ON ANALYTIC STRUCTURE IN THE MAXIMAL IDEAL SPACE OF $H_\infty(D^n)$

BY

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Let $H_\infty(D^n)$ denote the complex Banach algebra of bounded holomorphic functions on the open unit polydisc

$$D^n = \{(z_1, \cdots, z_n) \in \mathbf{C}^n : |z_1| < 1, \cdots, |z_n| < 1\}.$$

The map $(z_1, \cdots, z_n) \to f(z_1, \cdots, z_n)$ imbeds D^n as an open subset of the maximal ideal space of $H_\infty(D^n)$; so we let $M(H_\infty(D^n))$ denote the closure of D^n in this space. By an analytic map into $M(H_\infty(D^n))$ we mean a function

$$F : D^m \to M(H_\infty(D^n))$$

such that $\hat{f} \circ F$ is analytic in D^m for every f in $H_\infty(D^n)$, where $\hat{f}$ is the Gélfand extension of f to $M(H_\infty(D^n))$. The image of F is called an analytic set in $M(H_\infty(D^n))$. If F is one-one, then $F(D^n)$ is a m-dimensional analytic polydisc.

In this paper we construct various dimensional analytic polydiscs in $M(H_\infty(D^n))$ as limits of analytic maps into D^n and compare these in a natural way with the analytic structure in $M(H_\infty(D))^n$, the n-fold Cartesian product of $M(H_\infty(D))$. We also show that only points belonging to the closure of zero sets of functions in $H_\infty(D^n)$ can belong to analytic sets obtained in this manner.

The maximal ideal space of the algebra $H_\infty(D)$ has been extensively studied, beginning with I. J. Schark [13], and continuing with D. Newman [12], A. Gleason and H. Whitney [5], L. Carleson [3, 4], A. Kerr-Lawson [11], K. Hoffman [8, 10], and others. In the paper of I. J. Schark, it was shown that there exist non-trivial analytic mappings from D into $M(H_\infty(D))\backslash D$. Angus Kerr-Lawson [11] extended the Schark idea and showed that "non-tangential" and "oricycular" points in $M(H_\infty(D))$ lie in non-trivial analytic sets. By an algebraic argument, K. Hoffman [8] showed that each non-trivial Gleason part in $M(H_\infty(D))$ is a 1-dimensional analytic disc. Shortly thereafter Professor Hoffman [10] gave a "geometric" method for obtaining the coordinate maps for the analytic discs in $M(H_\infty(D))$.

The natural inductive vehicle for generalization to higher dimensional polydiscs is the topological tensor product $\otimes_\lambda^n H_\infty(D)$, where $\otimes_\lambda^n$ is the completion of the algebraic tensor product $\otimes^n$ in the uniform norm. However, it is now well known (see [1]) that $\otimes_\lambda^n H_\infty(D) \neq H_\infty(D^n)$. Hence, the lifting of 1-dimensional results becomes more than routine.

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I. Preliminaries

$M(H_\infty(D^n))$ is a compact Hausdorff space with topology as follows: a net $\{\varphi_\alpha\}$ converges to φ_0 if and only if $\varphi_\alpha(f)$ converges to $\varphi_0(f)$ for all f in $H_\infty(D^n)$. If $\varphi \in M(H_\infty(D^n))$, then the Gleason part containing φ, denoted $P(\varphi)$, is defined as

$$P(\varphi) = \{\psi \in M(H_\infty(D^n)) : \rho(\varphi, \psi) < 1\}$$

where

$$\rho(\varphi, \psi) = \sup \{|\hat{f}(\psi)| : f \in H_\infty(D^n), |f| \leq 1, \hat{f}(\varphi) = 0\}$$

is the pseudo-hyperbolic distance from φ to ψ. This defines an equivalence relation on $M(H_\infty(D^n))$. Parts are important in the investigation of analytic structure since any analytic set through φ is contained in the Gleason part $P(\varphi)$ (see [2, p. 130]).

Using the Schwarz inequality it is easy to show that ρ restricted to D^n has the following useful formula.

LEMMA 1.1. *For* $(z_1, \cdots, z_n), (w_1, \cdots, w_n) \in D^n$,

$$\rho((z_1, \cdots, z_n), (w_1, \cdots, w_n)) = \max_{1 \leq k \leq n} \left\{ \left| \frac{z_k - w_k}{1 - \bar{z}_k w_k} \right| \right\}.$$

Another application of the Schwarz inequality, in combination with the preceding lemma, gives a generalization of Pick's theorem to higher dimensions (see [7, p. 239]).

THEOREM 1.2 (Pick). *If* $f \in H_\infty(D^n)$ *with* $|f| \leq 1$, *then*

$$\rho(f(z_1, \cdots, z_n), f(w_1, \cdots, w_n)) \leq \rho((z_1, \cdots, z_n), (w_1, \cdots, w_n))$$

for all $(z_1, \cdots, z_n), (w_1, \cdots, w_n) \in D^n$.

We shall often use the resulting corollary.

COROLLARY 1.3. *Let* $\{\alpha_\lambda\}$ *and* $\{\beta_\lambda\}$ *be nets in* D^n *indexed by the same set and converging in* $M(H_\infty(D^n))$ *to* φ *and* ψ *respectively. If* $\rho(\alpha_\lambda, \beta_\lambda) \to 0$, *then* $\varphi = \psi$.

Proof. $|f(\alpha_\lambda) - f(\beta_\lambda)| \leq 2\rho(f(\alpha_\lambda), f(\beta_\lambda))$.

In [10] the problem of determining which subsets of $M(H_\infty(D))$ support analytic structure is shown to be directly related to the concept of an interpolating sequence. A countable subset, $\{\alpha_n\}_1^\infty$, of D is called an interpolating sequence if there exists $\delta > 0$ such that

$$\prod_{k \neq n} \left| \frac{\alpha_k - \alpha_n}{1 - \alpha_k \bar{\alpha}_n} \right| = \prod_{k \neq n} \rho(\alpha_k, \alpha_n) \geq \delta$$

for all k. In particular, an interpolating sequence is a Blaschke sequence (See [9, p. 197].)

THEOREM 1.4 (Hoffman [10]). *For* $\alpha \in D$, *let* $L_\alpha(z) = (z + \alpha)/(1 + \bar{\alpha}z)$. *As a net* $\{\alpha_i\}$ *in* D *converges to a point* φ *in* $M(H_\infty(D))$ *the corresponding maps* L_{α_i} *converge in* $M(H_\infty(D))^D$ *to a map* L_φ, *which is analytic, and maps* D *onto the part* $P(\varphi)$. $P(\varphi)$ *is non-trivial if and only if* φ *belongs to the closure of an interpolating sequence. In this case* L_φ *is one-one and* $\rho(z, w) = \rho(L_\varphi(z), L_\varphi(w))$. *Finally if* $\varphi \in M(H_\infty(D))$ *is a point of a non-trivial part and* S *and* T *are subsets of* D *such that* φ *is an accumulation point of both, then*

$$\inf [\rho(s, t) : s \in S, t \in T] = 0.$$

We shall need the following results on interpolating sequences. The first is a special case of a more general result on Banach algebras (see [9, p. 205]). A more direct proof is indicated in [10, p. 89].

THEOREM 1.5. *Let* $S = \{\alpha_n\}_1^\infty \subset D$ *be an interpolating sequence for* $H_\infty(D)$. *Then the closure of* S *in* $M(H_\infty(D))$ *is homeomorphic to the Čech compactification of the natural numbers.*

THEOREM 1.6 (Hayman [6]). *Let* $\{\alpha_n\}_1^\infty$ *be an interpolating sequence for* $H_\infty(D)$. *Then there exists a sequence of functions* $\{f_j\}_1^\infty$ *in* $H_\infty(D)$ *and a constant* $c > 0$ *such that* $f_j(\alpha_n) = \delta_j^n$ *(Kronecker delta) and* $\sum_{j=1}^\infty |f_j(z)| < c$ *for all* $z \in D$.

By $\otimes_\lambda^n H_\infty(D)$ we denote the smallest closed subalgebra of $H_\infty(D^n)$ which contains all functions of the form $F(z_1, \cdots, z_n) = f(z_j)$ for some j and some choice of f in $H_\infty(D)$. Since the maximal ideal space of $\otimes_\lambda^n H_\infty(D)$ is $M(H_\infty(D))^n$, there is a natural continuous map

$$\pi : M(H_\infty(D^n)) \to M(H_\infty(D))^n$$

defined by $\pi(\varphi) = \varphi$ restricted to $\otimes_{\lambda k=1}^n H_\infty(D)$. It is easy to see that parts in $M(H_\infty(D))^n$ are products of parts from $M(H_\infty(D))$, and if $\psi \in P(\varphi)$, then $\pi(\psi) \in P(\pi(\varphi))$. We shall use the π map to relate $M(H_\infty(D^n))$ and $M(H_\infty(D))^n$.

II. Analytic Structure over $D^{k-1} \times M(H_\infty(D)) \times D^{n-k}$

Following the lead of Hoffman, it is natural to search for analytic maps into $M(H_\infty(D^n))$ as limits of analytic maps into D^n. Since the family of analytic functions on D^n is closed under bounded pointwise convergence, the set of all analytic maps from D^n into $M(H_\infty(D^n))$ is a closed subset of $M(H_\infty(D^n))^{D^n}$; therefore, any map obtained as a limit of such mappings is analytic.

In this section we completely settle the question of analytic structure over $D^{k-1} \times M(H_\infty(D)) \times D^{n-k}$, $1 \le k \le n$, with the aid of the following theorem.

THEOREM 2.1. π is one-one over $D^{k-1} \times M(H_\infty(D)) \times D^{n-k}$, $1 \le k \le n$.

Proof. Let $\varphi \in M(H_\infty(D^n))$ and

$$\pi(\varphi) = (z_0, m, z_0') \in D^{k-1} \times M(H_\infty(D)) \times D^{n-k}.$$

Let $\{\alpha_\lambda\} \to m$. It suffices to show that

$$\{(z_0, \alpha_\lambda, z_0')\} \to \varphi.$$

Let $\{(z_0, \alpha_j, z_0')\}$, $j \in A$, be a converging subnet of $\{(z_0, \alpha_\lambda, z_0')\}$ and choose

$$\{((w_1)_i, (w_2)_i, \cdots, (w_n)_i)\}, \quad i \in B,$$

converging to φ. By considering the product ordering on $\Omega = A \times B$, we can assume that we have a common indexing set. Then $\{(w_k)_\delta\} \to m$ and $\{\alpha_\delta\} \to m$ for $\delta \in \Omega$. Let $f \in H_\infty(D^n)$ with $\|f\| \leq 1$, and assume that $\varepsilon > 0$. Then there exists $\delta_0 \in \Omega$ such that $\delta \geq \delta_0$ implies

$$\rho(((w_1)_\delta, \cdots, (w_{k-1})_\delta), z_0) < \varepsilon/6,$$

$$\rho(((w_{k+1})_\delta, \cdots, (w_n)_\delta), z_0') < \varepsilon/6,$$

$$|f(z, (w_k)_\delta, z_0') - m(f(z_0, \cdot, z_0'))| < \varepsilon/6,$$

and

$$|m(f(z_0, \cdot, z_0')) - f(z_0, \alpha_\lambda, z_0')| < \varepsilon/3.$$

It follows that for $\delta \geq \delta_0$,

$$\begin{aligned} &|f((w_1)_\delta, \cdots, (w_n)_\delta) - f(z_0, \alpha_\delta, z_0')| \\ &\quad \leq 2\rho(f((w_1)_\delta, \cdots, (w_n)_\delta), f(z_0, (w_k)_\delta, z_0')) + 2\varepsilon/3 \\ &\quad \leq 2\rho(((w_1)_\delta, \cdots, (w_{k-1})_\delta, (w_{k+1})_\delta, \cdots, (w_n)_\delta), (z_0, z_0')) + 2\varepsilon/3 \\ &\quad < \varepsilon \end{aligned}$$

by Theorem 1.2. Thus each converging subnet of $\{(z_0, \alpha_\lambda, z_0')\}$ converges to φ; therefore,

$$\{(z_0, \alpha_\lambda, z_0')\} \to \varphi,$$

and π is one-one over $D^{k-1} \times M(H_\infty(D)) \times D^{n-k}$.

COROLLARY 2.2. *Each $f \in H_\infty(D^n)$ has a bounded continuous extension to*

$$D^{k-1} \times M(H_\infty(D)) \times D^{n-k}.$$

COROLLARY 2.3. *Let $\varphi \in M(H_\infty(D^n))$ and*

$$\pi(\varphi) = (z_0, m, z_0') \in D^{k-1} \times M(H_\infty(D)) \times D^{n-k}.$$

Then $P(\varphi)$ is an n-dimensional ($(n-1)$-dimensional) analytic polydisc whenever $P(m)$ is non-trivial (trivial).

Proof. Assume $P(m)$ is non-trivial and let $\{\alpha_\lambda\} \to m$. Then

$$\{(z_0, \alpha_\lambda, z_0')\} \to \varphi$$

by Theorem 2.1. Define $L_\lambda : D^{k-1} \times D \times D^{n-k} \to D^n$ by

$$L_\lambda(z, w, z') = (L_{z_0}(z), L_{\alpha_\lambda}(w), L_{z_0'}(z')),$$

where

$$L_{z_0}(z) : D^{k-1} \to D^{k-1} \quad \text{and} \quad L_{z_0'}(z') : D^{n-k} \to D^{n-k}$$

are defined in each coordinate by the maps of Theorem 1.4. Since π is one-one over

$$D^{k-1} \times M(H_\infty(D)) \times D^{n-k},$$

it follows that $L_\lambda \to L_\varphi : D^n \to M(H_\infty(D^n))$, where L_φ is analytic,

$$L_\varphi(D^n \subset P(\varphi),$$

and L_φ is one-one since $\otimes_\lambda^n H_\infty(D)$ separates points on $L_\varphi(D^n)$. If $\psi \in P(\varphi)$ and $\pi(\psi) = (w_0, m^*, w_0')$, then $m^* \in P(m)$ and

$$(w_0, w_0') \in D^{k-1} \times D^{n-k}.$$

By Theorem 1.4 there exist $(z, z') \in D^{k-1} \times D^{n-k}$ and $w^* \in D$ such that $L_{z_0}(z) = w_0$, $L_{z_0'}(z') = w_0'$ and $L_{\alpha_\lambda}(w^*) \to m^*$. Hence, $L_\varphi(z, w^*, z') = \psi$ since π is one-one. Thus L_φ is onto $P(\varphi)$. In the case $P(m)$ is trivial, repeat the argument using α_λ in the *k-th* coordinate of L_λ.

We remark here that the L_φ maps in the preceding corollary are homeomorphisms into the metric topology of $M(H_\infty(D^n))$. From this fact it can be shown that $M(H_\infty(D^n))\backslash D^n$ contains homeomorphic copies of $M(H_\infty(D^k))$, $k \leq n$.

III. Analytic structure over non-trivial parts

In this section we show that in general there is analytic structure over non-trivial parts in $M(H_\infty(D))^n$. In particular, it is shown that π is not one-one over non-trivial parts in $M(H_\infty(D))^n$. This results in a sheeting of analytic sets over these parts.

THEOREM 3.1. *Let $\varphi \in M(H_\infty(D^n))$ and $\pi(\varphi) = (m_1, \cdots, m_n)$ where k of the parts $P(m_1), \cdots, P(m_n)$ are non-trivial. Then $P(\varphi)$ contains a k-dimensional analytic polydisc.*

Proof. Let $\{((\alpha_1)_\lambda, (\alpha_2)_\lambda, \cdots, (\alpha_n)_\lambda)\} \to \varphi$ and suppose $P(m_1)$, $P(m_2), \cdots, P(m_k)$ are non-trivial. Define

$$L_\lambda(z_1, \cdots, z_k) = (L_{(\alpha_1)_\lambda}(z_1), \cdots, L_{(\alpha_k)_\lambda}(z_k)),$$

where $L_{(\alpha_j)_\lambda} \to L_{m_j}$ as discussed in Theorem 1.4. By choosing an appropriate subnet, there exists

$$L_\varphi : D^k \to P(\varphi)$$

with L_φ analytic and $L_\varphi = \lim_\lambda L_\lambda$. We see that L_φ is one-one by considering the tensor algebra $\otimes_\lambda^n H_\infty(D)$ on $L_\varphi(D^k)$.

THEOREM 3.2. *Let $\varphi \in M(H_\infty(D^n))$ and $\pi(\varphi) = (m_1, m_2, \cdots, m_n)$ where each part $P(m_k)$ is non-trivial.* Then if $m_k \in \text{cl}\{(\alpha_k)_n\}_{n=1}^\infty$, *with each sequence*

$\{(\alpha_k)_n\}_1^\infty$ *interpolating, it follows that* φ *belongs to the closure of*

$$\{((\alpha_1)_{l_1}, \cdots, (\alpha_n)_{l_2})\},$$

where $(l_1, \cdots, l_n) \in Z_+^n$ *and* Z_+ *denotes the set of all nonnegative integers.*

Proof. Let

$$\{((\gamma_1)_\lambda, \cdots, (\gamma_n)_\lambda)\} \to \varphi.$$

Then $\{(\gamma_k)_\lambda\} \to m_k$ and $m_k \in \mathrm{cl}\{(\alpha_k)_n\}_1^\infty$ with $\{(\alpha_k)_n\}_1^\infty$ interpolating. By Theorem 1.4 we can choose a subnet $\{(\alpha_k)_\lambda\} \to m_k$ such that $\rho((\alpha_k)_\lambda, (\gamma_k)_\lambda) \to 0$ for each k. Then

$$\rho(((\alpha_1)_\lambda, \cdots, (\alpha_n)_\lambda), ((\gamma_1)_\lambda, \cdots, (\gamma_n)_\lambda)) = \max_k \rho((\alpha_k)_\lambda, (\gamma_k)_\lambda) \to 0.$$

Hence, $\{((\alpha_1)_\lambda, (\alpha_2)_\lambda, \cdots, (\alpha_n)_\lambda\} \to \varphi$ by Corollary 1.3.

In order to show that π is not one-one over $(m_1, m_2, \cdots, m_n)$ where at least two parts $P(m_1), P(m_2), \cdots, P(m_n)$ are non-trivial, we turn our attention to sets of the form

$$\{((\alpha_1)_{l_1}, (\alpha_2)_{l_2}, \cdots, (\alpha_n)_{l_n})\}$$

where $(l_1, l_2, \cdots, l_n) \in Z_+^n$ and each sequence $\{(\alpha_k)_n\}_{n=1}^\infty$ is interpolating. The following theorem is a higher dimensional analogue of Theorem 1.5.

THEOREM 3.3. *If* $\{(\alpha_k)_l\}_{l=1}^\infty$ *is an interpolating sequence in* D *for each* $k = 1, \cdots, n$, *then the closure of* $\{((\alpha_1)_{l_1}, (\alpha_2)_{l_2}, \cdots, (\alpha_n)_{l_n})\}$ *in* $M(H_\infty(D^n))$ *is homeomorphic to the Cech compactification of* Z_+^n.

Proof. It suffices to show that disjoint subsets of

$$\{((\alpha_1)_{l_1}, (\alpha_2)_{l_2}, \cdots, (\alpha_n)_{l_n})\}$$

have disjoint closures in $M(H_\infty(D^n))$. To show this let S be any subset of

$$\{((\alpha_1)_{l_1}, (\alpha_2)_{l_2}, \cdots, (\alpha_n)_{l_n})\}.$$

By Theorem 1.6 there exist sequences $\{(f_k)_n\}_{n=1}^\infty$ *in* $H_\infty(D)$ such that $(f_k)_n(\alpha_j) = \delta_j^n$ for each $k = 1, \cdots, n$ and $\sum_{n=1}^\infty |(f_k)_n(z)| < c_k$ for all $z \in D$. Now if $\mu : Z_+^n \to \mathbf{C}$ is any bounded function, then

$$f(z_1, \cdots, z_n) = \sum_{l \in Z_n^+} \mu(l)(f_1)_{l_1}(z_1) \cdots (f_n)_{l_n}(z_n)$$

belongs to $H_\infty(D^n)$ since bounded pointwise convergence gives uniform convergence on compact subsets of D^n. But

$$f((\alpha_1)_{l_1}, \cdots, (\alpha_n)_{l_n}) = \mu(l_1, \cdots, l_n).$$

Therefore by a suitable choice of μ, f is 0 on S and 1 on the complement of S. Hence, S and $\{((\alpha_1)_{l_1}, \cdots, (\alpha_n)_{l_n})\} \backslash S$ have disjoint closures in $M(H_\infty(D^n))$.

COROLLARY 3.4. *Let* $(m_1, \cdots, m_n) \in M(H_\infty(D))^n$ *with at least two parts* $P(m_1), \cdots, P(m_n)$ *non-trivial and outside of* D. *Then* π *is not one-one over* $(m_1, \cdots, m_n)$.

Proof. Suppose $P(m_1)$ and $P(m_2)$ are non-trivial with $m_1 \in \text{cl}\{\alpha_n\}_1^\infty$, $m_2 \in \text{cl}\{\beta_n\}_1^\infty$ where both sequences are interpolating. Let $\{\alpha_{n_\lambda}\} \to m_1$ with $\lambda \in \Gamma$ and $\{\beta_{m_\gamma}\} \to m_2$ with $\gamma \in \Lambda$. It suffices to show that $\{(\alpha_{n_\lambda}, \beta_{m_\gamma})\}$, $(\lambda, \gamma) \in \Gamma \times \Lambda$, does not converge in $M(H_\infty(D^2))$. Since

$$\text{cl}\{\alpha_n\}_1^\infty \cong \beta N, \qquad \text{cl}\{\beta_n\}_1^\infty \cong \beta N$$

by Theorem 1.5, and $\text{cl}\{(\alpha_n, \beta_m)\}_{N \times N} \cong \beta(N \times N)$ by Theorem 3.3, this amounts to showing that $\{(n_\lambda, m_\gamma)\}$, $(\lambda, \gamma) \in \Gamma \times \Lambda$, does not converge in $\beta(N \times N)$. To see this it suffices to exhibit two subnets of $\{(n_\lambda, m_\gamma)\}$ which have disjoint closures in $\beta(N \times N)$. For the subnets take

$$S = \{(n_\lambda, m_\gamma) : (\lambda, \gamma) \in \Gamma \times \Lambda \text{ but } n_\lambda > m_\gamma\}$$

and

$$T = \{(n_\lambda, m_\gamma) : (\lambda, \gamma) \in \Gamma \times \Lambda \text{ but } n_\lambda < m_\gamma\}.$$

Then $T \cap S = \emptyset$, and hence S and T have disjoint closures in $\beta(N \times N)$. But the limit of any converging subnet of $\{[\alpha_{n_\lambda}, \beta_{m_\gamma})\}$ maps under π to (m_1, m_2).

Theorem 3.1 showed that we always have analytic structure over non-trivial parts in $M(H_\infty(D))^n$. The preceding corollary shows that for parts $Q = P(m_1) \times \cdots \times P(m_n)$ of dimension $k \geq 2$, where at least two non-trivial parts are outside of D, there are many analytic polydiscs P of dimension k with $\pi(P) = Q$. It is a conjecture that these analytic polydiscs are actually parts in $M(H_\infty(D^n))$.

The following theorem will show that in general π is not one-one over one-dimensional parts in $M(H_\infty(D))^n$.

THEOREM 3.5. *Let $\varphi \in M(H_\infty(D^n))$ with φ belonging to the closure of the sequence $\{((\alpha_1)_l, \cdots, (\alpha_n)_l)\}_{l=1}^\infty$ where at least one of the sequences $\{(\alpha_i)_l\}_{l=1}^\infty$ is interpolating in D. Then $P(\varphi)$ contains an n-dimensional analytic polydisc.*

Proof. Let $\{((\alpha_1)_\lambda, \cdots, (\alpha_n)_\lambda)\} \to \varphi$ and suppose $\{(\alpha_1)_l\}_{l=1}^\infty$ is interpolating. Define $L_\lambda : D^n \to D^n$ by

$$\begin{aligned} L_\lambda(z_1, \cdots, z_n) &= (L_{(\alpha_1)_\lambda}(z_1), \cdots, L_{(\alpha_n)_\lambda}(z_n)) \\ &= \left(\frac{z_1 + (\alpha_1)_\lambda}{1 + \overline{(\alpha_1)}_\lambda z_1}, \cdots, \frac{z_n + (\alpha_n)_\lambda}{1 + \overline{(\alpha_n)}_\lambda z_n}\right). \end{aligned}$$

We can assume that Γ is such that

$$L_\lambda \to L_\varphi : D^n \to M(H_\infty(D^n)).$$

Then $L_\varphi = \lim_\lambda L_\lambda$ is analytic with $L_\varphi(D^n) \subset P(\varphi)$ It is clear that L_φ is one-one in the first coordinate because the tensor algebra separates points in this coordinate. Thus fix $z_1 = w$. Then it is easy to check that $L_{(\alpha_1)_l}(w)$ is interpolating and by Theorem 1.6, there exists a sequence $\{f_l\}_1^\infty$ in $H_\infty(D)$ such that

$$f_l(L_{(\alpha_1)_m}(w)) = \delta_l^m \quad \text{and} \quad \textstyle\sum_{l=1}^\infty |f_l(z)| < c,$$

for all $z \in D$. Define

$$h_k(z_1, \cdots, z_n) = \sum_{l=1}^{\infty} f_l(z_1)\left(\frac{z_k - (\alpha_k)_l}{1 - \overline{(\alpha_k)}_l z_k}\right).$$

Then $h_k \in H_\infty(D^n)$ and

$$h_k(L_{(\alpha_1)_\lambda}(w), \cdots, L_{(\alpha_k)_\lambda}(z_k), \cdots, L_{(\alpha_n)_\lambda}(z_n)) = \frac{L_{(\alpha_k)_\lambda}(z_k) - (\alpha_k)_\lambda}{1 - \overline{(\alpha_k)}_\lambda L_{(\alpha_k)_\lambda}(z_k)} = z_k.$$

Hence, L_φ is one-one.

Suppose $P(\psi)$ is trivial, $\psi \in \mathrm{cl}\{\beta_m\}_1^\infty$, and $\{\beta_\lambda\} \to \psi$. If $\{\alpha_m\}_1^\infty$ is any interpolating sequence, then $\{(\alpha_\lambda, \beta_\lambda, \beta_\lambda \cdots, \beta_\lambda)\}$ is a subnet of $\{(\alpha_m, \beta_m, \cdots, \beta_m)\}_1^\infty$. Let Φ be any cluster point of $\{(\alpha_m, \beta_m, \cdots, \beta_m)\}$. Then the previous theorem applies to show that $P(\Phi)$ contains a n-dimensional polydisc. In particular, the map

$$(z_2, \cdots, z_n) \to \lim_\lambda (\alpha_\lambda, L_{\beta_\lambda}(z_2), \cdots, L_{\beta_\lambda}(z_n))$$

is one-one and $\pi(\lim_\lambda (\alpha_\lambda, L_{\beta_\lambda}(z_2), \cdots, L_{\beta_\lambda}(z_n)) = (\varphi, \psi, \psi, \cdots, \psi)$, where $\{\alpha_\lambda\} \to \varphi$. Thus π is in general not one-one over one-dimensional parts in $M(H_\infty(D))^n$, and in this particular example π collapses an n-dimensional analytic polydisc onto a one-dimensional analytic disc. In fact, since every neighborhood of ψ contains a disc whose hyperbolic radius can be made arbitrarily large (see [11, p. 754]), it is not difficult to see that there is a sheeting of n-dimensional analytic polydiscs over this one-dimensional part.

IV. Analytic structure over trivial parts

In this section we give a condition for analyticity in $M(H_\infty(D^n))$ which permits the construction of a one-dimensional analytic disc whose projection under π is a point in $M(H_\infty(D))^n$.

THEOREM 4.1. *If $\{\beta_n\}_1^\infty$ is a sequence in D and $\beta_n \to e^{i\theta}$, $\theta \neq \pm\pi/2$, then every φ belonging to the closure of*

$$\{((\alpha_1)_l, \cdots, \beta_l, \cdots, \bar{\beta}_l, \cdots, (\alpha_n)_l)\}_{l=1}^\infty$$

belongs to a one-dimensional analytic disc.

Proof. Assume that the sequences $\{\beta_l\}_1^\infty$ and $\{\bar{\beta}_l\}_1^\infty$ are in the u and v coordinate positions respectively. Let

$$\{((\alpha_1)_\lambda, \cdots, \beta_\lambda, \cdots, \bar{\beta}_\lambda, \cdots, (\alpha_n)_\lambda)\}$$

converge to φ. Since $(z_1, \cdots, z_n) \to z_u z_v$ is holomorphic on D^n, $\{\beta_\lambda \bar{\beta}_\lambda = |\beta_\lambda|^2\}$ converges in $M(H_\infty(D))$, to say ψ. In [11] it is shown that such an angle of approach to the unit circle requires that $P(\psi)$ be non-trivial. Thus, there exists an interpolating sequence $\{\gamma_n\}_1^\infty$ such that $\psi \in \mathrm{cl}\{\gamma_n\}_1^\infty$. Let

$$A(z) = \prod_{n=1}^{\infty} \frac{\bar{\gamma}_n}{|\gamma_n|} \frac{\gamma_n - z}{1 - \bar{\gamma}_n z}, \quad f(z_1, \cdots, z_n) = A(z_u z_v)$$

and

$$L_m(z) = ((\alpha_1)_m, \cdots, L_{\beta_m}(z), \cdots, L_{\bar\beta_m}(z), \cdots, (\alpha_n)_m).$$

Then

$$[f \circ L_\lambda(z)]'(0) = A'(|\beta_\lambda|^2)(L_{\beta_\lambda}(z)L_{\bar\beta_\lambda}(z))'(0)$$
$$= A'(|\beta_\lambda|^2)[(\beta_\lambda + \bar\beta_\lambda)/2]2(1 - |\beta_\lambda|^2).$$

We proceed to show that this expression is bounded away from zero independent of λ. We have $L_{\gamma_n}(z) = (z + \gamma_n)/(1 + \bar\gamma_n z)$. Let $f_n(z) = A \circ L_{\gamma_n}(z)$. Then

$$\{\gamma_{n(\lambda)}\} \to \psi \quad \text{and} \quad L_{\gamma_{n(\lambda)}} \to L_\psi.$$

Thus

$$f_\lambda(z) \to f(z) = \hat{A} \circ L_\psi(z),$$

and

$$|f_n'(0)| = (1 - |\gamma_n|^2)|A'(\gamma_n)| = \textstyle\prod_{k\neq n} |(\gamma_n - \gamma_k)/(1 - \bar\gamma_n\gamma_k)|$$
$$= \textstyle\prod_{k\neq n} \rho(\gamma_n, \gamma_k) \geqq \delta$$

since $\{\gamma_n\}_1^\infty$ is interpolating. It follows that

$$|f'(0)| = |\lim_\lambda f_\lambda'(0)| \geq \delta > 0.$$

Now choose a disc $V = \{z : |z| < \varepsilon^*\}$ such that $|f'(z)| \geq \delta/2$ and $f_\lambda' \to f'$ uniformly on V. Therefore, there exists λ_0 such that $\lambda \geq \lambda_0$ implies $|f'(z) - f_\lambda'(z)| < \delta/4$ for all $z \in V$. It follows that $|f_\lambda'(z)| \geq \delta/4$ for all $z \in V$ and $\lambda \geq \lambda_0$. Now consider

$$U = \{m : |\hat{A}(m)| < \varepsilon\} \quad \text{where } \varepsilon < \varepsilon^*.$$

In [10, p. 86] it is shown that $\{z : |A(z)| < \varepsilon\} \subset U$ is the union of pairwise disjoint domains $R_1, R_2, R_3, \cdots$ with A mapping R_n biholomorphically onto the disc of radius ε about the origin. Also

$$R_n \subset \Delta(\gamma_n ; \eta) = \{z : \rho(z, \gamma_n) < \eta\}$$

where $\eta < (\delta - \eta)/(1 - \delta\eta)$. Thus choosing $\eta < \varepsilon^*$ implies

$$R_n \subset \Delta(\lambda_n ; \varepsilon^*) = L_{\lambda_n}(D_{\varepsilon^*}).$$

But U is a neighborhood of ψ. Therefore, for large λ, $|\beta_\lambda|^2 \in U$. In particular, $|\beta_\lambda|^2 \in R_{n(\lambda)}$. This means there exists $z_\lambda \in D_{\varepsilon^*}$ such that

$$L_{\gamma_{n(\lambda)}}(z_\lambda) = |\beta_\lambda|^2.$$

Then

$$|f_{n(\lambda)}'(z_\lambda)| = |A'(|\beta_\lambda|^2)| \frac{1 - |\gamma_{n(\lambda)}|^2}{|1 + \overline{\gamma_{n(\lambda)}} z_\lambda|^2} \geq \delta/4.$$

It follows that

$$|(f \circ L_\lambda(z))'(0)| \geq \frac{|1 + \bar\gamma_{n(\lambda)} z_\lambda|^2}{1 - |\gamma_{n(\lambda)}|^2} \cdot \frac{\delta}{4} \cdot \frac{\beta_\lambda + \bar\beta_\lambda}{2} \cdot 2(1 - |\beta_\lambda|^2).$$

Now $|z_\lambda| < \varepsilon^*$ and $|\beta_\lambda|^2 = L_{\gamma_{n(\lambda)}}(z_\lambda)$ gives

$$\frac{1 - |\beta_\lambda|^2}{1 - |\gamma_{n(\lambda)}|} \geq \frac{1 - \dfrac{|z_\lambda| + |\gamma_{n(\lambda)}|}{1 + |\gamma_{n(\lambda)}|\,|z_\lambda|}}{1 - |\gamma_{n(\lambda)}|} \geq \frac{1 - \varepsilon^*}{2}.$$

Therefore,

$$|(\hat{f} \circ L_\varphi(z))'(0)| \geq c^* > 0$$

which means that L_φ is one-one in a neighborhood of the origin.

THEOREM 4.2. *Suppose* $\{\beta_\lambda\}_{\lambda \in \Gamma} \to \psi = P(\psi)$ *and* Γ *is such that* $\{\bar{\beta}_\lambda\} \to \varphi$. *Then* $\{L_{\bar{\beta}_\lambda}(z)\} \to \varphi$ *for all* $z \in D$. *In particular, if* $\psi \in \partial(H_\infty(D))$ *(Šilov boundary of* $H_\infty(D)$*), then* $\varphi \in \partial(H_\infty(D))$.

Proof. If $\{L_{\bar{\beta}_\lambda}(z)\} \nrightarrow \varphi$ for some $z \in D$, then $\{L_{\bar{\beta}_\lambda}(z)\} \nrightarrow \varphi$ for all $z \in D \backslash \{0\}$. Thus there exists $f \in H_\infty(D)$ such that $\{f(\bar{\beta}_\lambda)\} \to r$ and $\{f(L_{\bar{\beta}_\lambda}(\frac{1}{2}))\} \to s$ where $r \neq s$. Let

$$F(z) = \overline{f(\bar{z})}.$$

Then $F \in H_\infty(D)$ and $\{F(\beta_\lambda)\} \to \bar{r}$ and $\{F(L_{\beta_\lambda}(\frac{1}{2}))\} \to \bar{s}$. This contradicts the fact that $\{L_{\beta_\lambda}(z)\} \to \psi$ for all $z \in D$. In [9, p. 179] it is shown that $\psi \in \partial(H_\infty(D))$ if and only if $\psi(B) \neq 0$ for every Blaschke product B. From the above construction, it is clear that $\varphi(B) \neq 0$ for every Blaschke product B; hence, $\varphi \in \partial(H_\infty(D))$. Notice that this procedure also gives the result that $\{\bar{\beta}_\lambda\}$ converges without taking a subnet.

Theorems 4.1 and 4.2 show how to construct a one-dimensional analytic disc in $M(H_\infty(D^n))$ which maps under π to a point in $M(H_\infty(D))^n$ with a trivial part. Moreover, we can choose this point to lie in the Šilov boundary $\partial(M(H_\infty(D))^n) = (\partial(H_\infty(D)))^n$.

V. A necessary condition

In the preceding sections we saw examples of analytic sets in $M(H_\infty(D^n))$ obtained as the limit of analytic mappings into D^n. We present here a necessary condition for a point of $M(H_\infty(D^n))$ to belong to an analytic set obtained in this manner. This is a modification of an argument employed in both [10] and [11].

THEOREM. *Let* F *be any non-constant map from* D^m *into* $M(H_\infty(D^n))$ *which lies in the closure of the set of analytic maps from* D^m *into* D^n. *Then each point in the range of* F *lies in the closure of the zero set of a function in* $H_\infty(D^n)$.

Proof. Let $\varphi = F(0)$ and $\{F_\lambda\}$ be a net of analytic maps from D^m into D^n such that $\lim_\lambda F_\lambda = F$. Then F is analytic and hence $F(D^m) \subset P(\varphi)$. Since F is non-constant, there exists $f \in H_\infty(D^n)$ such that $\hat{f}(\varphi) = 0$ and $\hat{f} \circ F \not\equiv 0$. Let

$$U = \{\psi \in M(H_\infty(D^n)) : |\hat{f}_j(\psi)| < \varepsilon, j = 1, \cdots, l \text{ and } \hat{f}_j(\varphi) = 0\},$$

where $f_j \in H_\infty(D^n)$. Then there exists r, $0 < r < 1$, and net index λ_0 such that

$$F_\lambda(D_r^m) \subset U \quad \text{for all } \lambda \geq \lambda_0$$

and

$$D_r^m = \{(z_1, \cdots, z_m) : |(z_1, \cdots, z_m)| < r\}.$$

Choose $(z_1^0, \cdots, z_m^0) \in D_r^m$ with $z_1^0 \neq 0$ and $\hat{f} \circ F(z_1^0, \cdots, z_m^0) \neq 0$. Let

$$T(z) = \left(z, \frac{z}{z_1^0} z_2^0, \cdots, \frac{z}{z_1^0} z_m^0\right)$$

and

$$V = \{z \in D : T(z) \in D_r^m\}.$$

Then V is an open connected subset of D with $f \circ F_\lambda \circ T$ converging uniformly to $\hat{f} \circ F \circ T$ on compact subsets of V. Then since $\hat{f} \circ F \circ T$ has a zero at 0 and is not identically zero, it must be that $f \circ F_\lambda \circ T$ has a zero on V for all sufficiently large indices λ. The image of these zeros are zeros of f and they lie in U which is a basic neighborhood of φ.

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