ON GENERALISED FREE PRODUCTS OF TORSION-FREE NILPOTENT GROUPS I

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1. Introduction

1.1. This note may be viewed as a continuation of the study of the generalised free products of two finitely generated nilpotent groups (see [1] and Joan Landman-Dyer [2]). Here we shall concern ourselves with the question of zero divisors in the group rings of such generalised free products.

1.2. We recall that a group G is *indicable* if G is either trivial or else has a homomorphism onto the infinite cyclic group. Hence, according to current terminology, G is *locally indicable* if every finitely generated subgroup of G is indicable (G. Higman [3]). In his important paper [3], G. Higman proved that the integral group ring of a locally indicable group has no zero divisors. Thus it is of interest to consider which groups are locally indicable. Until very recently the known locally indicable groups were those described in Higman's paper [3]. More specifically, in [3], Higman either proves or points out that ordered groups, poly-locally-indicable groups (see [4] for terminology) and ordinary free products of locally indicable groups are locally indicable. Latterly torsion-free groups with a single positive (i.e., no negative exponents occur in some) defining relation were added to the list [5] as well as every generalised free product of two locally indicable groups with a cyclic subgroup amalgamated (A. Karrass and D. Solitar [6]). Here, by mimicking part of the argument in [6], we shall prove the following

Theorem. *Every generalised free product of any two finitely generated torsion-free nilpotent groups is locally indicable.*

It is worth pointing out that although the proof of Theorem 1 is not difficult the class of groups considered is quite complicated. For example there exists a generalised free product of two finitely generated torsion-free nilpotent groups which contains a non-trivial subgroup which coincides with its derived group (see [2]). For another illustration of the complicated nature of such generalised free products see [7].

2. The proof of Theorem 1

2.1. The key step in the proof of Theorem 1 is the following simple

Lemma 1. *Let A $(\neq 1)$ be an indicable group, B a finitely generated torsion-free nilpotent group and let*

$$P = \{A * B; U\}$$

Received May 8, 1970.

be a generalised free product of A and B amalgamating the subgroup U. Then P is indicable.

Proof. It is convenient to divide the proof into two cases.

First, we consider the case where U is of finite index in B. Let θ be a homomorphism of A onto an infinite cyclic group C. We may think of C additively as a subgroup of the additive group Q of rational numbers. Let $m(B)$ be a Mal'cev completion of B (see e.g. [8, page 256] or [9, page 32]). Then $m(B)$ is also a Mal'cev completion of U. Let θ' be the restriction of θ to U. Then θ' can be extended to a homomorphism θ'' of $m(B)$ into Q (see e.g., [8, page 256] or [9, page 32]). Let $\tilde{\theta}$ be the restriction of θ'' to B. Then θ and $\tilde{\theta}$ agree on U and hence, by a characteristic property of generalised free products, they can simultaneously be extended to a homomorphism τ of P into Q. Now $P\tau$ is a non-trivial, finitely generated subgroup of Q (since $P\tau$ is generated by the images of A and B). So $P\tau$ is infinite cyclic as desired.

In order to complete the proof of Lemma 1 we have to consider the case where U is of infinite index in B. Now it is easy to prove that a subgroup of a finitely generated nilpotent group N is of finite index if and only if it is of finite index modulo the derived group of N. It follows that there is a homomorphism σ of B onto the infinite cyclic group C which contains U in its kernel. The mapping of A onto the identity subgroup of C and σ can simultaneously be extended to a homomorphism of P onto C. This then completes the proof of Lemma 1.

2.2. We are now in a position to prove Theorem 1. Thus let A and B be finitely generated torsion-free nilpotent groups and let

$$P = \{A * B;\, U\}$$

be a generalised free product of A and B amalgmating a subgroup U. Our objective is to prove that P is locally indicable. Thus let H ($\neq 1$) be a finitely generated subgroup of P. According to a theorem of A. Karrass and D. Solitar [6], H contains a normal subgroup S of a very special type with free factor group H/S. Thus if $H/S \neq 1$, H has an infinite cyclic factor group. If $S = H$, then its turns out that H is a so-called tree product of finitely many factors. Each such factor is isomorphic to a subgroup of either A or B and H is built up by a succession of generalised free products of two groups with a single amalgamation. This building process starts with one of these factors and consists of repeatedly forming a generalised free product of a group already constructed with one of the factors. So a repeated application of Lemma 1 shows that H has an infinite cyclic factor group. This completes the proof of Theorem 1.

2.3. We show now that not every subgroup of a generalised free product of two finitely generated torsion-free nilpotent groups is indicable. To this end let A be a free nilpotent group of class two on a and x and let B be a free nilpotent group of class three on b and y. Let

$$H = gp(a, x^{-1}ax), \qquad K = gp(b, [b, y^{-1}by]).$$

Then H and K are free abelian of rank two. Let P be the generalised free product of A and B amalgamating H with K according to the following prescription:

$$P = \{A * B; a = b, x^{-1}ax = [b, y^{-1} by]\}.$$

Therefore

$$a = x[a, y^{-1} ay]x^{-1}. \tag{1}$$

Let H be the normal subgroup of P generated by a. Then it is clear from (1) that H coincides with its derived group. Since $a \neq 1$, H is the desired subgroup of P which is not indicable.

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