INTEGRAL EQUATIONS ON A HILBERT SPACE[1]

BY

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The purpose of the following comments is to describe a more general setting in which the techniques and theorems discovered by J. S. Mac Nerney in [2] remain valid.

Let (H, Q) be a complete inner product space with norm N corresponding to the inner product Q, and let $B(H)$ denote the set of all linear and continuous functions from H to H. If each of U and V is a member of $B(H)$, then we say $U \ll V$ provided that, if x is in H, then $Q(Ux, x) \leq Q(Vx, x)$.

Let P be an algebra over the real numbers of Hermitian (a member U of $B(H)$ is *Hermitian* provided U^* is U, where U^* is the *adjoint* of U) members of $B(H)$ such that I, the identity function on H, is in P; and P is closed in the topology of point-wise convergence on H. Note that, if each of U and V is in P, then (letting the "product" UV denote the function $U[V]$)

$$UV = (UV)^* = V^*U^* = VU$$

and P is commutative. Thus (see, for example, p. 265 of [4]), if each of U and V is in P with $O \ll U$ and $O \ll V$, then $O \ll UV$, and the following lemma is true.

Lemma 1. *If each of U, V, A, and B is in P, $O \ll U$, $O \ll V$, $-U \ll A \ll U$, and $-V \ll B \ll V$, then*

$$-UV \ll AB \ll UV.$$

In light of this lemma, if each of U and V is in P with $O \ll U \ll V$, then $O \ll U^2 \ll V^2$, and from this it follows that, if x is in H, then

$$N(Ux) \leq N(Vx).$$

Let S be a linearly ordered set with order relation $\mathfrak{O}$. If each of x and y is in S, then an *$\mathfrak{O}$-subdivision* of $\{x, y\}$ is a sequence $\{t_p\}_0^n$ such that t_0 is x, t_n is y and the following hold:

(i) if $\{x, y\}$ is in $\mathfrak{O}$, then $\{t_{p-1}, t_p\}$ is in $\mathfrak{O}$ for $p = 1, \cdots, n$;
(ii) if $\{y, x\}$ is in $\mathfrak{O}$, then $\{t_p, t_{p-1}\}$ is in $\mathfrak{O}$ for $p = 1, \cdots, n$.

A *refinement* of the $\mathfrak{O}$-subdivision t, of the member $\{x, y\}$ of $S \times S$, is an $\mathfrak{O}$-subdivision of $\{x, y\}$ of which t is a subsequence.

If A is a sequence with values in a ring, then $\prod_1^1 A_p$ is A_1, and for each positive integer n, $\prod_1^{n+1} A_p$ is $(\prod_1^n A_p)A_{n+1}$. Suppose f is a function from

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$S \times S$ to a ring. If $\{x, y\}$ is in $S \times S$ and $\{t_p\}_0^n$ is an $\mathcal{O}$-subdivision of $\{x, y\}$, then $\prod_t f$ denotes $\prod_1^n f(t_{p-1}, t_p)$, while $\sum_t f$ denotes $\sum_1^n f(t_{p-1}, t_p)$. Furthermore, f is said to be *$\mathcal{O}$-additive* provided that, if each of $\{x, y\}$ and $\{y, z\}$ is in $\mathcal{O}$, then

$$f(x, y) + f(y, z) = f(x, z) \quad \text{and} \quad f(z, y) + f(y, x) = f(z, x),$$

while f is said to be *$\mathcal{O}$-multiplicative* provided that, if each of $\{x, y\}$ and $\{y, z\}$ is in $\mathcal{O}$, then

$$f(x, y)f(y, z) = f(x, z) \quad \text{and} \quad f(z, y)f(y, x) = f(z, x).$$

Let $\mathcal{O}\mathcal{A}^+$ denote the set of all $\mathcal{O}$-additive functions α, from $S \times S$ to P, such that $0 \ll \alpha$, and let $\mathcal{O}\mathcal{M}^+$ denote the set of all $\mathcal{O}$-multiplicative functions μ, from $S \times S$ to P, such that $0 \ll \mu - I$.

If α is in $\mathcal{O}\mathcal{A}^+$, μ is in $\mathcal{O}\mathcal{M}^+$, $\{a, b\}$ is in $S \times S$, t is an $\mathcal{O}$-subdivision of $\{a, b\}$, and s is a refinement of t, then the following hold:

(i) $\prod_t (I + \alpha) \ll \prod_s (I + \alpha) \ll \text{Exp}\,\{\alpha(a, b)\}$;
(ii) $0 \ll \sum_s [\mu - I] \ll \sum_t [\mu - I]$.

Since P is a *complete lattice* (see Theorem 4.23.4 and its proof on p. 163 of [1]), there exists a unique member ${}_a\prod^b (I + \alpha)$ of P and a unique member ${}_a\sum^b [\mu - I]$ of P such that

(i) if u is an $\mathcal{O}$-subdivision of $\{a, b\}$ then

$$\prod_u (I + \alpha) \ll {}_a\prod^b (I + \alpha) \ll \text{Exp}\,\{\alpha(a, b)\},$$

and

$$0 \ll {}_a\sum^b [\mu - I] \ll \sum_u [\mu - I],$$

(ii) if c is a positive number and x is in H then there is an $\mathcal{O}$-subdivision u of $\{a, b\}$ with the property that, if v is a refinement of u, then

$$N({}_a\prod^b (I + \alpha)x - \prod_v (I + \alpha)x) < c$$

and

$$N({}_a\sum^b [\mu - I]x - \sum_v [\mu - I]x) < c.$$

The following theorem has been proved by J. S. Mac Nerney in [3, p. 328].

THEOREM 1. *There is a reversible function $\mathcal{E}^+$, from $\mathcal{O}\mathcal{A}^+$ onto $\mathcal{O}\mathcal{M}^+$, such that the following statements are equivalent*:
(i) *μ is in $\mathcal{O}\mathcal{M}^+$, α is in $\mathcal{O}\mathcal{A}^+$, and μ is $\mathcal{E}^+(\alpha)$;*
(ii) *α is in $\mathcal{O}\mathcal{A}^+$ and $\mu(a, b) = {}_a\prod^b (I + \alpha)$ for each $\{a, b\}$ in $S \times S$;*
(iii) *μ is in $\mathcal{O}\mathcal{M}^+$ and $\alpha(a, b) = {}_a\sum^b [\mu - I]$ for each $\{a, b\}$ in $S \times S$.*

Let R be a ring with multiplicative identity element denoted by 1, which has the following two properties:

(i) there is a function $|\cdot|$ from R to P such that
(a) $|x + y| \ll |x| + |y|$ for each x and y in R,

(b) $|xy| \ll |x|\,|y|$ for each x and y in R,
(c) $0 \ll |x|$ for each x in R, and $|x|$ is 0 only in case x is 0, and
(d) $|1| = |-1| = I$;

(ii) The ring R is *complete* in the sense that if $\{M, \leq\}$ is a directed set, f is a function from M to R, g is a function from M to P such that, if each of p and q is in M with $p \leq q$, then $0 \ll g(q) \ll g(p)$ and

$$|f(p) - f(q)| \ll g(p) - g(q);$$

then there is a member Z of R with the property that, if p is in M, then

$$|f(p) - Z| \ll g(p) - L,$$

where L is the member of P that is the point-wise limit of the net g. In this sense we also say f *converges in* R and has *limit* Z *in* R.

Let $\mathcal{OA}$ denote the set of all $\mathcal{O}$-additive functions V, from $S \times S$ to R, for which there is a member α of $\mathcal{OA}^+$ such that, if $\{a, b\}$ is in $S \times S$, then $|V(a, b)| \ll \alpha(a, b)$, and let $\mathcal{OM}^+$ denote the set of all $\mathcal{O}$-multiplicative functions W, from $S \times S$ to R, for which there is a member μ of $\mathcal{OM}^+$ such that, if $\{a, b\}$ is in $S \times S$, then $|W(a, b) - 1| \ll \mu(a, b) - I$.

Suppose α is in $\mathcal{OA}^+$, μ is in $\mathcal{OM}^+$, V is in $\mathcal{OA}$ and $|V| \ll \alpha$, W is in $\mathcal{OM}$ and $|W - 1| \ll \mu - I$, $\{a, b\}$ is in $S \times S$, t is an $\mathcal{O}$-subdivision of $\{a, b\}$ and s is a refinement of t. Using the techniques developed by Mac Nerney in [2], one can show that the following hold:

(i) $|\prod_s (1 + V) - \prod_t (1 + V)| \ll \prod_s (I + \alpha) - \prod_t (I + \alpha)$;
(ii) $|\sum_t [W - 1] - \sum_s [W - 1]| \ll \sum_t [\mu - I] - \sum_s [\mu - I]$.

In view of the completeness of R, let ${}_a\prod^b (1 + V)$ and ${}_a\sum^b [W - 1]$ denote, respectively, the unique members X and Y of R, such that

(i) $|X - \prod_t (1 + V)| \ll {}_a\prod^b (I + \alpha) - \prod_t (I + \alpha)$, and
(ii) $|Y - \sum_t [W - 1]| \ll \sum_t [\mu - I] - {}_a\sum^b [\mu - I]$.

In the above setting, with the above definition and descriptions of the classes $\mathcal{OA}^+$, $\mathcal{OA}$, $\mathcal{OM}^+$, and $\mathcal{OM}$, the entire theory developed by Mac Nerney in [2] can be duplicated. For example, if $\mathcal{OB}$ is the set of all functions A from S to R such that $dA(dA(a, b) = A(b) - A(a)$ for all $\{a, b\}$ in $S \times S)$ is in $\mathcal{OA}$, and the integrals mentioned are the limits in R, of appropriate sums, through successive refinements of $\mathcal{O}$-subdivisions of members of $S \times S$, then the following theorem can be proved.

THEOREM 2. *There is a reversible function* $\mathcal{E}$, *from* $\mathcal{OA}$ *onto* $\mathcal{OM}$, *such that the following statements are equivalent*:

(i) *W is in $\mathcal{OM}$, V is in $\mathcal{OA}$, and W is $\mathcal{E}(V)$;*
(ii) *V is in $\mathcal{OA}$, and $W(a, b) = {}_a\prod^b (1 + V)$ for each $\{a, b\}$ in $S \times S$;*
(iii) *W is in $\mathcal{OM}$, and $V(a, b) = {}_a\sum^b [W - 1]$ for each $\{a, b\}$ in $S \times S$;*

(iv) *V is in $\mathcal{OA}$, W is from $S \times S$ to R such that, if $\{a, b\}$ is in $S \times S$, then $W(a, \ )$ is in $\mathcal{OB}$ and*

$$W(a, b) = 1 + (L) \int_a^b W(a, \ )V;$$

(v) *V is in $\mathcal{OA}$, W is from $S \times S$ to R such that, if $\{a, b\}$ is in $S \times S$, then $W(\ , b)$ is in $\mathcal{OB}$ and*

$$W(a, b) = 1 + (R) \int_a^b VW(\ , b);$$

(vi) *W is in $\mathcal{OM}$, V is in $\mathcal{OA}$, and there is a member $\{\alpha, \mu\}$ of $\mathcal{E}^+$ such that*

$$| W(a, b) - 1 - V(a, b) | \ll \mu(a, b) - I - \alpha(a, b)$$

for each $\{a, b\}$ in $S \times S$.

This leads to the following theorem on the solutions of integral equations.

THEOREM 3. *Suppose a is a member of S, $\{V, W\}$ belongs to $\mathcal{E}$, and U is a function from S to R. The following two statements are equivalent:*

(i) *U is a member of $\mathcal{OB}$, and for each b in S*

$$U(b) = U(a) + (L) \int_a^b UV;$$

(ii) *for each b in S, $U(b) = U(a)W(a, b)$.*

Furthermore, the following two statements are also equivalent:

(iii) *U is a member of $\mathcal{OB}$, and for each b in S*

$$U(b) = U(a) + (R) \int_b^a VU;$$

(iv) *for each b in S, $U(b) = W(b, a)U(a)$.*

THEOREM 4. *Suppose V if in $\mathcal{OA}$ and W is $\mathcal{E}(V)$. Let G be a sequence such that, if $\{x, y\}$ is in $S \times S$, then $G_0(x, y)$ is 1, while, if n is a positive integer, then $G_n(x, y)$ is $(L)_x\int^y G_{n-1}(x, \ )V$. Then, for each $\{a, b\}$ in $S \times S$, $W(a, b)$ is the limit, in R, of the sequence $\sum_0^n G_p(a, b)$ for $n = 0, 1, \cdots$.*

THEOREM 5. *If R is torsion free, g is a member of $\mathcal{OB}$ that has commuting values, and W is $\mathcal{E}(dg)$, then the following statements are equivalent:*

(i) *$W(a, b)W(b, a) = 1$ for all $\{a, b\}$ in $S \times S$;*

(ii) *${}_a\sum^b | [dg]^2 | = 0$ for all $\{a, b\}$ in $S \times S$—in the sense that, if x is in H and c is a positive number, then there is an $\mathcal{O}$-subdivision t of $\{a, b\}$ such that, if s is a refinement of t, then $N(\sum_s | [dg]^2 | x) < c$;*

(iii) *W is* Exp (dg).

An example of the above setting is as follows. Let R_0 denote a commutative subring of $B(H)$ such that I is a member of R_0 ; if T is a member of R_0 , then T^* also belongs to R_0; and R_0 is closed in the strong operator topology for $B(H)$. Let P_0 be the closed (in the strong operator topology) real algebra generated by the Hermitian members of R_0. For a member T of R_0, define $|T|$ to be $[TT^*]^{1/2}$ (the unique member A of P_0 such that $0 \ll A$ and A^2 is TT^*). Using the fact that, if each of A and B is in P_0 and $0 \ll A \ll B$, then $0 \ll A^{1/2} \ll B^{1/2}$, we have the following theorem.

THEOREM 6. *If each of A and B belongs to R_0 and $0 \ll B$, then the following statements are equivalent:*

(i) $0 \ll |A|^2 \ll B^2$;
(ii) $0 \ll |A| \ll B$;
(iii) $N(Ax) = N(|A|x) \ll N(Bx)$ *for each x in H.*

Using the above facts, we have the next theorem.

THEOREM 7. *The ring R_0, with the function $|\cdot|$ (from R_0 to P_0), satisfies all the hypotheses imposed on the ring R and the function $|\cdot|$ (from R to P).*

The above constitutes a commutative example of the preceding theory. A non-commutative example is furnished by the following.

If n is a positive integer, let R_0^n denote the set of all $n \times n$ matrices with entries in R_0, and, for A in R_0^n, define $|A|_n$ to be the smallest member C of P_0 such that, if p is a positive integer in $[1, n]$, then $\sum_1^n |A_{pq}| \ll C$.

The following setting illustrates one advantage of the above treatment. Suppose S is the real line and F is a *non-decreasing* function from S to the set of projections on H to H. Let P denote the smallest algebra that is closed in the topology of point-wise convergence on H and also contains the range of F. Since F is non-decreasing, the projections in the range of F commute, and hence P is commutative. In this case, dF is a member of $\mathcal{O}\mathcal{A}^+$ even though F is not of bounded variation with respect to the usual norm on $B(H)$ (thereby, not included by the theory in [2]).

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