LOCALIZING CW-COMPLEXES

BY

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In the Sullivan-Quillen proof of the Adams conjecture on the image of the J-homomorphism, and in Sullivan's work on BPL and F/PL, it has become necessary to systematically exclude p-primary information about CW-complexes for certain primes p. The method of doing this is clear for loop spaces and for suspensions. Sullivan has used "local cells and local spheres" to do this for arbitrary complexes. Sullivan's construction suffers from the fact that it is not functorial, but is defined only up to homotopy type.

We give a construction below, inspired by the paper [2] of F. P. Peterson and the construction of Eilenberg-MacLane complexes given by E. Spanier in his book [3]. This construction is functorial, and has all of the desirable properties of Sullivan's construction.

1. R-groups

If R is a subring of the rational numbers containing 1, an abelian group A is called an R-group if the map $A = A \otimes Z \to A \otimes R$ is an isomorphism. Since all subrings of the rational numbers are free of torsion, $\text{Tor}(A, R) = 0$. Thus the exact sequence

$$0 \to \text{Tor}(A, R/A) \to A \otimes Z \to A \otimes R \to A \otimes R/Z \to 0$$

shows that A is an R-group if and only if $\text{Tor}(A, R/Z) = 0$, $A \otimes R/Z = 0$.

Let $M = M(R) = \{m \mid m \text{ is an integer}, m \text{ a unit of } R\}$. Then $R = Z[M^{-1}]$, $M^{-1} = \{m^{-1} \mid m \in M\}$, so that M and R determine one another. Since M is countable, let $m_1, m_2, \cdots$ be an indexing of the elements of M. Let $R_i = Z$, $r_i : R_i \to R_{i+1}$ be multiplication by m_i. Then $R = \lim R_i$.

Proposition 1.1. *If A is an R-group,* $\text{Hom}(Z, A) = \text{Hom}(R, A)$, $\text{Hom}(R/Z, A) = 0$.

Proof. $\text{Hom}(R, A) = \text{lim inv}(\text{Hom}(R_i, A))$. If A is an R-group, it is an R-module, so that $m_i : A \to A$ is an isomorphism. Thus every map in the inverse system $\text{Hom}(R_i, A)$ is an isomorphism, so that $\text{Hom}(R, A) = \text{Hom}(R_1, A) = \text{Hom}(Z, A)$. Thus $\text{Hom}(R/Z, A) = 0$.

Proposition 1.2. *If A is an R-group,* $\text{Ext}(R, A) = 0 = \text{Ext}(R/Z, A)$.

Proof. From the fact that $\text{Hom}(R, A) \to \text{Hom}(Z, A)$ is surjective, and $\text{Ext}(Z, A) = 0$, we see that $\text{Ext}(R/Z, A) = \text{Ext}(R, A)$.

Let $F_0 \to F_1 \to R/Z$ be a free resolution of R/Z. Since R is an R-group,

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$F_0 \otimes R \to F_1 \otimes R$ is an isomorphism. Now there is an exact sequence

$$0 \to \text{Hom}\,(R/Z, A) \to \text{Hom}\,(F_1, A) \to \text{Hom}\,(F_0, A) \to \text{Ext}\,(R/Z, A).$$

Since $\text{Hom}\,(F_i, A) = \text{Hom}\,(F_i, \text{Hom}\,(R, A)) = \text{Hom}\,(F_i \otimes R, A)$, and since

$$\text{Hom}\,(F_1 \otimes R, A) \to \text{Hom}\,(F_0 \otimes R, A)$$

is an isomorphism, $\text{Ext}\,(R/Z, A) = 0$.

If m is an integer, $Z \to Z \to Z/m$ is a free resolution of Z/m. Since for any group A, $\text{Hom}\,(Z, A) = A = A \otimes Z$, we see that

$$\text{Hom}\,(Z/m, A) \cong \text{Tor}\,(Z/m, A), \qquad \text{Ext}\,(Z/m, A) \cong Z/m \otimes A.$$

If $n_i = m_1 m_2 \cdots m_i$, $R/Z = \lim Z/n_i$. Thus

$$\lim \text{Tor}\,(Z/n_i, A) = \text{Tor}\,(R/Z, A), \lim\,(Z/n_i \otimes A) = R/Z \otimes A.$$

Proposition 1.3. *A is an R-group if either* $\text{Tor}\,(Z/n_i, A) = 0$, $Z/n_i \otimes A = 0$, *all i or* $\text{Hom}\,(Z/n_i, A) = 0$, $\text{Ext}\,(Z/n_i, A) = 0$, *all i.*

Proposition 1.4. *If* $\text{Hom}\,(R/Z, A) = 0$, $\text{Ext}\,(R/Z, A) = 0$, *A is an R-group.*

Proof. There is an exact sequence, for $m \in M$, of the form

$$0 \to Z/m \to R/Z \to R/Z \to 0.$$

This gives an exact sequence

$$0 \to \text{Hom}\,(R/Z, A) \to \text{Hom}\,(R/Z, A) \to \text{Hom}\,(Z/m, A)$$
$$\to \text{Ext}\,(R/Z, A) \to \text{Ext}\,(R/Z, A) \to \text{Ext}\,(Z/m, A) \to 0$$

Theorem 1.5. *A is an R-group if and only if any of the following equivalent conditions hold:*

(1) $R/Z \otimes A = 0$, $\text{Tor}\,(R/Z, A) = 0$
(2) $Z/m \otimes A = 0$, $\text{Tor}\,(Z/m, A) = 0$, *all* $m \in M$
(3) $\text{Ext}\,(R/Z, A) = 0$, $\text{Hom}\,(R/Z, A) = 0$
(4) $\text{Ext}\,(Z/m, A) = 0$, $\text{Hom}\,(Z/m, A) = 0$, *all* $m \in M$.
(5) $\text{Hom}\,(R, A) \to \text{Hom}\,(Z, A)$ *is an isomorphism and* $\text{Ext}\,(R, A) = 0$.
(6) $\text{Ext}\,(Z/p, A) = 0$, $\text{Hom}\,(Z/p, A) = 0$, *all primes* $p \in M$.

Proposition 1.6. *If a group is isomorphic to a subgroup of an R-group and is isomorphic to a quotient group of an R-group, it is an R-group.*

Proof. If A is isomorphic to a subgroup of an R group, $\text{Tor}\,(R/Z, A) = 0$. If A is isomorphic to a factor group of an R-group, $R/Z \otimes A = 0$.

Proposition 1.7. *If* $0 \to A \to B \to C \to 0$ *is a short exact sequence of groups, if any two groups are R-groups, so is the third.*

Proof. Look at the exact sequence for $R/Z \otimes$ and Tor $(R/Z, \ )$ for the three groups.

THEOREM 1.8. *The homology groups of a complex of R-groups are again R-groups.*

Proof. By 1.6, the boundary groups are R-groups. By 1.7, this implies that the cycle groups and thus the homology groups are R-groups.

PROPOSITION 1.9. *If A is a group, $R \otimes A = 0$ if $Q \otimes A = 0$,* Tor $(Q/R, A) = 0$.

The group Q/R is isomorphic to $\bar{R}/Z$, where $\bar{R}$ is the ring generated by those t^{-1} with $t \in Z$, $t^{-1} \notin R$. Notice that $R \cap \bar{R} = Z$, $R \otimes \bar{R} = Q$, and that these two properties characterize $\bar{R}$ as a subring of Q.

THEOREM 1.10. *If $\alpha: A \to B$ is such that $Q \otimes \alpha: Q \otimes A \to Q \otimes B$ is an isomorphism, and the kernel and cokernel of α are $\bar{R}$-groups, then*

$$R \otimes \alpha: R \otimes A \to R \otimes B$$

is an isomorphism.

Proof. $R \otimes A \to R \otimes B$ will be an isomorphism if and only if

$$R \otimes \ker(\alpha) = 0 \quad \text{and} \quad R \otimes \operatorname{coker}(\alpha) = 0,$$

since R is torsion free.

If $Q \otimes \ker(\alpha) = 0$, $R \otimes \ker(\alpha)$ will vanish if Tor $(Q/R, \ker(\alpha) = 0$. However,

$$\operatorname{Tor}(Q/R, \ker(\alpha)) = \operatorname{Tor}(\bar{R}/Z, \ker(\alpha)).$$

Thus, if $\ker(\alpha)$ is an $\bar{R}$-group, Tor $(Q/R, \ker(\alpha)) = 0$.

2. Homotopy theory with coefficients

In [2], Peterson defined cohomotopy groups with coefficients, and exposed the main properties of these groups. Our discussion here is analogous to his, with slight modifications to suit R-groups.

If A is an abelian group, $F_0 \to F_1 \to A$ a free resolution of A, define a space $M(A, 1)$ as follows. Let $M(F_i, 1)$ be a bouquet of circles, with one circle for each generator of F_i. Let $M(F_0, 1) \to M(F_1, 1)$ realise $F_0 \to F_1$ in homology. Let $M(A; 1)$ be the mapping cone of this map. As a complex, $M(A, 1)$ depends upon the resolution of A, but Peterson showed that its staple homotopy type is uniquely determined.

If X is a space, we let $\pi_n(X; A) = [\Sigma^{n-1}M(A, 1), X]$ for $n \geq 2$. Since $[M(F_i, 1), X] = \operatorname{Hom}(F_i, \pi_1(X))$, and more generally,

$$[\Sigma^{n-1}M(F_i, 1), X] = \operatorname{Hom}(F_i, \pi_n(X)),$$

we see that there is an exact sequence

$$\cdots \to \pi_{n+1}(X; F_0) \to \pi_n(X; A) \to \pi_n(X; F_1) \to \pi_n(X_1 F_0) \to \cdots$$

which gives us short exact sequences

$$0 \to \text{Ext}\,(A, \pi_{n+1}(X)) \to \pi_n(X; A) \to \text{Hom}\,(A, \pi_n(X)) \to 0$$

THEOREM 2.1. *If $\pi_1(X)$ is abelian and* Ext $(R/Z, \pi_1(X)) = 0$, *$\pi_*(X)$ is an R-group if and only if $\pi_*(X; R/Z) = 0$.*

If $\pi_1(X)$ is abelian, and $Z/m \otimes \pi_1(X) = 0$, all $m \in M$, $\pi_*(X)$ is an R-group if and only if $\pi_*(X, Z/m) = 0$, all $m \in M$.

We observe that if $m \in M$, m can be written uniquely as a product of powers of primes, all of which are in M. Thus $m \in M$ above could be replaced by $p \in M$, p prime.

The space $M(R/Z, 1)$ is necessarily infinite, so that it is not compact. However, if we choose the proper resolution, $M(Z/m, 1)$ will be compact. Thus, if X is the union of an increasing sequence of subcomplexes X_i, $\pi_*(X; Z/m) = \lim \pi_*(X_i; Z/m)$.

If X is a complex, let $R^1(X)$ be the complex obtained from X by attaching to X a cone on each map into X of any $\Sigma^k M(Z/m, 1)$ for $m \in M$. Then X is a subcomplex of $R^1(X)$, and contains the 0- and 1-skeleton of $R^1(X)$. Thus $\pi_1(X) \to \pi_1(R^1(X))$ is onto. It is not difficult to see that the kernel consists of the element of order m for some $m \in M$.

Thus $\pi_1(R^1(X)) = \pi_1(X)/\text{Tor}\,(R/Z, \pi_1(X))$. Notice that

$$\pi_n(X; Z/m) \to \pi_n(R^1(X); Z/m)$$

is the zero map all $m \in M$, $n \geq 1$. This gives us the following result.

THEOREM 2.2. *If $R^{i+1}(X) = R^1(R^i(X))$, and $R(X)$ is the union of the $R^i(X)$,*

$$\pi_1(R(X)) = \pi_1(X)/\text{Tor}\,(R/Z, \pi_1(X))$$

$$\pi_n(R(X); Z/m) = 0 \quad \textit{for } m \in M,\ n \geq 1.$$

COROLLARY 2.3. *$\pi_n(R(X))$ is an R-group for $n > 1$. If $\pi_1(X)$ is abelian and $R/Z \otimes \pi_1(X) = 0$, $\pi_1(R(X))$ is an R-group.*

THEOREM 2.4. *$H_*(X; R) \to H_*(R(X); R)$ is an isomorphism.*

Proof. $R^1(X)/X$ is a one point union of spaces $\Sigma^k M(Z/m, 1)$ for $m \in M$. Since $Z/m \otimes R = 0$, Tor $(Z/m, R) = 0$,

$$H_*(X; R) \to H_*(R^1(X); R)$$

is an isomorphism. Singular homology takes increasing unions of CW-complexes into direct limits of the singular homology groups of the individual complexes, since the n-simplex is compact for all n.

PROPOSITION 2.5. *If $X_1 \subset X_2 \subset \cdots$ is an increasing sequence of CW* *complexes,* $X = UX_i$, *then if for some space* Y,

$$[X_{i+1}, Y] \to [X_i, Y]$$

is surjective for all i, $[X, Y] \to [X_i, Y]$ *is surjective for all* i.

Proof. Since $[X_{n+1}, Y] \to [X_n, Y]$ is surjective for every n, given f_i ; $X_i \to Y$ it follows from the homotopy extension property of the pair (X_{n+1}, X_n) that there is a sequence of maps $f_n : X_n \to Y$ for $n \geq i$ such that $f_{n+1} \mid X_n = f_n$. Then, defining $f : X \to Y$ so that $f \mid X_n = f_n$ for $n \geq i$, yields a map whose restriction to X_i is f_i. Since X has the weak topology, this map is continuous. Thus $[X, Y] \to [X_i, Y]$ is surjective.

Suppose that $f : X \to Y$ as above, and each $f \mid X_i$ is homotopic to the constant (basepoint) map. Because pairs of CW-complexes have the homotopy extension property, any homotopy of f to a constant on X_i can be extended to a homotopy from f to a map factoring through X_{i+1}/X_i. Thus, if every map X_{i+1}/X_i into Y is null homotopic, we can compose this homotopy with a second homotopy, which is constant at the end.

Changing parameters so that the null homotopy of $f \mid X_i$ is constant from $1\text{-}\frac{1}{2}^i$ to 1, we can extend this to a null homotopy of $f \mid X_{i+1}$ which is constant from $1\frac{1}{2}^{i+1}$ to 1.

PROPOSITION 2.6. *If* $[X_{i+1}/X_i, Y]$ *consists of only the homotopy class of the basepoint map for all* i, $[X, Y] \to [X_i, Y]$ *is injective for all* i.

THEOREM 2.7. *If* $\pi_n(Y)$ *is an R-group for* $n > 1$, $\text{Tor}\,(R/Z, \pi_1(Y)) = 0$, *then for any* X,

$$[R(X), Y] \to [X, Y]$$

is bijective.

Proof. Because $\pi_n(Y; Z/m) = 0$ for $m \in M$, $n \geq 1$, we have

$$[R^i(X), Y] \to [R^{i-1}(X), Y]$$

a bijection for all i. Thus the map in the theorem is surjective by 2.5. By 2.6, $[R^i(X)/R^{i-1}(X), Y] = 0$ implies that the map is one to one.

3. The relationship between H_* and Π_*

If A is an abelian group, p a prime, $H_*(K(A, 1); Z/p)$ can be described in terms of $A \otimes Z/p$ and $\text{Tor}\,(A, Z/p)$. Thus, if $p \in M$, and A is an R-group, $\tilde{H}_*(K(A, 1); Z/p) = 0$.

From this, either the comparison theorem for the Serre spectral sequence, or the direct calculation of the groups $H_*(K(A, n); Z/p)$ in the Cartan seminar [1] shows that $\tilde{H}_*(K(A, n); Z/p) = 0$ for A an R-group.

PROPOSITION 3.1. $\tilde{H}_*(K(A, n); Z)$ *is an R-group if and only if* A *is an R-group.*

Proof. $\tilde{H}_n(K(A, n); Z) = A$, so the only if part is true. The other implication is shown above.

PROPOSITION 3.2. *If $F \to E \to B$ is a fibration with $\pi_1(B)$ acting simply on $H_*(F; Z/p)$ for $p \in M$, p prime, then if two of the three $\tilde{H}_*(F; Z)$, $\tilde{H}_*(E; Z)$, $\tilde{H}_*(B; Z)$ are R-groups so is the third.*

Proof. For any space X, $\tilde{H}_*(X; Z)$ is an R-group if and only if $\tilde{H}_*(X; Z/p) = 0$, all primes $p \in M$. The proposition follows from the comparison theorem for the Serre spectral sequence.

Combining these results, we obtain the following.

PROPOSITION 3.3. *If X has a finite number of non-vanishing homotopy groups, $\pi_1(X) = 0$, then $\pi_*(X)$ is an R-group if and only if $\tilde{H}_*(X; Z)$ is.*

This result allows us to prove the "mod R" Hurewicz theorem. Since R-groups do not form a Serre class, this is not a "mod C" theorem.

THEOREM 3.4. *If $\pi_1(X) = 0$, $\pi_i(X)$ is an R-group for $i \leq n$ if and only if $\tilde{H}_i(X; Z)$ is an R-group for $i \leq n$.*

Proof. Let $X_{(n)}$ be obtained from X by killing off all homotopy groups above dimension n, so that $X_{(n)}$ has only a finite number of homotopy groups.

Now $\tilde{H}_i(X; Z) = \tilde{H}_i(X_{(n)}; Z) = \tilde{H}_i(X_{(n)}; R) = \tilde{H}_i(X; R)$ for $i \leq n$ if the homotopy groups of $X_{(n)}$ are all R-groups. If the homotopy groups of $X_{(n)}$ are not all R-groups,

$$\tilde{H}_i(X, Z) = \tilde{H}_i(X_{(n)}; Z) \neq \tilde{H}_i(X_{(n)}; R) = \tilde{H}_i(X; R)$$

for some $i \leq n$, so $\tilde{H}_i(X; Z)$ is not an R-group.

If $\pi_1(X) \neq 0$, we will say that X is R-simple if $\pi_1(X)$ is abelian, $\pi_1(X)$ is an R-group, and $\pi_1(X)$ acts trivially on the mod p homology for $p \in M$, of the fiber of $X \to X_{(1)}$. This is probably the same as acting trivially on the mod p homotopy of X. Our techniques so far would allow us to replace "$\pi_1(X) = 0$" in 3.4 by "X is R-simple".

COROLLARY 3.5. *If $\pi_1(X)$ is R-simple, $\tilde{H}_*(R(X); Z)$ is an R-group, and $\tilde{H}_*(X; R) = \tilde{H}_*(R(X); Z)$.*

COROLLARY 3.6. *If $\pi_*(Y)$ is an R-group,*

$$\pi_i(Y) = [R(S^i), Y]$$

Proof. Apply 2.7, 2.2, and 3.5.

4. Properties of R

Below is a listing of the elementary properties of R. Proofs are left to the reader.

It is clear that any map $f : X \to Y$ induces a map $R(f) : R(X) \to R(Y)$. Further, $R(fg) = R(f)\, R(g)$.

If f is an inclusion, so is $R(f)$. We have a map $R(Y/X) \to R(Y)/R(X)$ which clearly induces an isomorphism on homotopy.

If f is any map, $C(f)$ the mapping cone of f, there is an inclusion

$$C(f) \to C(R(f)).$$

This induces $R(C(f)) \to R(C(R(f)))$. However, clearly $C(R(f)) \to R(C(R(f)))$ induces an isomorphism on homotopy, so that there is a map $R(C(f)) \to C(R(f))$, defined up to homotopy, which is a homotopy equivalence. In particular, we have a map $R(\Sigma(f)) \to \Sigma R(f)$, where Σ denotes suspension, defined up to homotopy, which is a homotopy equivalence.

Given two spaces, X, Y, $R^1(X) \times R^1(Y)$ contains $R^1(X \times Y)$. Thus $R(X \times Y)$ is a subcomplex of $R(X) \times R(Y)$. The inclusion map clearly is an isomorphism on homotopy groups, and thus is a homotopy equivalence.

Dually, there is an inclusion of $R(X) \vee R(Y)$ in $R(X \vee Y)$, which induces an isomorphism on homology groups.

If $f \simeq g$, $R(f) \simeq R(g)$, and the homotopy is determined by the homotopy from f to g in a functorial way.

Given a Serre fibration,

$$F \to E \xrightarrow{\pi} B,$$

we have an inclusion in the quasi-fibring $R(F) \to R(E) \to R(B)$. $R(F)$ is homotopy equivalent to the fiber of $R(\pi)$.

Since X is filtered by its n-skeletons X^n, $R(X)$ is filtered by the $R(X^n)$, and $R(X^{n+1})/R(X^n)$ is homotopy equivalent to $R(X^{n+1}/X^n)$. Thus it is, up to homotopy equivalence, a bouquet of $R(S^{n+1})$'s. Thus $R(X)$ has an "R cell decomposition".

This observation can be made stronger. Since X^{n+1} is the mapping cone of a map of a disjoint union of n-spheres into X^n, thus $R(X^{n+1})$ is, up to a homotopy equivalence which is itself determined up to homotopy, the mapping cone of a disjoint union of $R(S^n)$'s into $R(X^n)$.

If X is an H-space, $R(X)$ is also an H-space, with multiplication determined up to homotopy. If X has a classifying space Y, so that X is homotopy equivalent to ΩY as an H-space, then $R(X)$ is homotopy equivalent to $\Omega(RY)$ as an H-space if $\pi_1(X) = 0$, so that $R(X)$ also has a classifying space if $\pi_1(X) = 0$.

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