CORE THEOREMS FOR COREGULAR MATRICES

BY
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1. Introduction

The relation between the core of a complex sequence and the core of its transform by a regular matrix has been studied by Knopp and others [1, Ch. 6]. In this paper it is shown that core theorems for coregular matrices can be obtained rather readily from known core theorems for regular matrices by means of a decomposition of the coregular matrices. These new core theorems contain several results of B. E. Rhoades [3] for coregular matrices.

Let $A = (a_{nk})$ be an infinite matrix of complex numbers and let (s_n) be a complex sequence such that $A_n(s) = \sum_k a_{nk} s_k$ exists for every n. The sequence $(A_n(s))$ is called the transform of (s_n) by the matrix A. When $a_{nk} = 0$ for $k > n$, A is said to be triangular. Clearly, when A is triangular, $A_n(s)$ always exists. The matrix A is said to be conservative if $\lim_n A_n(s)$ exists whenever $\lim_n s_n$ exists. Necessary and sufficient conditions that A be conservative are well known [2, Th. 1]. When A is conservative, one defines $X(A)$, the characteristic of A, as $X(A) = t - \sum_k a_k$, where $t = \lim_n \sum_k a_{nk}$ and $a_k = \lim_n a_{nk}$. If $X(A) \neq 0$, A is said to be coregular. The matrix A is said to be regular if and only if $\lim_n A_n(s) = \lim_n s_n$ whenever $\lim_n s_n$ exists. Necessary and sufficient conditions that A be regular are also well known [2, Th. 2].

The core of a complex sequence (s_n) is defined by Cooke [1, p. 137] to be the intersection of the sets R_n, where R_n is the convex hull of the points $[s_n, s_{n+1}, \cdots]$, $n = 0, 1, \cdots$.

2. The main theorems

For complex sequences (s_n) and complex matrices A, the following assertion will be investigated:

(I) The core of $(A_n(s))$ is a subset of the image of the core of (s_n) under the linear transformation $w = z \cdot X(A) + \sum a_k s_k$.

Since the core of a real sequence (s_n) is the closed interval [lim inf s_n lim sup s_n], the real counterpart of (I) for real matrices with $X(A) \geq 0$ is the following:

(II)

$$\sum a_k s_k + X(A) \cdot \liminf s_n \leq \liminf A_n(s)$$

and

$$\limsup A_n(s) \leq \sum a_k s_k + X(A) \cdot \limsup s_n .$$

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THEOREM 2.1. *Let A be a complex coregular matrix and let (s_n) be a bounded complex sequence. A necessary and sufficient condition for* (I) *is*

$$\lim_n \sum_k |a_{nk} - a_k| = |X(A)|.$$

COROLLARY 2.2. *Let A be a real coregular matrix with $X(A) > 0$ and let (s_n) be a bounded real sequence. Then a necessary and sufficient condition for* (II) *is*

(a) $\lim_n \sum_k |a_{nk} - a_k| = X(A)$.

THEOREM 2.3. *Let A be a complex coregular matrix and let (s_n) be a complex sequence such that $A_n(s)$ exists for every n and $\sum a_k s_k$ converges. A sufficient condition for* (I) *is that there exists a number K such that for all n and all $k \geq K$,*

$$(a_{nk} - a_k)/X(A) = \text{Re}\,[(a_{nk} - a_k)/X(A)] \geq 0,$$

where Re $[z]$ *denotes the real part of z.*

COROLLARY 2.4. *Let A be a real coregular matrix with $X(A) > 0$. Let (s_n) be a real sequence such that $A_n(s)$ exists for every n and $\sum a_k s_k$ converges. A sufficient condition for* (II) *is*

(b) *there exists a number K such that $a_{nk} \geq a_k$ for all n and all $k \geq K$.*

The proofs of these theorems require the following lemma.

LEMMA 2.5. *Let A be a coregular matrix and define $B = (b_{nk})$ where $b_{nk} = (a_{nk} - a_k)/X(A)$. Then B is regular and*

$$A_n(s) = X(A) \cdot B_n(s) + \sum a_k s_k$$

for all sequences (s_n) for which $\sum a_k s_k$ converges and $A_n(s)$ exists.

Proof. B is regular, since $\lim_n b_{nk} = 0$ for every k,

$$\sum_k |b_{nk}| \leq (\sum_k |a_{nk}| + \sum_k |a_k|)/|X(A)| \text{ for every } n,$$

and $\lim_n \sum_k b_{nk} = 1$. Clearly, $A_n(s) = X(A) \cdot B_n(s) + \sum a_k s_k$.

Proof of Theorem 2.1. Agnew proved that if B is a regular matrix and (s_n) is bounded, then the core of $(B_n(s))$ is contained in the core of (s_n) if and only if $\lim_n \sum_k |b_{nk}| = 1$ [1, Th. 6.4 II]. Theorem 2.1 follows from Agnew's result, by means of the decomposition of Lemma 2.5.

Proof of Theorem 2.3. Cooke [1, p. 145] remarks that the condition that there exists a K such that $b_{nk} = \text{Re}\,[b_{nk}] \geq 0$ for every n and for $k \geq K$ is a sufficient condition that the core of $(B_n(s))$ be contained in the core of (s_n) when B is regular and (s_n) is arbitrary. Theorem 2.3 follows from this result by the use of Lemma 2.5.

THEOREM 2.6. *In order that the triangular coregular matrix A be such that* (I) *holds for those sequences (s_n) for which $\sum a_k s_k$ converges, it is necessary* *and sufficient that there exists a number K such that for all $n \geq k \geq K$,*

$$(a_{nk} - a_k)/X(A) = \text{Re}\,[(a_{nk} - a_k)/X(A)] \geq 0.$$

Proof. Define a triangular matrix $B = (b_{nk})$ as follows: let $b_{nk} = (a_{nk} - a_k)/X(A)$ if $n \geq k$, $b_{nk} = 0$ otherwise. Then, as in the proof of Lemma 2.5, B is regular and $A_n(s) = X(A) \cdot B_n(s) + \sum_{k=0}^{n} a_k s_k$. By a result of Agnew [1, Th. 6.4 I], the core of $(B_n(s))$ is contained in the core of (s_n) if and only if there exists a K such that $b_{nk} = \text{Re}\,[b_{nk}] \geq 0$ for all n and for all $k \geq K$. Hence, if $W_n(s) = X(A) \cdot B_n(s) + \sum_{k=0}^{\infty} a_k s_k$, the core of $(W_n(s))$ is contained in the image of the core of (s_n) under the transformation

$$w = z \cdot X(A) + \sum a_k s_k .$$

Now

$$| W_n(s) - A_n(s) | = | \sum_{k \geq n+1} a_k s_k | \to 0,$$

so, the cores of $(A_n(s))$ and $(W_n(s))$ are identical [1, Th. 6.3 II].

COROLLARY 2.7. *In order that the real coregular triangular matrix A, with $X(A) > 0$, be such that* (II) *holds for real sequences (s_n) such that $\sum a_k s_k$ converges, it is necessary and sufficient that*

(c) *there exists a K such that $a_{nk} \geq a_k$ for all $n \geq k \geq K$.*

3. Related results

Recently, B. E. Rhoades [3] investigated statement (II) under various combinations of conditions on the real matrix A. It will be shown that the corollaries of Section 2 above imply some of his results. His conditions are the following.

(d) There exists an integer p such that $a_k = 0$ for all $k \geq p$.
(e) There exists an integer q such that $a_{nk} \geq 0$ for all $k \geq q$.
(f) $\lim_n \sum_k | a_{nk} | = t$.

Rhoades' results for coregular matrices may be stated as follows:

THEOREM 3.1 [3, Th. 4]. (e) *is sufficient for* (II) *for those sequences (s_n) for which $\sum a_k s_k$ converges.*

THEOREM 3.2 [3, Th. 5]. *If A is triangular and satisfies* (d), *then* (e) *is necessary and sufficient for* (II).

THEOREM 3.3 [3, Th. 6]. (f) *is sufficient for* (II) *for bounded sequences.*

THEOREM 3.4 [3, Th. 7]. *If $a_k = 0$ for all k and A is triangular, then* (f) *is necessary and sufficient for* (II) *for bounded sequences.*

In proving these theorems, Rhoades used the following lemma.

LEMMA 3.5 [3, Lemma 1]. *If A is coregular and satisfies* (e), *then $X(A) > 0$.*

Theorem 3.2 follows readily from Corollary 2.7. For sufficiency, Lemma 3.5 assures that $X(A) > 0$. Choose $K = \max(p, q)$. Then (c) holds, and by the corollary, (II) follows. For the necessity part, (d) implies that $\sum a_k s_k$ converges for any sequence. Hence, if (s_n) is divergent, (II) and coregularity imply that $X(A) > 0$. Let $q = \max(K, p)$. Then condition (e) holds.

In order to show that Section 2 implies Theorems 3.3 and 3.4, an additional lemma is required. It may be of some independent interest.

LEMMA 3.6. *Condition* (f) *is equivalent to the assertion that* $a_k \geq 0$ *for all* k *and condition* (a) *holds.*

Proof. Condition (f) implies that $\lim_n \sum_k (|a_{nk}| - a_{nk}) = 0$. If

$$\lim_n (|a_{np}| - a_{np}) = a > 0$$

for some p, then

$$\lim_n \sum_k (|a_{nk}| - a_{nk}) = \lim_n (|a_{np}| - a_{np}) + \lim_n \sum_{k \neq p} (|a_{nk}| - a_{nk}) \geq a > 0.$$

Hence, for all k, $a_k = \lim_n |a_{nk}| \geq 0$. Since A is conservative, $\sum |a_k| = \sum a_k$ converges. Given $\varepsilon > 0$, there is an N such that $\sum_{k>N} a_k < \varepsilon/2$. Thus,

$$\limsup_n \sum_{k \geq 0} |a_{nk} - a_k| \leq \lim_n \sum_{k=0}^{N} |a_{nk} - a_k| + \lim_n \sum_{k>N} |a_{nk}| + \sum_{k>N} a_k \leq \lim_n \sum_{k>N} |a_{nk}| + \varepsilon/2.$$

Now,

$$\lim_n \sum_{k>N} |a_{nk}| = \lim_n \sum_{k\geq 0} |a_{nk}| - \sum_{k=0}^{N} |a_k| < \lim_n \sum_{k\geq 0} a_{nk} - \sum_{k \geq 0} a_k + \varepsilon/2,$$

using (f) and the definition of N. Hence,

$$\limsup_n \sum_k |a_{nk} - a_k| < \lim_n \sum_k (a_{nk} - a_k) + \varepsilon = X(A) + \varepsilon.$$

On the other hand,

$$\liminf_n \sum_k |a_{nk} - a_k| \geq \lim_n \sum_k (a_{nk} - a_k) = X(A).$$

If $a_k \geq 0$ for all k and condition (a) holds, then

$$\limsup_n \sum_k |a_{nk}| \leq \lim_n \sum_k |a_{nk} - a_k| + \sum a_k = \lim_n \sum_k (a_{nk} - a_k) + \sum a_k = \lim_n \sum_k a_{nk}.$$

Also, $\liminf_n \sum_k |a_{nk}| \geq \lim_n \sum_k a_{nk}$, so condition (f) holds.

It is to be noted that since (f) implies (a), if A is coregular and satisfies (f), then $X(A) > 0$. In the light of this remark and Lemma 3.6, Theorem 3.3 is a consequence of Corollary 2.2.

Using Lemma 3.6 and Corollary 2.2, it is seen that Theorem 3.4 may be strengthened to the following:

THEOREM 3.7. *If* $a_k \geq 0$ *for all* k, *then* (f) *is necessary and sufficient for* (II) *for bounded sequences.*

In order to see that Theorem 3.1 for bounded sequences is a consequence of Theorem 3.3, one uses a decomposition of the matrix A. Let A satisfy (e). Define matrices $C = (c_{nk})$ and $D = (d_{nk})$ as follows: let $c_{nk} = 0$ for $k < q$, $c_{nk} = a_{nk}$ for $k \geq q$; let $d_{nk} = a_{nk}$ for $k < q$, $d_{nk} = 0$ for $k \geq q$. Then C is conservative and satisfies (f) since $c_{nk} \geq 0$. Furthermore, C is coregular, since by Lemma 3.5, $X(A) > 0$, and clearly, $X(C) = X(A)$. Now, if (s_n) is bounded, then $A_n(s)$ exists and equals $C_n(s) + D_n(s)$. Also, $\lim_n D_n(s) = \sum_{k=0}^{q-1} a_k s_k$. Hence,

$$\limsup A_n(s) = \limsup C_n(s) + \sum_{k=0}^{q-1} a_k s_k ,$$

and a similar result holds for the inferior limits. Theorem 3.1 for bounded sequences now follows from an application of Theorem 3.3 to C.

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