FACTORIZATION OF LOCALLY COMPACT ABELIAN GROUPS

BY

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1. Introduction

In the theory of abstract abelian groups, divisible groups play the role of universal direct factors. That is, if a divisible group is embedded as a subgroup of another group, it is a direct factor of that group. Furthermore, a group having the property that it is a direct factor of any group in which it is embedded must be divisible. We will consider the corresponding situation for locally compact abelian groups. We will assume all groups mentioned to be locally compact abelian groups. If G and H are groups then $G \times H$ will denote the product group with the product topology. If A and B are subgroups of G and the map $(a, b) \rightarrow a + b$ from $A \times B$ to G is one to one and onto we write $G = A \dotplus B$; if this map is also a homeomorphism we write $G = A \oplus B$. If G and H are groups we say that G is embedded in H if we have a topological isomorphism φ of G into H. It follows that the image $\varphi(G)$ is a closed subgroup of H, [4]. We may sometimes identify G with its image $\varphi(G)$ and think of G as a subgroup of H with the topology on G being that induced by the topology on H. We say that H is a quotient group of G if there is a continuous open homomorphism of G onto H. If G is a group then $\hat{G}$ will denote the dual group. We use the fact that if G is embedded in H then $\hat{G}$ is a quotient group of $\hat{H}$, and the dual fact that if H is a quotient group of G then $\hat{H}$ is embedded in $\hat{G}$. Our Theorem 1 should be compared with a similar result due to J. Dixmier, namely Theorem 5, in [1]. We begin the discussion with a few definitions.

DEFINITION 1. A group G is said to be a *universal internal direct factor* if whenever G is embedded as a subgroup of a group H, then $H = G \oplus K$ for some subgroup K of H.

DEFINITION 2. A group G is said to be a *universal external direct factor* if whenever G is embedded as a subgroup of a group H, then $H = G' \oplus K$ for some subgroups G' and K of H, where G and G' are topologically isomorphic.

The distinction between universal external and universal internal direct factors is not necessary in the abstract case, since it is easily seen that the two concepts are equivalent. However, in our case they are not; in fact, we will prove the following theorems.

THEOREM I. *A group is a universal internal direct factor if and only if it is of the form $R^n \times T^m$ where R is the real numbers and T is the circle group both*

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with the usual topology, n is a non-negative integer and $\mathfrak{m}$ is an arbitrary cardinal.

THEOREM II. *A group is a universal external direct factor if and only if it is of the form $R^n \times T^{\mathfrak{m}} \times D_1 \times D_2$ where R^n, $T^{\mathfrak{m}}$ are as in Theorem I, D_1 is a discrete divisible torsion group, and D_2 is either $\{0\}$ or a discrete divisible torsion-free group of infinite rank.*

2. Proofs

The authors express their gratitude to the referee for suggesting the following lemma which simplifies some of the other proofs.

LEMMA 1. *Let X, Y, and Z be topological spaces and let $f : X \to Y$, and $g : X \to Z$ be continuous. Suppose that Z is discrete and that for each $z \in Z$, f restricted to $g^{-1}(z)$ is a homeomorphism onto a closed subset of Y. Then the map $h : X \to Y \times Z$ given by $h(x) = (f(x), g(x))$ is a homeomorphism.*

Proof. Clearly h is continuous and the fact that for each $z \in Z$, f restricted to $g^{-1}(z)$ is one to one makes h one to one. It is enough to show that h is a closed map of X onto $h(X)$. Suppose $C \subseteq X$ is closed and $(f(x), g(x)) = (y, z)$ is in the closure of $h(C)$. We want to show that $x \in C$. For any neighborhood U of y, $U \times \{z\}$ is a neighborhood of (y, z), so $(U \times \{z\}) \cap h(C)$ is not empty; hence there exists $x' \in C \cap g^{-1}(z)$ such that $f(x') \in U$. From the hypotheses it follows that $f(C \cap g^{-1}(z))$ is a closed subset of Y and hence $x \in C \cap g^{-1}(z) \subseteq C$.

Both of the theorems will be proved by reducing the problem to the consideration of certain simple groups. We will use the following three facts.

(1) *A group of the form $R^n \times T^{\mathfrak{m}}$ is a universal internal direct factor (we call $R^n \times T^{\mathfrak{m}}$ cylindric).*

(2) *Any group can be embedded as a closed subgroup in a group of the form $R^n \times T^{\mathfrak{m}} \times D$ where D is discrete and divisible (we call $R^n \times T^{\mathfrak{m}} \times D$ a d-group).*

(3) *A quotient group of a d-group is a d-group.*

Proof of (1). This is essentially given in [2] but we repeat. Let G be a closed subgroup of H and let $G = \times G_i$ where each G_i is either R or T and the product is over some index set I. Let f_i be the *i-th* projection from G onto G_i. Each f_i has a extension $\bar{f}_i$ which is a continuous homomorphism from H onto G_i since each f_i is either a character or a real character on G [2, pp. 380, 391]. If we set $f = \times \bar{f}_i$ then f is a continuous homomorphism from H onto G and f is the identity on G. Thus, by a theorem of Weil [4], $H = G \oplus K$ where K is the kernel of f.

Proof of (2). Let G be given. From [4], we see that $G = R^n \times H$ where H has a compact open subgroup C. There is a continuous isomorphism φ of H into $T^{\mathfrak{m}}$ for some power $\mathfrak{m}$, [2]. Since H/C is discrete there is an embedding ψ of H/C into a divisible group D, [3]. Define $h : G \to R^n \times T^{\mathfrak{m}} \times D$ by

$$h(x, y) = (x, \varphi(y), \psi(y + C)).$$

If we let $f(x, y) = (x, \varphi(y))$ and $g(x, y) = \psi(y + C)$, then we see that $h = (f, g)$ and that the conditions of Lemma 1 are met. The map h is the required embedding. One can readily conclude by means of the duality theorems that any group is a quotient group of a group of the form $R^n \times F \times C$ where F is a discrete free abelian group and C is a compact torsion-free group.

Proof of (3). Let $G = R^n \times T^{\mathfrak{m}} \times D$ be a d-group and f a continuous open homomorphism from G onto H. Now $H_1 = f(R^n \times T^{\mathfrak{m}})$ is an open divisible subgroup of H and so there is a subgroup D_1 of H such that H is the (abstract) internal direct sum of H_1 and D_1. But because H_1 is open, D_1 is discrete and closed, and $H = H_1 \oplus D_1$. We need only show that H_1 is a d-group. Since the dual of a free group is a group of the form $T^{\mathfrak{m}}$ for some power $\mathfrak{m}$, and a subgroup of a free group is free, it follows by duality that a factor group of $T^{\mathfrak{m}}$ is of the form $T^{\mathfrak{n}}$ for some power $\mathfrak{n}$. Therefore $H_2 = f(T^{\mathfrak{m}}) = T^{\mathfrak{n}}$. By (1), H_2 is an internal direct factor of H_1, so $H_1 = H_2 \oplus K$ where K must be a quotient group of R^n. It is known, [2], that K is of the form $R^m \times T^p$, for integers $m, p \geq 0$; and the proof is complete.

LEMMA 2. *Suppose G is of the form $D_1 \times D_2$ where D_1 is a discrete divisible torsion-free group of finite non-zero rank and D_2 is a discrete divisible torsion group; then G is not a universal external direct factor.*

Proof. We may assume $G = Q_1 \times \cdots \times Q_n \times D_2$ where each Q_i is a copy of the discrete rationals. Let

$$H = R_1 \times \cdots \times R_n \times Q_1/Z_1 \times \cdots \times Q_n/Z_n \times D_2$$

where each R_i is a copy of the real numbers with the usual topology and Z_i is a copy of the integers in Q_i; Q_i/Z_i has the discrete topology. Let $h : G \to H$ be given by

$$h(r_1, \cdots, r_n, d) = (r_1, \cdots, r_n, r_1 + Z_1, \cdots, r_n + Z_n, d).$$

Define also f and g by

$$f(r_1, \cdots, r_n, d) = (r_1, \cdots, r_n)$$

$$g(r_1, \cdots, r_n, d) = (r_1 + Z_1, \cdots, r_n + Z_n, d).$$

Again $h = (f, g)$ and conditions of Lemma 1 are met. Hence h embeds G into H. We show that H cannot have a factor isomorphic to G by showing that H cannot have a factor isomorphic to the discrete rationals, Q. For suppose $H = Q \oplus K$ for some subgroup K. Since Q is discrete, the connected component of the identity of H is contained in K, so

$$R_1 \times \cdots \times R_n \times (0) \times \cdots \times (0) \subseteq K.$$

Also

$$(0) \times \cdots \times (0) \times Q_1/Z_1 \times \cdots Q_n/Z_n \times D_2 \subseteq K$$

since it is a torsion group. This is a contradiction.

LEMMA 3. *For any prime p, the group Z_{p^∞} with the discrete topology is not a universal internal direct factor.*

Proof. We assume Z_{p^∞} embedded (not topologically, of course) in T as the group of all p^n-*th* roots of unity, for all n, but we use the additive notation. Let $H = T \times Z_{p^\infty}$ where Z_{p^∞} has the discrete topology. Define $h : Z_{p^\infty} \to H$ by

$$h(x) = (x, px).$$

Lemma 1 shows that this map is an embedding. Let

$$G = \{(x, px) \mid x \in Z_{p^\infty}\} = h(Z_{p^\infty}).$$

Suppose $H = G \oplus K$ for some K. By connectivity $T \times (0)$ would be contained in K and hence we would have $G \cap (T \times (0)) = 0$. However there exists a non-zero x in Z_p such that $px = 0$, hence $(x, px) \in G \cap (T \times (0))$ which gives a contradiction.

Proof of Theorem 1. We have already seen that a group of the form $R^n \times T^m$ is a universal internal direct factor. If G is a universal internal direct factor then by (2) we can embed G in a d-group, so it must be a factor of this d-group and hence a quotient of this d-group and so by (3) G itself is a d-group. It follows from Lemmas 2 and 3 that neither the discrete rationals nor a discrete Z_{p^∞} can be a universal internal direct factor. Since every discrete divisible group is a sum of such groups, [3], the result will follow if we show that a direct factor of a universal internal direct factor is again a universal internal direct factor.

Suppose $A \oplus B$ is a universal internal direct factor and $A \subseteq H$ as a closed subgroup. Then we may embed $A \times B$ as a closed subgroup of $H \times B$. Hence $H \times B = (A \times B) \oplus D$. Now consider the map

$$H \xrightarrow[i]{} H \times B \xrightarrow[\pi]{} A \times B \xrightarrow[p]{} A,$$

where $i(h) = (h, 0)$, π is the projection onto $A \times B$, and $p(a, b) = a$. It is easily seen that this map is a continuous homomorphism of H onto A leaving A fixed, so by a well-known theorem [4] A is a direct factor of H. This proves the theorem.

The next lemma shows that the discussion of universal external direct factors can be reduced to consideration of discrete groups.

LEMMA 4. *Let G_0 be of the form $R^n \times T^m$ and G_1 be discrete; then a group of the form $G_0 \times G_1$ is a universal external factor if and only if G_1 is a universal external direct factor.*

Proof. Suppose G_1 is a universal external direct factor and we have $G_0 \oplus G_1 \subseteq H$, as a closed subgroup. Then we have $H = G_1' \oplus K$ where G_1' and G_1 are topologically isomorphic. Since $G_0 \subseteq H$ is connected and G_1' is discrete we have $G_0 \subseteq K$ and hence $K = G_0 \oplus K_1$ since G_0 is a universal internal direct factor. So $H = G_1' \oplus G_0 \oplus K_1$.

Suppose $G_0 \oplus G_1$ is a universal external direct factor and $G_1 \subseteq H$ as a closed subgroup. Then $G_0 \oplus G_1 \subseteq H \oplus G_0$ as a closed subgroup. Therefore $H \oplus G_0 = G_0' \oplus G_1' \oplus K$, where G_0', G_1' are topologically isomorphic with G_0, G_1 respectively. Again since G_0 is connected and G_1' discrete we have $G_0 \subseteq G_0' \oplus K$, hence $G_0' \oplus K = G_0 \oplus K_1$. So $H \oplus G_0 = G_1' \oplus G_0 \oplus K$ and it follows that H has a factor topologically isomorphic with G_1' hence with G_1.

LEMMA 5. *Suppose H has a compact open subgroup and $D \subseteq H$ is discrete and divisible; then $H = D' \oplus K$ where D and D' are topologically isomorphic. Moreover, if D is torsion free then D' may be chosen equal to D.*

Proof. If H_1 is any compact open subgroup then $H_1 \cap D$ is finite, since D is discrete. Since the intersection of all compact open subgroups is the component of the identity C_0, [2], we must have a compact open subgroup H_0 such that $H_0 \cap D \subseteq C_0$. Since C_0 is connected it is divisible and we have $D + C_0 = D' \dotplus C_0$. Therefore,

$$D + H_0 = D + C_0 + H_0 = D' + C_0 + H_0 = D' + H_0 .$$

Now, $D' \cap H_0 \subseteq (D + C_0) \cap H_0 \subseteq C_0$ because $D \cap H_0 \subseteq C_0$ and $C_0 \subseteq H_0$, so we have $D' \cap H_0 \subseteq C_0 \cap D' = \{0\}$. Hence $D + H_0 = D' \dotplus H_0$. Furthermore,

$$D' \cong (D' \dotplus H_0)/H_0 = (D + H_0)/H_0 \cong D/(D \cap H_0);$$

now since D is divisible and $D \cap H_0$ is finite, it follows from the structure theorems for divisible groups, [3], that $D/(D \cap H_0) \cong D$ and hence that $D' \cong D$. Since $H/H_0 \supseteq (D' \dotplus H_0)/H_0$ and $(D' \dotplus H_0)/H_0$ is divisible we have

$$H/H_0 = (D' \dotplus H_0)/H_0 \dotplus H_2/H_0$$

and it follows that $H = D' \dotplus H_2$. Since $H_2 \supseteq H_0$, H_2 is open and therefore we have $H = D' \oplus H_2$. The last assertion of the lemma follows from the observation that if D is torsion free then $D \cap H_1 = \{0\}$ for any compact open subgroup H_1.

Proof of Theorem II. By Lemma 4 we need only consider the discrete part $D_1 \times D_2$. If D_2 has finite rank, then by Lemma 2, $D_1 \times D_2$ cannot be a universal external direct factor. Suppose $D_2 = \{0\}$ or has infinite rank and $D_1 \oplus D_2 \subseteq G$. By [4], we can factor $G = R^n \oplus H$, where H has a compact open subgroup H_0. First we notice that $D_1 \subseteq H$, so that if $D_2 = \{0\}$ we are done by Lemma 5. If D_2 has infinite rank consider the mapping ψ:

$$D_2 \rightarrow R^n \oplus H \rightarrow H \rightarrow H/H_0 ,$$

where the first map is an inclusion and the next two are the natural projections. The kernel of this mapping is $D_2 \cap (R^n \oplus H_0) = K$. The natural projection p from K into R^n is one to one and we want to show that $p(K)$ is discrete. If $x_k + y_k \in D_2$, $x_k \in R^n$, $y_k \in H_0$ and $\{x_k\}$ has a non-trivial limit point, then $\{x_k + y_k\}$ will have a non-trivial limit point by compactness of H_0 but this contradicts the discreteness of D_2. It follows that $K \cong Z^m$ for some $m \leq n$. Therefore the mapping ψ is 1-1 on a divisible subgroup D_0 of D_2 that has the same rank as D_2. The map $D_2 \to H$ is also 1-1 on D_0 and the image D' of D_0 under this map is discrete since the elements lie in different cosets modulo H_0. Clearly $D' \cap D_1 = \{0\}$, so H has a subgroup $D' \oplus D_1 \cong D$ and by Lemma 4, H has a direct factor topologically isomorphic with D.

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