will always be assumed in what follows.

In E_2 a sectorial norm-distance gives rise to a classification of triangles into

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two types as observed by N. Smith [3] in his investigation of packings with respect to the particular sectorial norm-distance

$$F((x_1, y_1), (x_2, y_2)) = |(x_1 - x_2)(y_1 - y_2)|^{1/2}.$$

He calls a triangle PQR type I if one of each of the pairs of vectors $\pm\overrightarrow{PQ}$, $\pm\overrightarrow{QR}$ and $\pm\overrightarrow{PR}$ belongs to the same sector. All other triangles are called type II.

If PQR is of type I with, say, $\overrightarrow{PQ}$, $\overrightarrow{QR}$ and $\overrightarrow{PR}$ belonging to the same sector then

$$F(P, Q) + F(Q, R) \leq F(P, R).$$

This distinction amongst triangles has been exploited by Smith in showing that the slackness (see [1] and [4]) of a packing with respect to the sectorial norm-distance above is non-negative:

$$\frac{A(\Pi)}{\Delta} + \frac{F(\Pi)}{2} + N - 1 \geq 0$$

where $A(\Pi)$ is the area of the domain bounded by Π, $F(\Pi)$ is the length of Π measured by F, N is the number of points of E and Δ is the mesh of the critical lattice relative to F.

In the expression for the slackness function corresponding to a sectorial norm-distance Zassenhaus replaces $\frac{1}{2}\Delta$ by the greatest lower bound of the areas of type II triangles when this is positive. He shows that the resulting modified slackness function is non-negative. His proof depends upon the following lemma.

Let F be a sectorial norm-distance and (Π, E) an F-distribution. If no side of Π is maximum side of a type I triangle with vertices in E then there exists a triangulation of the domain with boundary Π by means of type II triangles whose vertices are in E.

It is this lemma which is our chief concern here, the object of this paper being to present a new proof. Moreover the lemma will be taken out of the above context to the extent that it holds for any sectorial covering of the plane such as the above but which need not be attached to a sectorial norm-distance function.

2. Triangulations with a sectorial condition

An orientation of E_2 being fixed let $L_1, \cdots, L_{2r}$ $(r > 1)$ be distinct half-lines each with end point 0 which are so indexed that the angle from L_i to L_{i+1} $(i = 1, \cdots, 2r; L_{2r+1} = L_1)$ is positive and, further, L_{k+r} is the reflexion in 0 of L_k $(k = 1, \cdots, r)$.

Denote by S_i the half-open sector bounded by L_i and L_{i+1}, which includes L_i but not L_{i+1}.

Such a sectorial covering determines a partial ordering of the symmetric pairs of distinct points in E_2 in the following way.

For any two distinct points P and Q in E_2 denote the symmetric pair they determine by PQ or QP. Define PQ and RS to be comparable if and only if at least one of R and S is P or Q, say $S = P$, and either $\overrightarrow{PR}$ or $\overrightarrow{RP}$ belongs to the same sector as $\overrightarrow{PQ}$. Furthermore $PQ \geq PR$ if and only if the rotation of $\overrightarrow{PR}(\overrightarrow{RP})$ to $\overrightarrow{PQ}$ is non-negative, $\overrightarrow{PR}(\overrightarrow{RP})$ and $\overrightarrow{PQ}$ belonging to the same sector. Accordingly $PQ = PR$ if and only if P, Q and R are collinear.

Call a triangle PQR type I if PQ, PR and QR are comparable; otherwise call PQR type II. If PQR is type I and, say, $PQ < PR < QR$ then call PR the distinguished side of PQR.

Let E be a finite set of points and Π a Jordan polygon whose vertices are in E and which contains the remaining points of E in its interior. Denote by T the set of triangles with vertices in E such triangles being contained in the closed domain Π' bounded by Π. By an E-triangulation of Π' will be meant a triangulation by triangles in T the set of whose vertices is precisely E.

LEMMA. *Let Π, E and T be as above. If no type I triangle in T has distinguished side a side of Π then there exists an E-triangulation of Π' no triangle of which is type I.*

Proof. We shall show that, under the circumstances of the lemma, with any E-triangulation of Π' in which there are type I triangles there exists an E-triangulation of Π' in which there are fewer such.

Let ABC be a type I triangle in a particular E-triangulation of Π'. Relabeling if need be we can assume that $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{AC}$ belong to the same sector, S, ($S = S_i$ for some i, $1 \leq i \leq 2r$). Then AC is the distinguished side of ABC.

According to the hypothesis, AC is not a side of Π and so must be a side of another triangle ACD in that triangulation.

There are the following possibilities in regard to the quadrilateral $ABCD$.

(1) The angle at C in $ABCD$, $\angle C \geq \pi$. Then $\overrightarrow{CD}$ and $\overrightarrow{AD}$ belong to S, ACD is type I with distinguished side AD.

(2) $\angle A \geq \pi$. Then $\overrightarrow{DA}$ and $\overrightarrow{DC}$ belong to S, ACD is type I with distinguished side DC.

If $\angle A < \pi$ and $\angle C < \pi$ then either

(3) ABD and BCD are both type II

or (4) ABD or BCD is type I.

Suffice to consider the case in which ABD is type I when the possibilities are:

(4a) $\overrightarrow{AD}$ and $\overrightarrow{BD}$ belong to S. Then AD is distinguished side of ABD and, if BCD is type I, then DC is not its distinguished side.

(4b) $\overrightarrow{AD}$ and $\overrightarrow{BD}$ belong to S. Then AD is not distinguished side of ABD; $\overrightarrow{DC}$ belongs to S, BCD is type I with DC its distinguished side.

(4c) $\overrightarrow{AD}$ belongs to S. Then $\overrightarrow{DB}$ must belong to S; AD is not the distinguished side of ABD; $\overrightarrow{DC}$ belongs to S, BCD is type I with DC its distinguished side.

Summarizing the above possibilities, precisely one of the following holds:

(i) $ABCD$ is convex but neither ABD nor BCD is type I.
(ii) AD is distinguished side of ACD or ABD.
(iii) CD is distinguished side of ACD or BCD.

If (i) is the case then replacing AC by BD in the given E-triangulation of Π' yields one which has fewer type I triangles.

In either of the cases (ii) and (iii), say in (ii), the hypothesis requires that AD be a side of another triangle, ADE, in the triangulation. If AD is distinguished side of ADC then consideration of $ACDE$ similar to that given to $ABCD$ leads either to the possibility of a suitable retriangulation and the conclusion we seek or to a further repetition of the argument. If AD is distinguished side of ABD then replacing AC by BD does not increase the number of type I triangles since if BDC is also type I then so is ADC. We then apply our argument to $ABDE$.

The above procedure, if it does not terminate with a suitable retriangulation, gives rise to a sequence, $P_1 Q_1 (=AC)$, $P_2 Q_2, \cdots$ of distinguished sides of type I triangles in T. There being but finitely many triangles in T and $P_i Q_i$ being completely determined by $P_{i-1} Q_{i-1}$ $(i > 1)$ this sequence must be periodic. The proof will be complete upon showing that this is not possible.

Since the number of sectors is greater than 2 we can and shall choose a rectangular coordinate system in which $S - \{0\}$ lies in the right half-plane. Then if $\overrightarrow{PQ}$ belongs to S we shall say that $Q(P)$ is to the right (left) of $P(Q)$.

Looking again at the way in which the sequence $(P_i Q_i)$ arises we observe that (P_i) and (Q_i) are alternately stationary:

$$P_1 = P_2 = \cdots = P_{i_1} \qquad (i_1 \geq 1)$$

$$Q_{i_1} = Q_{i_1+1} = \cdots = Q_{i_2} \qquad (i_2 \geq i_1)$$

and so on where either $P_{j+1} \neq P_j$ and $Q_{j+1} = Q_j$ or $P_{j+1} = P_j$ and $Q_{j+1} \neq Q_j$ $(j = 1, 2, \cdots)$.

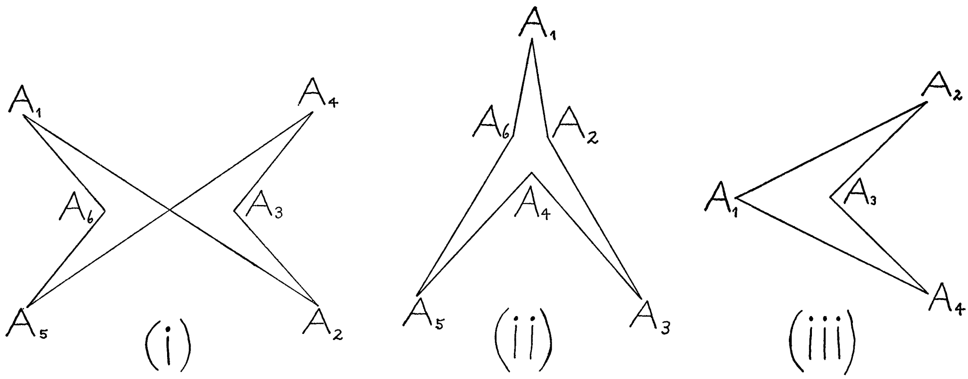
We observe further that if Q_1 is to the right (left) of P_1 then Q_i is to the right (left) of P_i for each i. Moreover the orientation of $P_i Q_i X$ ($X = Q_{i+1}$ if $Q_{i+1} \neq Q_i$, $X = P_{i+1}$ if $P_{i+1} \neq P_i$) is the same for each i. Indeed it is the same for each $P_i Q_i X$ where $X = Q_j$ if $P_j = P_i$ $(j > i)$, $X = P_j$ if $Q_j = Q_i$ $(j > i)$.

The subsequence $P_{i_1} Q_{i_1}$, $P_{i_2} Q_{i_2}, \cdots$ in which $i_1, i_2, \cdots$ are as above determines a polygonal path $P_{i_1} Q_{i_2} P_{i_3} \cdots$ which does not cross itself. Thus if it has multiple points then these are either vertices or belong to sides whose end points are multiple points while it is such that by a local variation of its vertices it can be made simple, i.e. it is the limit of simple paths.

Our concern then is with a class, K, of polygonal paths: namely, $A_1 A_2 \cdots$ belongs to K if and only if

(a) it is the limit of simple paths;

(b) A_{2k} is to the right of A_{2k-1} and A_{2k+1} for $k = 1, 2, \cdots$ or A_{2k} is to the left of A_{2k-1} and A_{2k+1} for $k = 1, 2, \cdots$;



(c) the orientations of $A_i\, A_{i+1}\, A_{i+2}$ and $A_{i+1}\, A_{i+2}\, A_{i+3}$ are opposite for $i = 1, 2, \cdots$;

We wish to show that K contains no closed path $\alpha = A_1 A_2 \cdots$,

$$A_{n+r} = A_r\,, \qquad r = 1, 2, \cdots .$$

In any such path either of the conditions (b) and (c) requires that n be even. In view of (a) there are interior angles defined at the vertices of α and (c) is equivalent to their being alternately greater than and less than π. We may suppose that the first alternative in (b) holds: each vertex with even index is to the right of the vertex preceding and of the vertex following. Then there exists a vertex of odd index say A_{2l+1} which is not to the right of any other and one of even index say A_{2r} which is not to the left of any other. Thus α is contained in a strip bounded on the left by a vertical line through A_{2l+1} and on the right by a vertical line through A_{2r}. But this implies that the interior angles at A_{2l+1} and A_{2r} are each less than π violating condition (c). Thus K contains no closed path and the proof is complete.

We conclude by remarking that each of the above conditions is necessary. The accompanying figures illustrate (i), conditions (b) and (c) but not (a); (ii), (c) and (a) but not (b); (iii), (a) and (b) but not (c).

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