LOCAL BOUNDARY BEHAVIOR OF HARMONIC FUNCTIONS

BY
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1. Introduction

The solution to the Dirichlet poblem on the unit disk, that is, the problem of finding a harmonic function $f(r, \theta)$ in the interior of the disk corresponding to a given function $f(\theta)$ on the boundary, has become part of the folk knowledge of mathematics. It is common knowledge also that $\lim_{r\to 1} f(r, \theta) = f(\theta)$ at each point of continuity of f. The solution to the converse problem, that of finding a boundary function (or some generalization of function) corresponding to a given harmonic function in the interior, is not so well known, but nevertheless has been extensively studied in the last decade. A solution always exists in the space of hyperfunctions H' on the boundary. In fact, these hyperfunctions are exactly the objects giving a solution to the converse problem. Moreover, the original Dirichlet problem has a unique solution when f is a hyperfunction instead of a point function. However, the statement about limits at points of continuity has no meaning for f in H'. It is the purpose of this report to give it meaning and to prove this theorem for f in H'.

Hyperfunctions have been characterized in a number of different ways. Two of them are as equivalence classes of pairs of holomorphic functions and as continuous linear functionals on a space of holomorphic test functions. See e.g. Sato [1], Köthe [2], [3], Lions and Magenes [4], and Schapira [5]. The former characterization enables one to consider them as types of generalized boundary values of harmonic functions and the latter as generalized functions in the sense of Gelfand-Shilov [6]. On the boundary Γ of the unit disk hyperfunctions correspond to exponential trigonometric series $\sum C_n e^{in\theta}$ whose coefficients satisfy

$$\limsup | C_n |^{1/|n|} \leq 1.$$

Thus the space H' contains all distributions on Γ (whose coefficients satisfy $C_n = O(| n |^p))$ and is contained in the space Z' of ultradistributions (since every trigonometric series, no matter what its coefficients are, converges in Z'). See [11].

In the case of distributions, there is an already available concept which corresponds to continuity at a point of a continuous function. It is the concept of point value introduced by Lojasiewicz [9]. A recent characterization of the elements of H' by Johnson [7] as series of distributions allows this concept to be extended in a natural way to hyperfunctions. It is then possible

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to ask whether $f(r, \theta)$ converges to $f(\theta)$ at points at which f has a value. We shall answer this by using Johnson's characterization:

To each hyperfunction $f \in H'$, there corresponds a sequence $\{g_n\}$ of functions continuous on Γ which satisfy the condition

$$(1.1) \qquad \lim_{n\to\infty} (n! \, \| g_n \|_\infty)^{1/n} = 0$$

and such that the series of distributions $\sum g_n^{(n)}$ converges to f in the sense of H'. Any such series whose terms satisfy (1.1) *converges in H' to some hyperfunction.*

We shall also need certain non standard properties of the Poisson kernel. These are derived in the Appendix.

Unfortunately there doesn't seem to be a universally used notation for the space of hyperfunctions. Since we use Johnson's characterization we shall also use his notation. In particular we will continue to denote by H' the space of hyperfunctions on the boundary of the unit disk. This is consistent with Schapira [5] since the only hyperfunctions on Γ are analytic functionals.

2. Point Values

Lojasiewicz [9] has shown the following definitions of value of a distribution at a point to be equivalent:

Let f be a distribution on $\mathbf{R}^1$. Then f has a value at x_0 if and only if either

(i) $\lim_{\lambda\to 0} f(\lambda x + x_0)$ *exists in the sense of distributions, or*

(ii) *for each continuous function g such that $f = g^{(n)}$ locally, there exists a polynomial P of degree $< n$ such that the point limit*

$$\lim_{x\to x_0} n! \left(\frac{g(x) - P(x)}{(x - x_0)^n} \right)$$

exists. The value γ at x_0 of f is the common value of these two limits.

Of course this definition refers to distributions on $\mathbf{R}^1$ and we are interested in the boundary Γ of the unit disk. However it is well known (and easy to show) that the distributions on Γ are algebraically and topologically isomorphic to periodic distributions on $\mathbf{R}^1$. As is customary we shall make no distinction between the two.

Johnson's characterization of a hyperfunction as a series of distributions enables us, in a natural way, to extend the definition of value at a point to hyperfunctions.

Definition. The hyperfunction f on Γ has a value at x_0 if there exists a representation $\sum g_n^{(n)}$ of f satisfying (1.1), a sequence of polynomials $\{P_n\}$ with P_n of degree $<n$, and a sequence of complex numbers $\{\gamma_n\}$, such that for each $\varepsilon > 0$, there exists a δ such that

$$\left| \frac{g_n(x) - P_n(x)}{(x - x_0)^n} - \frac{\gamma_n}{n!} \right| \leq \frac{\varepsilon^{n+1}}{n!} \quad \text{for} \quad 0 < |x - x_0| < \delta, n = 1, 2, \cdots,$$

and $g_0(x_0) = \gamma_0$. The value of f at x_0 is given by $\gamma = \sum_{n=0}^{\infty} \gamma_n$.

Briefly, this definition says that each distribution $g_n^{(n)}$ has a value γ_n at x_0 under Lojasiewicz's definition (ii) and that the limits in this definition converge to γ_n faster for larger n.

In order to assure that this definition makes sense we must check that the series $\sum \gamma_n$ converges and that the limit is independent of the representation $\sum g_n^{(n)}$ of f.

PROPOSITION 1. *Let $\{\gamma_n\}$ be the sequence of complex numbers in the definition. Then $\sum \gamma_n$ converges.*

Since γ_n is the value of the distribution $g_n^{(n)}$ at x_0, it follows from Lojasiewicz's definition (i) that $\lim_{\lambda\to 0} g_n^{(n)}(\lambda x + x_0) = \gamma_n$ in the sense of distributions, i.e. that $\lim_{\lambda\to 0} \langle \varphi, g_n^{(n)}(\lambda x + x_0)\rangle \to \bar{\gamma}_n\langle\varphi, 1\rangle$ for each $\varphi \in \mathfrak{D}(\mathbf{R}^1)$. Since g_n is periodic and hence bounded, $g_n^{(n)}$ is in S' (the space of tempered distributions) and since $\mathfrak{D}(\mathbf{R}^1)$ is dense in S, we have this convergence holding for each $\varphi \in S$ as well. In particular it holds for $\varphi(x) = e(-x^2/2)$, for which we get

$$| \langle \varphi, g_n^{(n)}(\lambda x + x_0)\rangle | = | \langle \varphi^{(n)}, \lambda^n g_n(\lambda x + x_0)\rangle | \leq | \lambda |^n \| \varphi^{(n)} \|_1 \| g_n \|_\infty .$$

Now $\| \varphi^{(n)} \|_1 \leq Cn!$ where C is a constant, whence by using (1.1) we see that corresponding to each sequence $\{\varepsilon_n\}$ of positive numbers and $\varepsilon > 0$, there is a sequence $\{\lambda_n\}$, $| \lambda_n | \leq 1$, such that

$$| \gamma_n | \leq K\varepsilon_n + | \lambda_n |^n(1/n!)\varepsilon^n B(\varepsilon)n!CK$$

where $K = | \langle \varphi, 1\rangle |^{-1}$. Therefore if for example, $\varepsilon_n = 1/n^2$, the series $\sum \gamma_n$ easily converges absolutely.

PROPOSITION 2. *Let f have two representations $\sum g_n^{(n)}$ and $\sum \tilde{g}_n^{(n)}$ both of which satisfy the conditions of the definition. Then the value of f calculated with either is the same.*

Since both representation converge to f the difference of the two, say $\sum h_n^{(n)}$ converges to 0 in H'. Since the test function space H, composed of holomorphic functions on Γ, is dense in $\mathfrak{D}(\Gamma)$, $\sum h_n^{(n)}$ converges to 0 in $\mathfrak{D}'(\Gamma)$ as well. Also each $h_n^{(n)}$ has a value, say γ_n', at x_0 and hence

$$h_n^{(n)}(\lambda x + x_0) \to \gamma_n'$$

in the sense of $\mathfrak{D}'(\mathbf{R}^1)$ as $\lambda \to 0$. Since $\sum h_n^{(n)}$ converges to 0 in $\mathfrak{D}'(\Gamma)$, there exists an m such that $h_n^{(n-m)}$ is a continuous function and $\sum h_n^{(n-m)}(x)$ converges to 0 uniformly (see [8, p. 87]). Thus $\sum h_n^{(n-m)}(\lambda x + x_0)$ converges to 0 uniformly for all x and λ and

$$\sum \langle \varphi^{(m)}, h_n^{(n-m)}(\lambda x + x_0)\rangle$$

converges to 0 uniformly in λ for any $\varphi \in \mathfrak{D}(\mathbf{R}^1)$. Hence

$$\sum_n \langle \varphi, h_n^{(n)}(\lambda x + x_0)\rangle$$

converges to 0 uniformly in λ. Corresponding to any $\varepsilon > 0$, one can choose an N first and then a λ such that

$$\begin{aligned}
| \textstyle\sum_{n=0}^{N} \gamma_n' | \, | \langle \varphi, 1 \rangle | \\
&= | \textstyle\sum_{n=0}^{N} \langle \varphi, \gamma_n' \rangle | \\
&\leq \textstyle\sum_{n=0}^{N} | \langle \varphi, h_n^{(n)}(\lambda x + x_0) - \gamma_n' \rangle | + | \textstyle\sum_{n=0}^{N} \langle \varphi, h_n^{(n)}(\lambda x + x_0) \rangle | \\
&< \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}$$

From this it follows that $\sum_{n=0}^{\infty} \gamma_n' = 0$, and hence that the value of f at x_0 is well defined.

Now we are at the stage where we can derive some of the properties of these values. The first is given by

PROPOSITION 3. *Let f have value γ at x_0. Then*

$$\sum_{n=0}^{\infty} n! \, \frac{g_n(x) - P_n(x)}{(x - x_0)^n}$$

converges to a continuous function $F(x)$ in some deleted neighborhood of x_0 and $\lim_{x \to x_0} F(x) = \gamma$.

If $g_n^{(n)}$ has a value at x_0, so does $g_n^{(n-j)}$ $j = 1, 2, \cdots, n$ (see [9, p. 15]). Moreover, denoting by γ_n^j the value of $g_n^{(n-j)}$ at x_0, we can show that γ_n^j satisfies an inequality similar to the one above for γ_n. In fact we have

$$\begin{aligned}
| \gamma_n^j | &\leq \varepsilon_n + | \lambda_n^{(n-j)} | \, \| g_n \|_\infty \, \| \varphi^{(n-j)} \|_1 \\
&\leq K((n - j)!/n!) \, B(\varepsilon)\varepsilon^n.
\end{aligned} \tag{2.1}$$

Now the polynomial P_n must be given by

$$P_n(x) = \textstyle\sum_{j=0}^{n-1} \gamma_n^{n-j}(x - x_0)^j/j! \tag{2.2}$$

whence

$$| P_n(x) | \leq \frac{B(\varepsilon)\varepsilon^n K}{n!} \sum_{j=0}^{n-1} | x - x_0 |^j \leq \frac{B(\varepsilon)\varepsilon^n K}{(1 - | x - x_0 |)} \tag{2.3}$$

for $| x - x_0 | < 1$. Since g_n satisfies the same sort of inequality, the series defining $F(x)$ converges uniformly on

$$[x_0 - 1/2, x_0 - 2\delta] \cup [x_0 + 2\delta, x_0 + 1/2].$$

Since δ is arbitrary the function $F(x)$ is continuous on $[x_0 - 1/2, x_0 + 1/2] - [x_0]$. However, we see immediately that

$$| F(x) - \gamma | = \left| \sum_{n=0}^{\infty} n! \, \frac{g_n(x) - P_n(x)}{(x - x_0)^n} - \sum_{n=0}^{\infty} \gamma_n \right| < \sum_{n=0}^{\infty} \varepsilon^{n+1}$$

Finally, we have the relation between the value of f at x_0 and the radial limit of the harmonic function corresponding to f inside the unit disk. (See [10] for a similar result for distributions.)

THEOREM. *Let f have a value γ at x_0. Then*

$$\lim_{r\to 1} (P_r * f)(x_0) = \gamma.$$

Here P_r denotes the Poisson kernel as usual and the limit is the pointwise limit. We have

$$\begin{aligned}
2\pi(P_r * f)(x_0) &= \sum_{n=0}^{\infty} 2\pi(P_r * g_n^{(n)})(x_0) \\
&= \sum_{n=0}^{\infty} 2\pi(P_r^{(n)} * g_n)(x_0) \\
&= \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} P_r^{(n)}(x_0 - t) g_n(t)\, dt \\
&= \sum_{n=0}^{\infty} \left\{ \int_{-\pi}^{x_0-\delta} + \int_{x_0+\delta}^{\pi} \right\} P_n^{(n)}(x_0 - t) g_n(t)\, dt \qquad (2.4)\\
&\quad + \sum_{n=0}^{\infty} \int_{x_0-\delta}^{x_0+\delta} \frac{P_r^{(n)}(x_0 - t)(t - x_0)^n}{n!}\, n!\, \frac{g_n(t) - P_n(t)}{(t - x_0)^n}\, dt \\
&\quad + \sum_{n=0}^{\infty} \int_{x_0-\delta}^{x_0+\delta} P_r^{(n)}(x_0 - t) P_n(t)\, dt \\
&= \sum_{n=0}^{\infty} q_1(n, r, \delta) + \sum_{n=0}^{\infty} q_2(n, r, \delta) + \sum_{n=0}^{\infty} q_3(n, r, \delta).
\end{aligned}$$

We shall show that the first and last series approach zero as $r \to 1$.

Let us first examine q_3. After n integrations by parts the integral becomes

$$q_3(n, r, \delta) = \{P_r^{(n-1)}(x_0 - t)P_n(t) - \cdots \pm P_r(x_0 - t)P_n^{(n-1)}(t)\}\Big|_{x_0-\delta}^{x_0+\delta}$$

since P_n is a polynomial degree $< n$. Here we have assumed that x_0 is in the interior of $(-\pi, \pi)$. If it were not we would merely take a different interval of length 2π.

Now by differentiating (2.2) we find that

$$\begin{aligned}
P_n^{(k)}(x) &= \sum_{j=k}^{n-1} \gamma_n^{n-j} \frac{j!}{(j-k)!} \frac{(x - x_0)^{j-k}}{j!} \\
&= \sum_{j=0}^{n-k-1} \gamma_n^{n-j-k} \frac{(x - x_0)^j}{j!}
\end{aligned}$$

whence

$$\begin{aligned}
|P_n^{(k)}(x)| &\le \sum_{j=0}^{n-k-1} K \frac{(j+k)!}{n!} B(\eta)\eta^n \frac{|x - x_0|^j}{j!} \qquad (2.5)\\
&\le KB(\eta)\eta^n \frac{k!}{n!} \frac{1}{(1 - |x - x_0|)^{k+1}} \quad \text{for} \quad |x - x_0| < 1.
\end{aligned}$$

Thus we see that

$$| P_r^{(n-k-1)}(\delta)P_n^{(k)}(x_0 \pm \delta) | \le (1-r)\frac{12^{n-k+1}}{\delta^{n-k+1}}\frac{1}{(n-k-1)!}KB(\eta)\eta^n\frac{k!}{n!}\left(\frac{1}{1-\delta}\right)^{k+1}$$

by Theorem a6 of the appendix and therefore that

$$\begin{aligned}| \textstyle\sum_{k=0}^{n-1} P_r^{(n-k-1)}(\delta)P_n^{(k)}(x_0 \pm \delta)(-1)^{k+1} | &\le \frac{(1-r)KB(\eta)}{n!}\left(\frac{\eta 12}{\delta}\right)^n \sum_{k=0}^{n-1}\frac{k!}{(n-k-1)!}\frac{\delta^{k-1}}{12^{k-1}}\frac{1}{(1-\delta)^{k+1}} \\ &\le \frac{(1-r)KB(\eta)}{n}\left(\frac{\eta 12}{\delta}\right)^n \frac{12}{\delta(1-\delta)}\frac{12(1-\delta)}{12-13\delta}, \qquad n = 1, 2, \cdots\end{aligned}$$

whence it follows that

$$\left|\sum_{n=0}^{\infty} q_3(n, r, \delta)\right| \le \frac{(1-r)KB(\eta)288}{(12-13\delta)(\delta-12\eta)} = (1-r)C_3(\delta, \eta), \tag{2.6}$$

$$0 < \delta < 1/2, 0 < \eta < \delta/12.$$

We now turn to the first series of integrals

$$\textstyle\sum_{n=0}^{\infty} q_1(n, r, \delta) = \sum_{n=0}^{\infty}\left\{\int_{-\pi}^{x_0-\delta} + \int_{x_0+\delta}^{\pi}\right\} P_r^{(n)}(x_0 - t)g_n(t)\, dt$$

which similarly can be shown by Theorem a6 (iv) again, to be dominated by a constant similar to that on the right side of (2.6). We denote by $(1 - r)C_1(\delta, \eta)$ this constant.

We need now show that the remaining series of integrals $q_2(n, r, \delta)$ converges to $2\pi\gamma$. We write the integral as

$$\begin{aligned}&\int_{x_0-\delta}^{x_0+\delta} \frac{P_r^{(n)}(x_0-t)(t-x_0)^n}{n!}\left\{n!\frac{g_n(t)-P_n(t)}{(t-x_0)^n} - \gamma_n\right\}dt \\ &\qquad + \int_{x_0-\delta}^{x_0+\delta}\frac{P_r^{(n)}(x_0-t)(t-x_0)^n}{n!}\gamma_n\, dt \\ &\qquad = I_1 + I_2.\end{aligned}$$

Let us look at I_2 first; we find that

$$\begin{aligned}\int_{x_0-\delta}^{x_0+\delta} & P_r^{(n)}(x_0-t)\frac{(t-x_0)^n}{n!}\,dt \\ &= \int_{-\delta}^{\delta}\frac{P_r^{(n)}(t)(-t)^n\, dt}{n!} \\ &= \int_{-\pi}^{\pi}\frac{P_r^{(n)}(t)(-t)^n}{n!}\,dt - \left\{\int_{\delta}^{\pi} + \int_{-\pi}^{-\delta}\right\}\frac{P_r^{(n)}(t)(-t)^n}{n!}\,dt \qquad (2.7) \\ &= 2\pi - \left\{\int_{\delta}^{\pi} + \int_{-\pi}^{-\delta}\right\}P_r(t)\, dt \\ &= \int_{-\delta}^{\delta}P_r(t)\, dt\end{aligned}$$

by Theorem a6 of the appendix. Thus

$$I_2 = 2\pi\gamma_n - \gamma_n\left\{\int_\delta^\pi + \int_{-\pi}^{-\delta}\right\} P_r$$

As for I_1, given any ε such that $0 < \varepsilon < 1/3$, we choose δ such that

$$\left| n!\,\frac{g_n(t) - P_n(t)}{(t - x_0)^n} - \gamma_n \right| < \varepsilon^{n+1}, \qquad | t - x_0 | < \delta. \tag{2.8}$$

Then we have, if $\delta < \pi/4$, by Theorem a6,

$$\begin{aligned} | I_1 | &\leq \int_{-\delta}^{\delta} \left| \frac{P_r^{(n)}(t)t^n}{n!} \right| dt\, \varepsilon^{n+1} \\ &\leq 2\int_0^{\pi/4} \left| \frac{P_r^{(n)}(t)t^n}{n!} \right| dt\, \varepsilon^{n+1} \\ &\leq 4(3\varepsilon)^{n+1}. \end{aligned} \tag{2.9}$$

Now we are able to estimate the difference and obtain

$$\begin{aligned} | q_2(n, r, \delta) - 2\pi\gamma_n | &= | I_2 - 2\pi\gamma_n + I_1 | \\ &\leq | \gamma_n | \left\{\int_\delta^\pi + \int_{-\pi}^{-\delta} P_r\right\} + 4(3\varepsilon)^{n+1} \\ &\leq | \gamma_n | \,(1 - r)\,\frac{40}{\delta^2} + 4(3\varepsilon)^{n+1}. \end{aligned} \tag{2.10}$$

By combining all the inequalities we have made so far, we finally get

$$\begin{aligned} | (P_r * f)(x_0) - \gamma | &= (1/2\pi)\, \Big| \textstyle\sum_{n=0}^\infty q_1(n, r, \delta) + \sum_{n=0}^\infty q_3(n, r, \delta) \\ &\qquad + \textstyle\sum_{n=0}^\infty \{q_2(n, r, \delta) - 2\pi\gamma_n\} \Big| \\ &\leq (1/2\pi)(1 - r)C_1(\delta, \eta) \\ &\quad + (1/2\pi)(1 - r)C_3(\delta, \eta) \\ &\quad + (1 - r)(7/\delta^2) \textstyle\sum_{n=0}^\infty | \gamma_n | + 2\varepsilon/(1 - 3\varepsilon). \end{aligned} \tag{2.11}$$

Thus if we first choose $\delta < 1/2$ such that (2.8) is satisfied, then choose $\eta = \delta/20$, and finally choose r sufficiently close to 1, we can make the left side of (2.11) less than some multiple of ε. This proves the theorem.

This theorem of course includes the case when f is a distribution with a value at x_0, since then the series representing f has only a finite number of terms. However it does include other hyperfunctions. In particular, if f is a hyperfunction with point support, it is easy to show that f has the form $\sum_{n=1}^\infty a_n\delta^{(n)}$ where $| n!a_n |^{1/n} \to 0$ and hence that it has a value everywhere except at 0.

The converse to the theorem is clearly not true. Any function with a jump discontinuity has a Fourier Series which is Abel-summable to the average of the left and right hand values at that point. (See [12].) However it doesn't have a value at that point in the sense of distributions.

We could have assumed that all the values of $g_n^{(n)}$ at x_0 except g_0 were equal to 0. Indeed we could have subtracted $\gamma_n e^{in(x-x_0)}$ from $g_n^{(n)}$ and then incorporated this in g_0.

Other properties of values are similar to the corresponding properties for distributions. We summarize in

PROPOSITION 4. (i) *If f_1 and f_2 have values γ_1 and γ_2 at x_0 respectively, then $a_1 f_1 + a_2 f_2$ has value $a_1\gamma_1 + a_2\gamma_2$ at x_0.*

(ii) *If f is equal to a continuous function g in some neighborhood of x_0, then f has a value at x_0 equal to $g(x_0)$.*

(iii) *If f' has a value at x_0 so does f.*

The proofs of these assertions follow easily from similar statements for distributions and will be omitted.

3. Appendix

We collect here some of the properties of the Poisson kernel $P_r(x)$ and its derivatives which we have used above. It is given by the formula

$$P_r(x) = \frac{1 - r^2}{1 - 2r\cos x + r^2}, \quad x \in \Gamma, r \in (0, 1). \tag{a1}$$

By the Cauchy formula we have

$$P_r^{(n)}(x) = \frac{n!}{2\pi i}\int_C \frac{P_r(z)}{(z - x)^{n+1}}\,dz, \tag{a2}$$

where C is any contour enclosing x and not enclosing the points at which $\cos z = 1/2r + r/2$. Taking C to be a circular contour of radius ρ and center x, we obtain

$$P_r^{(n)}(x) = \frac{n!}{2\pi}\int_0^{2\pi} \frac{P_r(x + \rho e^{i\theta})}{(\rho e^{i\theta})^n}\,d\theta. \tag{a3}$$

In order to obtain the necessary bounds on $P_r^{(n)}$ we shall need the following:

LEMMA a1. *The set*

$$\{(x, y) \mid |y| < x/2, x \in (0, \pi/2, x \in (0, \pi/2)\}$$

is a subset of $\{(x, y) \mid \sinh^2 y < \sin^2 x, x \in (0, \pi/2)\}$.

The curve $\sinh y = \sin x$ is monotonically increasing and convex from $x = 0$ to $x = \pi/2$ and agrees with $y = x/2$ at $x = 0$ and is above $y = x/2$ at $x = \pi/2$.

LEMMA a2. *The set of complex numbers* $\{z \mid |\cos z| < 1\}$ *contains the set*

$$\{z \mid z = x + (x/3)e^{i\theta}, x \in (0, \pi/2), \theta \in [0, 2\pi)\}.$$

Let z satisfy $z = x + x/3e^{i\theta}$. Then we have

$$| \operatorname{Im} z | = | (x/3) \sin \theta | < 1/2 \, | x + (x/3) \cos \theta | = \tfrac{1}{2} \, | \operatorname{Re} z |$$

since $| \sin \theta - \frac{1}{2} \cos \theta | \leq \frac{3}{2}$. But by Lemma a1, $\sinh^2 \operatorname{Im} z < \sin^2 \operatorname{Re} z$, which implies that

$$\begin{aligned} 1 &> 1 - \sin^2 \operatorname{Re} z + \sinh^2 \operatorname{Im} z \\ &= \cos^2 \operatorname{Re} z \cosh^2 \operatorname{Im} z + \sin^2 \operatorname{Re} z \sinh^2 \operatorname{Im} z \\ &= | \cos z |^2. \end{aligned}$$

LEMMA a3. *For* $z = x + (x/3)e^{i\theta}$, $| \cos z | \leq \cos (x - x/3)$, $x \in (0, \pi/4)$, $\theta \in [0, 2\pi)$.

We may calculate that

$$\begin{aligned} | \cos z |^2 &= \cos^2 (\operatorname{Re} z) + \sinh^2 (\operatorname{Im} z) \\ &= \cos^2 (x + (x/3) \cos \theta) + \sinh^2 ((x/3) \sin \theta) \\ &= f(\theta). \end{aligned}$$

Taking the derivative with respect to θ we obtain

$$\begin{aligned} &(x/3) \, 2 \cos (x + (x/3) \cos \theta) \sin (x + (x/3) \cos \theta) \sin \theta \\ &\qquad + (x/3) \, 2 \sinh ((x/3) \sin \theta) \cosh ((x/3) \sin \theta) \cos \theta \\ &= (x/3) \{\sin (2x + (2x/3) \cos \theta) \sin \theta + \sinh ((2x/3) \sin \theta) \cos \theta\} \\ &= f'(\theta). \end{aligned}$$

This expression is non-negative for $\theta \in [0, \pi/2]$; for $\theta \in (\pi/2, \pi)$ we use the fact that $\sinh \varphi \leq (6/5) \varphi$ for $\varphi \in [0, \pi/3]$ to obtain

$$\begin{aligned} f'(\theta) &\geq (x/3) \{\sin (2x + (2x/3) \cos \theta) \sin \theta + (2x/3) \sin \theta \cos \theta\} \\ &\geq (x/3) \sin \theta \{\sin 4x/3 - 4x/5\} \\ &\geq (x/3) \sin \theta \{(2/\pi)(4x/3) - 4x/5\} > 0. \end{aligned}$$

Thus $f(\theta)$ is increasing in $(0, \pi)$ and since $f(\theta) = f(-\theta)$ it must have its maximum at $\theta = \pi$.

LEMMA a4. *Let* $x \in (0, \pi/4)$; *then*

$$\frac{1}{2\pi} \int_0^{2\pi} \left| P_r \left(x + \frac{x}{3} e^{i\theta} \right) \right| d\theta \leq P_r \left(\frac{2}{3} x \right)$$

Let $z = x + (x/3) e^{i\theta}$; then by Lemma a2, and Lemma a3

$$\begin{aligned} | P_r(z) | &= (1 - r^2)/(1 - 2r \cos z + r^2) | \\ &\leq (1 - r^2)/(1 - 2r \, | \cos z | + r^2) \\ &= P_r(2x/3). \end{aligned}$$

For $x \notin (0, \pi/4)$ our calculation is much easier. Indeed we have

LEMMA a5. *Let $x \in [\pi/4, \pi]$, $0 < \rho \le \pi/12$; then*

$$\frac{1}{2\pi}\int_0^{2\pi} | P_r(x + \rho e^{i\theta}) | \, d\theta \le P_r\left(\frac{\pi}{6}\right).$$

We observe that

$$\begin{aligned} 1/| P_r(z) |^2 &= (1 + r^2)^2 - 2r(1 + r^2)(2 \operatorname{Re} \cos z) + 4r^2 | \cos z |^2 \\ &= (1 + r^2)^2 - 4r(1 + r^2) \cos (x + \rho \cos \theta) \cosh (\rho \sin \theta) \\ &\qquad + 4r^2 \{\cos^2 (x + \rho \cos \theta) + \sinh^2 (\rho \sin \theta\}. \end{aligned}$$

The derivative with respect to x is non-negative whence the function is non-decreasing in x in the interval $[\pi/4, 11\pi/12]$. The values it attains in $(11\pi/12, \pi]$ also are attained in this interval. Therefore its minimum is at $x = \pi/4$ and the maximum of $| P_r(x + \rho e^{i\theta}) |$ is at the same point. Then by Lemma a4 we reach our conclusion.

We now have all the inequalities we need to derive the properties of $P_r^{(n)}$ which we summarize in

THEOREM a6. *The function $x^n P_r^{(n)}(x)/n!$ satisfies the following conditions for $\frac{1}{2} \le r < 1$.*

$$\text{(i)} \qquad \int_0^{\pi} \left| \frac{x^n P_r^{(n)}(x)}{n!} \right| dx \le 5 \cdot 12^{n+1},$$

$$\text{(ii)} \qquad \frac{(-1)^n}{2\pi} \int_{-\pi}^{\pi} \frac{x^n P_r^{(n)}(x)}{n!} \, dx = 1,$$

$$\text{(iii)} \qquad \int_0^{\pi/4} \left| \frac{x^n P_r^{(n)}(x)}{n!} \right| dx \le 2 \cdot 3^{n+1},$$

$$\text{(iv)} \qquad | x^n P_r^{(n)}(x)/n! | \le 3^n \cdot 12(1 - r)/x^2, \quad 0 < x \le \pi/4,$$

$$\text{(v)} \qquad | x^n P_r^{(n)}(x)/n! | \le 12^{n+2}(1 - r), \quad \pi/4 \le x \le \pi.$$

To prove (i) and (iii) we use formula (a3) and Lemmas (a4) and (a5). In formula (a3) we first take $\rho = x/3$. Then we find

$$\begin{aligned} \int_0^{\pi/4} \left| \frac{x^n P_r^{(n)}(x)}{n!} \right| dx &\le \int_0^{\pi/4} \frac{3^n}{2\pi} \int_0^{2\pi} \left| P_r\left(x + \frac{x}{3} e^{i\theta}\right) \right| d\theta \, dx \\ &\le 3^n \int_0^{\pi/4} P_r(2x/3) \, dx \\ &\le \frac{3^{n+1}}{2} \int_0^{\pi} P_r(x) \, dx \\ &\le 2 \cdot 3^{n+1} < 6 \cdot 12^n. \end{aligned}$$

Taking $\rho = \pi/12$ we find

$$\int_{\pi/4}^{\pi} \frac{| x^n P_r^{(n)}(x) |}{n!} \, dx \le \int_{\pi/4}^{\pi} \frac{1}{2\pi} (12x/\pi)^n \int_0^{2\pi} \left| P_r\left(x + \frac{\pi}{12} e^{i\theta}\right| d\theta \, dx$$

$$\leq \int_{\pi/4}^{\pi} 12^n P_r(\pi/6)\, dx$$

$$\leq 3 \cdot 12^n P_r(\pi/6)$$

But as is well known (see [12, p. 96]),

$$P_r(x) < (\pi^2/2)(1-r)/(x^2+(1-r)^2) \tag{a4}$$

$$\text{for} \quad 0 < x < \pi, \quad \tfrac{1}{2} \leq r < 1,$$

so that $P_r(\pi/6) < 18$ whence we obtain

$$\int_0^{\pi} | x^n P_r^{(n)}(x)/n! \,|\, dx \leq (6 + 54)12^n.$$

To prove (ii) we integrate by parts n times. All the integrated terms are 0 since $x^k P_r^{(k-1)}(x)$ in an odd function for all positives integers k. What remains is

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r = 1.$$

Parts (iv) and (v) are straightforward calculations applying Lemmas (a4) and (a5) together with formula (a4) to the expression for $P_r^{(n)}$ given by formula (a3).

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