A PROPERTY OF A CLASS OF DISTRIBUTIONS ASSOCIATED WITH THE MINKOWSKI METRIC

BY

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It is a well-known fact that if a sufficiently differentiable function f on $R^n = \{\langle t_1, \cdots, t_n\rangle : t_1, \cdots, t_n \text{ real}\}$, $n \geqq 2$, satisfies the wave equation

$$\Box f = \partial^2 f/\partial t_1^2 - \partial^2 f/\partial t_2^2 - \cdots - \partial^2 f/\partial t_n^2 = 0$$

and $f = \partial f/\partial t_1 = 0$ on the *disk* $t_1 = a_1$, $(t_2 - a_2)^2 + \cdots + (t_n - a_n)^2 \leqq \beta^2$, when $a_1, \cdots, a_n$ are real and $\beta > 0$, then $f = 0$ throughout the double conical region

$$| t_1 - a_1 | + [(t_2 - a_2)^2 + \cdots + (t_n - a_n)^2]^{1/2} \leqq \beta.$$

The same conclusion holds if $P(\Box)f = 0$ where P is a polynomial of degree k with real roots and $f = \partial f/\partial t_1 = \cdots = \partial^{2k-1}f/\partial t_1^{2k-1} = 0$ on the disk.

The solutions of $P(\Box)f = 0$ which are tempered distributions can be characterized as the Fourier transforms of tempered distributions concentrated in the finitely many hyperboloids $x_1^2 - x_2^2 - \cdots - x_n^2 = (\text{root of } P)/4\pi^2$, which may involve derivatives perpendicular to a hyperboloid only to a degree up to one less than the multiplicity of the corresponding root of P.

The object of this paper is to prove that if a tempered distribution $T = T(x_1, \cdots, x_n)$ is, in a suitable sense, of faster than exponential decrease as $| x_1^2 - x_2^2 - \cdots - x_n^2 |^{1/2} \to \infty$, its Fourier transform is determined throughout each double conical region as described above by its values arbitrarily near the corresponding disk. A somewhat misformulated version of this result appeared in my doctoral dissertation at Princeton University, written while on a National Science Foundation Cooperative Fellowship (1961–62). Thanks are due to Professors G. A. Hunt and Edward Nelson for reading several earlier drafts and making helpful comments.

For any n-tuple $z = \langle z_1, \cdots, z_n\rangle$ of complex numbers, $n \geqq 2$, we will let

$${}_2| z |_n^2 = z_2^2 + \cdots + z_n^2, \quad \text{and} \quad \| z \|^2 = z_1^2 - {}_2| z |_n^2.$$

Let $Q(R^n)$, $n \geqq 2$, be the space of C^∞ complex-valued functions f on R^n such that for some $\beta > 0$, there is for every $m > 0$ a $K > 0$ such that

$$| f(x) | = | f(x_1, \cdots, x_n) | \leqq K \exp(\beta \,|\, \| x \|^2 |^{1/2})/(1 + x_1^2 + \cdots + x_n^2)^m$$

for all $x_1, \cdots, x_n$, with every partial derivative of f, of any order, satisfying the same conditions, possibly with different values of K. We define a pseudo-topology in Q as follows: $f_k \to 0$ in Q if and only if β and K can be chosen independently of k (the latter for each partial derivative and $m > 0$), and

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for each $q \geqq 0$, $M > 0$, and nonnegative integers $p_1, \cdots, p_n$,

$$(1 + x_1^2 + \cdots + x_n^2)^q \partial^{|p|} f_k / \partial x_1^{p_1} \cdots \partial x_n^{p_n} \to 0$$

ask $k \to \infty$, uniformly for $| \, \| x \|^2 \, | \leqq M$, where $| p | = p_1 + \cdots + p_n$. Let Q' be the space of linear functionals on Q continuous for the given pseudotopology.

Since L. Schwartz's space $\mathcal{S}$ is contained in Q and has a finer (pseudo) topology, and $\mathcal{S}$ is dense in Q for the latter's pseudotopology, each member of Q' defines a tempered distribution which in turn determines it uniquely, so that the two may be identified. Roughly speaking, a tempered distribution will belong to Q' if and only if it decays as $| \, \| x \|^2 \, | \to \infty$ faster than $\exp(-\beta \, | \, \| x \|^2 \, |^{1/2})$ for any $\beta > 0$.

THEOREM. *For any $T \in Q'$, $a_1, \cdots, a_n$ real numbers and $\beta > 0$, if $Tg = 0$ for all $g \in Q$ which are Fourier transforms of distributions with support in $t_1 = a_1$, $(t_2 - a_2)^2 + \cdots + (t_n - a_n)^2 \leqq \beta^2$, then $Tf = 0$ whenever $f \in Q$ is the Fourier transform of a distribution with support in*

$$| t_1 - a_1 | + [(t_2 - a_2)^2 + \cdots + (t_n - a_n)^2]^{1/2} \leqq \beta.$$

Proof. Since multiplication by $\exp(2\pi i(a_1 x_1 + \cdots + a_n x_n))$ takes Q' onto itself, it suffices to treat the case $a_1 = a_2 = \cdots = a_n = 0$. Let D_β be the set where $t_1 = 0$ and $t_2^2 + \cdots + t_n^2 \leqq \beta^2$, and C_β the set where $| t_1 | + (t_2^2 + \cdots + t_n^2)^{1/2} \leqq \beta$. Let F_β be the class of distributions with support in D_β, G_β the C^∞ functions with support in C_β, and $\tilde{F}_\beta$ and $\tilde{G}_\beta$ respectively their Fourier transforms.

We shall show that in the pseudotopology of Q, the closure of $\tilde{F}_\beta \cap Q$ contains $\tilde{G}_\beta$, and then that the closure of $\tilde{G}_\beta$ contains all members of Q which are Fourier transforms of distributions with support in C_β, by regularization.

LEMMA. *$\tilde{F}_\beta$ is the set of functions $g(x_1, \cdots, x_n)$ of the form*

$$\sum_{r=0}^{N} g_r(x_2, \cdots, x_n) x_1^r$$

for some $N < \infty$ (depending on g), where each g_r belongs to $\mathcal{S}'$ and can be extended to an entire function of $n - 1$ complex variables $z_j = x_j + iy_j$, $j = 2, \cdots, n$, such that

$$g_r(z_2, \cdots, z_n) \exp(-2\pi\beta(| z_2 |^2 + \cdots + | z_n |^2)^{1/2})$$

is uniformly bounded.

Proof. If $g_r(z_2, \cdots, z_n)$ is entire, belongs to $\mathcal{S}'$, and is bounded as indicated, then by the generalized Paley-Wiener theorem [1, tome II, Ch. VII, Section 8, p. 127] its inverse Fourier transform T_r has support in the cube $| t_j | \leqq \beta$, $j = 2, \cdots, n$. Taking orthogonal transformations of the x_j and t_j we obtain an intersection of cubes which is exactly D_β. Hence the finite sum of tensor products

$$\sum_{r=0}^{N} T_r(t_2, \cdots, t_n) \otimes (2\pi i)^{-r} d^r\delta(t_1)/dt_1^r$$

has support in D_β ; its Fourier transform is g.

The converse is an easy consequence of the characterization of distributions with support in a subspace [1, tome I, Ch. III, Section 10, Théorème XXXVI] and the generalized Paley-Wiener theorem. (It will not actually be used later.) This completes the proof of the lemma.

Given $f \in \tilde{G}_\beta$, let f_{jk}, $k = 0, \cdots, 2j-1$, $j = 1, 2, \cdots$, be such that

$$f_j(z_1, \cdots, z_n) = \sum_{k=0}^{2j-1} f_{jk}(z_2, \cdots, z_n) z_1^k$$

is equal to $f(z_1, \cdots, z_n)$ together with its first $j-1$ partial derivatives with respect to z_1 on the set $z_1^2 = {}_2|\, z\,|_n^2 \neq 0$, and with the first $2j-1$ derivatives with respect to z_1 on $z_1^2 = {}_2|\, z\,|_n^2 = 0$. f is an entire function of n complex variables, so that each $f_{jk}(z_2, \cdots, z_n)$ is uniquely defined for any complex $z_2, \cdots, z_n$. It follows from a known interpolation formula [2, Section 3.1, formula (5), p. 50] that

$$\begin{aligned} f_j(z) &= f_j(z_1, \cdots, z_n) \\ &= \frac{1}{2\pi i} \int_{\Gamma(z_2, \cdots, z_n)} \frac{[(\zeta^2 - {}_2|\, z\,|_n^2)^j - (\|\, z\,\|^2)^j] f(\zeta, z_2, \cdots, z_n)\, d\zeta}{(\zeta^2 - {}_2|\, z\,|_n^2)^j (\zeta - z_1)} \end{aligned}$$

where $\Gamma(z_2, \cdots, z_n)$ is any rectifiable simple closed curve in the complex plane with both points $\pm ({}_2|\, z\,|_n^2)^{1/2}$ in its interior. For small changes of $z_2, \cdots, z_n$, the curve Γ need not change, so that each f_{jk} is locally analytic and hence an entire function of $n-1$ complex variables.

To find the exponential type of the f_{jk}, let $R(z_2, \cdots, z_n)$ be a rectangle with sides 2 and $2 + 2\,|\, {}_2|\, z\,|_n^2\,|^{1/2}$ containing the points $\pm ({}_2|\, z\,|_n^2)^{1/2}$ and at distance at least 1 from both points. $R(z_2, \cdots, z_n)$ will clearly serve as $\Gamma(z_2, \cdots, z_n)$.

Since $f \in \tilde{G}_\beta$, there is a $K > 0$ such that

$$|\, f(z_1, \cdots, z_n)\,| \leqq K \exp(2\pi\beta \max(|\, z_1\,|, (|\, z_2\,|^2 + \cdots + |\, z_n\,|^2)^{1/2}))$$

for any $z_1, \cdots, z_n$. At each point ζ on any $R(z_2, \cdots, z_n)$,

$$|\, \zeta\,| \leqq 2 + (|\, z_2\,|^2 + \cdots + |\, z_n\,|^2)^{1/2},$$

so for some $L > 0$

$$|\, f(\zeta, z_2, \cdots, z_n)\,| \leqq L \exp[2\pi\beta(|\, z_2\,|^2 + \cdots + |\, z_n\,|^2)^{1/2}]$$

for any $z_2, \cdots, z_n$ and ζ on $R(z_2, \cdots, z_n)$.

Thus, since $(\zeta^2 - {}_2|\, z\,|_n^2)^j - (\|\, z\,\|^2)^j$ is divisible by $\zeta - z_1$ and

$$|\, \zeta - ({}_2|\, z\,|_n^2)^{1/2}\,|\,|\, \zeta + ({}_2|\, z\,|_n^2)^{1/2}\,| \geqq 1$$

for ζ on $R(z_2, \cdots, z_n)$, we find, after collecting terms in z_1^k and allowing for the length of $R(z_2, \cdots, z_n)$, that f_{jk} belongs to $\tilde{F}_{\beta+\delta}$ by the lemma for every $\delta > 0$ and $k = 0, \cdots, 2j - 1$, so $f_j \in \tilde{F}_\beta$.

Let us now show that the f_j converge to f in Q as $j \to \infty$. It will be convenient first of all to prove that for any $M \geqq 1$, $f_j(z_1, \cdots, z_n)$ converges uniformly to $f(z_1, \cdots, z_n)$ on the set V_M of all $z = \langle z_1, \cdots, z_n \rangle$ satisfying the following four conditions:

$$\left| \| z \|^2 \right| \leqq M; \qquad |\operatorname{Im} z_r| \leqq \sqrt{M}, \quad r = 1, \cdots, n;$$
$$\operatorname{Re}({}_2|z|_n^2) \geqq -M; \quad \text{and} \quad |\operatorname{Im}({}_2|z|_n^2)| \leqq M.$$

Note that for $\langle z_1, \cdots, z_n \rangle \in V_M$,

$$(f - f_j)(z_1, \cdots, z_n) = \frac{1}{2\pi i} \int_{S(z_2, \cdots, z_n)} \frac{(\| z \|^2)^j f(\zeta, z_2, \cdots, z_n)\, d\zeta}{(\zeta^2 - {}_2|z|_n^2)^j (\zeta - z_1)}$$

where $S(z_2, \cdots, z_n)$ is a rectangle with sides $4\sqrt{M}$ and $4\sqrt{M} + 2\,|{}_2|z|_n^2|^{1/2}$, containing the two points $\pm ({}_2|z|_n^2)^{1/2}$ and at distance at least $2\sqrt{M}$ from both points, since for ζ on or outside $S(z_2, \cdots, z_n)$,

$$|\zeta^2 - {}_2|z|_n^2| = |\zeta - ({}_2|z|_n^2)^{1/2}|\,|\zeta + ({}_2|z|_n^2)^{1/2}|$$
$$\geqq 2\sqrt{M} \max(2\sqrt{M}, |{}_2|z|_n^2|^{1/2}) \geqq 4M,$$

so that, for one thing, z_1 lies inside $S(z_2, \cdots, z_n)$ (which is necessary for the validity of the integral formula for $f - f_j$, although not for the previous formula for f_j). Also, for ζ on $S(z_2, \cdots, z_n)$ with $\langle z_1, \cdots, z_n \rangle \in V_M$ for some z_1,

$$|f(\zeta, z_2, \cdots, z_n)| \leqq H \exp(2\pi\beta(n + 4)\sqrt{M})$$

where H is the L_1 norm of the function whose Fourier transform is f, since for $\langle t_1, \cdots, t_n \rangle \in C_\beta$ we have

$$|\exp(2\pi i(\zeta t_1 + \textstyle\sum_{r=2}^n t_j z_j))| \leqq \exp(2\pi\beta(|\operatorname{Im} \zeta| + (n - 1)\sqrt{M}),$$

and $|\operatorname{Im} \zeta| \leqq 5\sqrt{M}$ since $\operatorname{Re}({}_2|z|_n^2) \geqq -M$ and $|\operatorname{Im}({}_2|z|_n^2)| \leqq M$ imply $|\operatorname{Im}({}_2|z|_n^2)^{1/2}| < 3\sqrt{M}$.

Furthermore, we have

$$|\zeta - z_1| \geqq |\zeta \mp ({}_2|z|_n^2)^{1/2}| - |\pm({}_2|z|_n^2)^{1/2} - z_1| \geqq 2\sqrt{M} - \sqrt{M} = \sqrt{M}.$$

Hence

$$|(f - f_j)(z_1, \cdots, z_n)|$$
$$\leqq \frac{M^j(16\sqrt{M} + 4\,|{}_2|z|_n^2|^{1/2}) H \exp(2\pi\beta(n + 4)\sqrt{M}}{2\pi[2\sqrt{M} \max(2\sqrt{M}, |{}_2|z|_n^2|^{1/2})]^j \sqrt{M}}$$
$$\leqq 16\sqrt{M}\, H \exp(2\pi\beta(n + 4)\sqrt{M})/4^j,$$

for any $\langle z_1, \cdots, z_n \rangle \in V_M$.

Thus for any $f \in \tilde{G}_\beta$ we have defined a specific sequence $\{f_j\}$ of members of $\tilde{F}_\beta$ converging to f uniformly on each set V_M. If P is a polynomial in $n - 1$ variables, $P(z_2, \cdots, z_n)f \in \tilde{G}_\beta$, and $[P(z_2, \cdots, z_n)f]_j = P(z_2, \cdots, z_n)f_j$. Hence these functions converge likewise on V_M as $j \to \infty$ to $P(z_2, \cdots, z_n)f$.

For any real $X = \langle x_1, \cdots, x_n\rangle \in V_M$ and $\alpha = 1, \cdots, n$ there is a circle $C_\alpha(x)$ in the complex plane of radius $\min(\sqrt{M/2n}, M/4n \mid x_\alpha \mid)$ centered at x_α such that if $z_r \in C_r(x_1, \cdots, x_n)$, $r = 1, \cdots, n$, then $\langle z_1, \cdots, z_n\rangle \in V_{2M}$. For z_α on $C_\alpha(x_1, \cdots, x_n)$ we have $| z_\alpha - x_\alpha | \leqq \sqrt{M/2n}$, and hence

$$0 \leqq x_\alpha^2 \leqq 4 \operatorname{Re}(z_\alpha^2) + 4M \quad \text{and} \quad | x_\alpha | \leqq 1 + 2M + 2 \operatorname{Re}(z_\alpha^2).$$

Since $\operatorname{Re} z_1^2 \leqq 2M + \operatorname{Re}({}_2| z |_n^2)$ everywhere in V_{2M}, it follows that for any $\alpha = 1, \cdots, n$,

$$| x_\alpha | \leqq 1 + 2 \operatorname{Re}({}_2| z |_n^2) + 2nM \leqq | 1 + 2nM + 2({}_2| z |_n^2) |,$$

so that

$$n \max(2/\sqrt{M}, 4 \mid x_\alpha \mid/M) \leqq n \mid 4 + 8nM + 8({}_2| z |_n^2) \mid$$

(recall that $M \geqq 1$). Since for any nonnegative integer s there is an $H_s > 0$ such that

$$| (4 + 8nM + 8({}_2| z |_n^2))n |^s \mid (f - f_j)(z_1, \cdots, z_n) \mid$$
$$\leqq 16\sqrt{(2M)}\, H_s \exp(2\pi\beta(n + 4)\sqrt{(2M)})/4^j$$

for any $\langle z_1, \cdots, z_n\rangle \in V_{2M}$, it follows using multiple Cauchy integrals over the circles C_α that for any nonnegative integer q and differential $D^p = \partial^{|p|}/\partial x_1^{p_1} \cdots \partial x_n^{p_n}$, $(1 + x_1^2 + \cdots + x_n^2)^q D^p(f - f_j)$ converges to 0 uniformly for $| \, \| x \|^2 \, | \leqq M$, taking $s = q + | p |$.

It also follows that for any $f \in \tilde{G}_\beta \subset Q$, $m > 0$, and $p = \langle p_1, \cdots, p_n\rangle$, there are constants A and B such that for $| \, \| x \|^2 \, | \geqq 1$,

$$| D^p(f - f_j)(x_1, \cdots, x_n) | \leqq A \exp(B| \, \| x \|^2 \, |^{1/2})/(1 + x_1^2 + \cdots + x_n^2)^m,$$

and a $C > 0$ such that for $| \, \| x \|^2 \, | < 1$,

$$| D^p(f - f_j)(x_1, \cdots, x_n) | \leqq C/(1 + x_1^2 + \cdots + x_n^2)^m.$$

Hence for some $D > 0$ (depending on f, p, and m)

$$| D^p(f - f_j)(x_1, \cdots, x_n) | \leqq D \exp(B| \, \| x \|^2 \, |^{1/2})/(1 + x_1^2 + \cdots + x_n^2)^m$$

for all $x_1, \cdots, x_n$, so that the required conditions of uniform boundedness are satisfied, and $f_j \to f$ in the pseudotopology of Q.

Now let $S(t_1, \cdots, t_n) = S(t)$ be any distribution with support in C_β whose Fourier transform $\tilde{S}(x) = \tilde{S}(x_1, \cdots, x_n)$ is a function belonging to Q. Let $S_k(t) = S((1 + 1/k)t)$, $k = 1, 2, \cdots$, so that $\tilde{S}_k(x) = \tilde{S}(kx/(k + 1))$. It is easily seen that $\lim_{k\to\infty} \tilde{S}_k = \tilde{S}$ in the pseudotopology of Q. Now let $\{h_m\}_{m=1}^\infty$ be a sequence of C^∞ functions with supports shrinking to $\{0\}$, converging to δ in the topology of $\mathfrak{D}'$. Then for any fixed k, $h_m * S_k \in G_\beta$ for m

large enough, so that $T(\tilde{h}_m \tilde{S}_k) = 0$. But $\tilde{h}_m \tilde{S}_k \to \tilde{S}_k$ in Q as $k \to \infty$ since for any $p = \langle p_1, \cdots, p_n \rangle$, $D^p(\tilde{h}_m \tilde{S}_k) \to D^p \tilde{S}_k$ uniformly on compact sets, and $D^q \tilde{h}_m$ is bounded uniformly in m for any $q = \langle q_1, \cdots, q_n \rangle$, so that $D^p(\tilde{h}_m \tilde{S}_k)$, after being expanded as a finite sum by Leibniz's rule, is seen to approach 0 at ∞ faster than $(1 + x_1^2 + \cdots + x_n^2)^{-r}$ for any $r > 0$ on each set $|\, \| x \|^2 \,| \leqq M$, uniformly in m. Thus $T(\tilde{S}_k) = 0$ for all k, so $T(\tilde{S}) = 0$, Q.E.D.

References

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