# ON THE CHAIN-COMPLEX OF A FIBRATION

BY

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It was proved in [7] that a Serre-fibration $\pi : E \to B$ with fibre $F$ can be replaced, as far as its singular complex is concerned, by a "twisted Cartesian Product" $B \times_\xi F$. In [1], [2], [9], [10] it was shown how, from this, the theorem of E. H. Brown on the structure of a suitable differential on $B \otimes F$ could be derived. Here, as throughout the present paper we use the same letter to denote a space, its singular complex and its *normalised* chain-complex.

At the time of these papers, the relevant algebraic ideas—in particular that of a coalgebra, a comodule and the cotensor product—were not well understood; due to this both the proofs given and the nature of the result obtained remained obscure. We hope to clarify these matters here.

The existence of the differential itself is established by a simple "perturbation argument", Chapter 3; essentially the same argument appears in [10]. Then, the $B$-comodule structure and the dual A-module structure, where $A$ denotes the group of the twisted Cartesian product $B \times_\xi F$, are investigated in Chapter 4. Here we follow the method of Weishu Shih [1]. The form of the differential given by E. H. Brown then follows from a simple, purely algebraic lemma, Chapter 2; also it follows that the appropriate chain complex for the fibration $E'$ induced by a map $\beta : B' \to B$ is the co-tensor product $B' \otimes^B (B \times_\xi F)$.

The result of Eilenberg and Moore, [4], namely $H(E') = \text{Cotor}^B\ (B', E)$ is now not hard to prove. We prove it here assuming only—as was done in [6]—that the action of $\pi_1(B)$ an the homology of $F$ is trivial. Chapter 5 merely summarises the necessary cohomological algebra from [4] and [6].

A point of notation: the symbol of any object also stands for the identity map on that object.

I am indebted to several conversations with John Moore.

## 1. Preliminaries

Let $R$ be a commutative ring with unit; $C^+$ denotes the category of positive complexes over $R$, i.e. the sequences

$$\to K_{n+1} \xrightarrow{d_{n+1}} K_n \xrightarrow{d_n} K_{n-1} \to \cdots \to K_1 \to K_0 \to 0$$

of $R$-modules and $R$-morphisms such that $d_n\, d_{n+1} = 0$. There is an evident forgetful functor $\square : C^+ \to G^+$, the category of positive graded $R$-modules;

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we shall also embed $G^+$ in $C^+$ by identifying an object of $G^+$ with the complex all of whose differentials are zero.

$(B, \psi, \varepsilon)$ will be called a coalgebra (over $C^+$) if $B$ is an object of $C^+$ and $\psi : B \to B \otimes B$, $\varepsilon : B \to R$ are maps of $C^+$ such that

$$(\varepsilon \otimes B)\psi = (B \otimes \varepsilon)\psi = B \quad \text{and} \quad (B \otimes \psi)\psi = (\psi \otimes B)\psi.$$

Here we assume the usual definitions of $B \otimes B \in C^+$, we regard $R \in C^+$ as the object $\to 0 \to 0 \to \cdots \to 0 \to R \to 0$ and we identify $B \otimes R$, $R \otimes B$ and $B$, as usual.

In the same way we apply the usual definitions of algebra, module, comodule, cf. [2], [4]. The following notations will be used throughout: the multiplication and unit of an algebra $A$ will be denoted by

$$\phi : A \otimes A \to A, \quad \eta : R \to A;$$

the operations of right or left modules will be denoted by

$$\tau : M \otimes A \to M, \quad \tau : A \otimes M \to M,$$

co-operations will be denoted by

$$\sigma : M \to M \otimes B, \quad \sigma : M \to B \otimes M$$

and $\phi$, $\psi$, $\tau$, $\sigma$ will be replaced by $\phi_A$, $\psi_{B'}$ $\tau_{M'}$ $\sigma_M$ when clarity will require it. The categories of right (left) $A$-modules or $B$-comodules (all over $C^+$) will be denoted by $\mathfrak{M}_A$, ${}_A\mathfrak{M}$, $\mathfrak{M}^B$, ${}^B\mathfrak{M}$ respectively. If $B'$, $B$ are coalgebras, ${}^{B'}\mathfrak{M}^B$ will denote the full subcategory of ${}^{B'}\mathfrak{M} \cap \mathfrak{M}^B$ for which

$$\begin{tikzcd} M \arrow[r, "\sigma"] \arrow[d, "\sigma'"] & M \otimes B \arrow[d, "\sigma' \otimes B"] \\ B' \otimes M \arrow[r, "B' \otimes \sigma"'] & B' \otimes M \otimes B \end{tikzcd}$$

is commutative. Similarly we define ${}_{A'}\mathfrak{M}_A$, $\mathfrak{M}_A^B$ etc.

We shall use the forgetful functor $\square$ for all the categories we have mentioned: if $X$ is one of our objects over $C^+$, $\square X$ is the same object over $G^+$, i.e. after the removal of the differentials.

If $M$ is any object of $C^+$ and $B$ a coalgebra, then $B \otimes M$ will denote the "extended co-module", cf. [4], [6], namely the object of ${}^B\mathfrak{M}$ with structure morphism $\psi_B \otimes M : B \otimes M \to B \otimes (B \otimes M)$. Similarly, for an algebra $A$ we define $M \otimes A \in \mathfrak{M}_A$.

## 2. Twisted objects

Throughout this chapter, let $(A, \phi, \eta)$, $(B, \psi, \varepsilon)$ denote an algebra, coalgebra over $C^+$.

2.1 Definition. An object $P$ of ${}^B\mathfrak{M}_A$ will be called a "principal twisted object" if $\square P = \square(B \otimes A)$; i.e. $P$ is $B \otimes A$ apart from the differentials. In particular, the structure morphisms are

$$\tau_P = B \otimes \phi_A : P \otimes A \to A, \quad \sigma_P = \psi_B \otimes A : P \to B \otimes P.$$

Thus, denoting the differential of $P$ by $D$, we shall refer to the principal twisted object $(B \otimes A, D)$.

The following notations will be used:

$$i_B : B \to B \otimes A, \quad j_A : B \otimes A \to A$$

are given by $i_B = B \otimes \eta_A$, $j_A = \varepsilon_B \otimes A$.

2.2 Proposition. *Let $(B \otimes A, D)$ be a principal twisted object. By $x : B \to A$ denote the composition*

$$B \xrightarrow{i_B} B \otimes A \xrightarrow{D} B \otimes A \xrightarrow{j_A} A.$$

*Then*

$$D = d_B \otimes A + B \otimes d_A + (B \otimes \phi)(B \otimes x \otimes A)(\psi \otimes A).$$

*Proof.* Since the differential is compatible with the structure morphisms we have

$$(1_*) \qquad D(B \otimes \phi) = (B \otimes \phi)(D \otimes A + B \otimes A \otimes d_A)$$

$$(1^*) \qquad (\psi \otimes A)D = (B \otimes D + d_B \otimes B \otimes A)(\psi \otimes A).$$

By $I : B \to A$ we denote the trivial map $\eta\varepsilon$. Note that

$$B \otimes A = (B \otimes \phi)(B \otimes I \otimes A)(\psi \otimes A).$$

Hence

$$\begin{aligned} D &= D(B \otimes \phi)(B \otimes I \otimes A)(\psi \otimes A) \\ &= (B \otimes \phi)(D \otimes A + B \otimes A \otimes d_A)(B \otimes I \otimes A)(\psi \otimes A) \end{aligned}$$

Here we note $\phi(A \otimes d_A + d_A \otimes A) = d_A \phi$ and $d_A I = 0$. Hence

$$\begin{aligned} D &= (B \otimes d_A \phi)(B \otimes I \otimes A)(\psi \otimes A) \\ &\qquad + (B \otimes \phi)(D \otimes A)(B \otimes I \otimes A)(\psi \otimes A) \\ (2_*) \qquad &= B \otimes d_A + (B \otimes \phi)(Di_B \otimes A). \end{aligned}$$

Dually, we obtain

$$(2^*) \qquad D = d_B \otimes A + (B \otimes j_A D)(\psi \otimes A)$$

whence, from $(2_*)$,

$$j_A D = d_A j_A + \phi(j_A D i_B \otimes A)$$

which we substitute in (2*):

$$\begin{aligned} D &= d_B \otimes A + (B \otimes (d_A j_A \phi(j_A D i_B \otimes A)))(\psi \otimes A) \\ &= d_B \otimes A + B \otimes d_A + (B \otimes \phi)(B \otimes x \otimes A)(\psi \otimes A), \quad \text{Q.E.D.} \end{aligned}$$

Note that $d_B \otimes A + B \otimes d_A = d_{B \otimes A}$ is exactly the differential of $B \otimes A$; the remaining term which "twists" the differential has a well-known form, cf. [1], [2]:

2.3 Definitions. Let $M \in \mathfrak{M}^B$, $N \in {}_A\mathfrak{M}$ and let $x : B \to A$ be a morphism. The composition

$$M \otimes N \xrightarrow{\sigma \otimes N} M \otimes B \otimes N \xrightarrow{M \otimes x \otimes N} M \otimes A \otimes N \xrightarrow{M \otimes \tau} M \otimes N$$

is called the "cap product" $x \cap : M \otimes N \to M \otimes N$. With this notation the conclusion of 2.2 can be written

$$D = d_{B \otimes A} + x\cap;$$

for this to be a differential, i.e. $DD = 0$, we must have

$$d_A x + x d_B + \phi(x \otimes x)\psi = 0,$$

cf. [2, 3.1]. In this case we call $x$ a "twisting cochain"; $d_{M \otimes N} + x\cap$ which we shall denote by $D^x$ is then a differential of $M \otimes N$. $M \otimes N$ with this differential will be denoted by $MxN \in C^+$; we call it a "twisted object". Note that $\square(MxN) = M \otimes N$.

2.4* Proposition. *The unique differential on* $M \otimes N$ *which makes*

$$(MxA) \otimes N \xrightarrow{M \otimes \tau} M \otimes N$$

*into a chain-map is* $D^x = d_{M \otimes N} + x\cap$.

*Proof.* This is immediate from the commutative diagram

$$\begin{tikzcd}
M \otimes A \otimes N \arrow[r, "\sigma \otimes A \otimes N"] \arrow[d, "M \otimes \tau"] & M \otimes B \otimes A \otimes N \arrow[r, "M \otimes x \otimes A \otimes N"] \arrow[d, "M \otimes B \otimes \tau"] & M \otimes A \otimes A \otimes N \arrow[r, "M \otimes \phi \otimes N"] \arrow[d, "M \otimes A \otimes \tau"] & M \otimes A \otimes N \arrow[d, "M \otimes \tau"] \\
M \otimes N \arrow[r, "\sigma \otimes N"'] & M \otimes B \otimes N \arrow[r, "M \otimes x \otimes N"'] & M \otimes A \otimes N \arrow[r, "M \otimes \tau"'] & M \otimes N.
\end{tikzcd}$$

2.4* PROPOSITION. *The unique differential on* $M \otimes N$ *which makes*

$$M \otimes N \xrightarrow{\sigma \otimes N} M \otimes (BxN)$$

*into a chain map is* $D^x = d_{M \otimes N} + x\cap$.

2.5* DEFINITION (cf. 2.1* in [3] or 2 in [4]). Let $M \in \mathfrak{M}^B$ and $L \in {}^B\mathfrak{M}$. The kernel in $C^+$ of the map

$$\sigma_M \otimes N - M \otimes \sigma_N : M \otimes N \rightarrow M \otimes B \otimes N$$

is called the *cotensor* product of $M$, $L$ over $B$ and denoted by $M \otimes^B N$ (or $M \square_B N$ in [4]). The sequence

$$0 \rightarrow M \xrightarrow{\sigma} M \otimes B \xrightarrow{\sigma \otimes B - M \otimes \psi} M \otimes B \otimes B$$

is split-exact (cf. 2.2* in [3]) and hence so is the sequence

$$0 \rightarrow M \otimes N \xrightarrow{\sigma \otimes N} M \otimes B \otimes N$$
$$\xrightarrow{\sigma \otimes B \otimes N - M \otimes \psi \otimes N} M \otimes B \otimes B \otimes N.$$

Hence, by 2.4* we have

2.6* PROPOSITION. *With the notations of* 2.3, $MxN = M \otimes^B (BxN)$.

The definition dual to 2.5* defining the tensor product over an algebra $A$ is classical.

2.6$_*$ PROPOSITION. *With the notations of* 2.3, $MxN = (MxA) \otimes_A N$.

Using well-known associativity properties we get

2.7 PROPOSITION. $MxN = M \otimes^B (BxA) \otimes_A N$.

## 3. A perturbation lemma

Let $X$, $Y$ be objects of $G^+$ and $f = \{f_n : X_n \rightarrow Y_{n+p}\}$ be a sequence of maps; we call $p$ the *grading* of $f$; if $f, g : X \rightarrow Y$ are maps of grading $p$, $q$ respectively, we define the commutator $[f, g]$ as $fg + (-1)^{pq+1}gf$.
Now, let $M$, $N$ be objects of $C^+$ and let

$$M \underset{f}{\overset{\nabla}{\rightleftarrows}} N$$

be chain maps such that

$$f\nabla = M, \quad \nabla f = N + d\Phi + \Phi d$$

where $\Phi : N \rightarrow N$ is a chain homotopy (i.e. a map of grading $+1$) satisfying

$$\Phi\nabla = 0, \quad f\Phi = 0, \quad \Phi\Phi = 0.$$

*Let* $D : N \rightarrow N$ be a (second) differential on $N$; we shall discuss the problem of modifying $\nabla, f$ and the differential on $M$ so as to obtain a chain equivalence between $M$ and $N$ with these new differentials. We write

$$D - d = t : N \to N;$$

then, since $D^2 = d^2 = 0$ we must have $[d, t] + t^2 = 0$. Next, we define

$$t_1 = t, \quad t_{n+1} = t\Phi t_n \quad (n \geq 1)$$

so that

$$t_n = t\Phi \cdots \Phi t$$

with $(n - 1)$ $\Phi$'s and $n$ $t$'s. Also, we write

$$\sum_n = t_1 + \cdots + t_n,$$

and we denote by $I_n$ the ideal of operators which contain $t$ at least $n$ times. Note that $t_n \in I_n$.

3.1 Lemma.

$$[d, \sum_n] + \sum_n \nabla f \sum_n \in I_{n+1} \quad (n \geq 1).$$

*Proof.* For $n = 1$, $\sum_n = t$ and $[d, t] + t \nabla ft = -t^2 + t \nabla ft \in I_2$. We continue by induction.

$$\begin{aligned} \sum_{n+1} \nabla f \sum_{n+1} &= \sum_{1 \leq i \leq n} t_{i+1} \nabla f \sum_n + t \nabla f \sum_{n+1} + \sum_{n+1} \nabla f t_{n+1} \\ &= t\Phi \sum_n \nabla f \sum_n + t \nabla f \sum_{n+1} + \sum_{n+1} \nabla f t_{n+1}. \end{aligned}$$

Now, by the inductive hypothesis,

$$\sum_n \nabla f \sum_n \equiv -[d, \sum_n] \mod I_{n+1}$$

and hence, calculating from now on mod $I_{n+2}$

$$\begin{aligned} \sum_{n+1} \nabla f \sum_{n+1} &\equiv -t\Phi[d, \sum_n] + t \nabla f \sum_{n+1} \\ &\equiv -t\Phi[d, \sum_n] + t \nabla f \sum_n. \end{aligned}$$

Now, $t \nabla f \sum_n = t(N + d\Phi + \Phi d) \sum_n$ whence

$$\sum_{n+1} \nabla f \sum_{n+1} \equiv t \sum_n - t\Phi \sum_n d + t\, d\Phi \sum_n.$$

Also,

$$\begin{aligned} [d, \sum_{n+1}] &= [d, t] + [d, t\,\Phi \sum_n] \\ &= [d, t] + dt\Phi \sum_n + t\Phi \sum_n d \end{aligned}$$

whence

$$\begin{aligned} [d, \textstyle\sum_{n+1}] + \sum_{n+1} \nabla f \sum_{n+1} \\ &\equiv t \sum_n + [d, t]\{N + \Phi \sum_n\} \\ &= t^2 + t^2\Phi \sum_{n-1} + [d, t] + [d, t]\Phi \sum_{n-1} + [d, t]\Phi t_n \\ &= [d, t]\Phi t_n \\ &= -t^2\Phi t_n \in I_{n+2} \end{aligned}$$

because $[d, t] + t^2 = 0$, Q.E.D.

We now define

$$D_1 = d : M \to M, \quad \nabla_1 = \nabla, \quad f_1 = f, \quad \Phi_1 = \Phi$$

and

$$D_{n+1} = D_n + f t_n \nabla = d + f \textstyle\sum_n \nabla$$

$$\nabla_{n+1} = \nabla_n + \Phi t_n \nabla = \nabla + \Phi \textstyle\sum_n \nabla$$

$$f_{n+1} = f_n + f t_n \Phi = f + f \textstyle\sum_n \Phi$$

$$\Phi_{n+1} = \Phi_n + \Phi t_n \Phi = \Phi + \Phi \textstyle\sum_n \Phi.$$

From 3.1 we now trivially deduce

3.2 Lemma.

$$D_n D_n \in I_n, \quad D \nabla_n - \nabla_n D_n \in I_n, \quad D_n f_n - f_n D \in I_n,$$

$$f_n \nabla_n = M, \quad \nabla_n f_n - N - D\Phi_n - \Phi_n D \in I_n,$$

$$\Phi_n \nabla_n = 0, \quad f_n \Phi_n = 0, \quad \Phi_n \Phi_n = 0.$$

## 4. The twisted Eilenberg Zilber theorem

Letters such as $A$, $B$, $E$, $F$ will stand, ambiguously for either simplicial sets (complete semisimplicial complexes) or the corresponding *normalised* chain-complexes over the ring $R$; these are of course, objects of $C^+$. The Eilenberg Zilber theorem then provides natural maps

$$B \otimes F \underset{f}{\overset{\nabla}{\rightleftarrows}} B \times F$$

where $B \times F$, $B \otimes F$ denote the Cartesian product and the tensor product respectively. There is also a chain-homotopy $\Phi : B \times F \to B \times F$; and the maps $\nabla$, $f$, $\Phi$ now have exactly the formal properties discussed in chapter 3 above. In this case we shall write

$$\nabla = \nabla(B, F), \quad f = f(B, F), \quad \Phi = \Phi(B, F)$$

when necessary.

We shall have to make considerable use of the associativity properties of these maps:

4.0 Lemma (cf. [1] for a proof). *The following diagrams are commutative*:

$$\begin{array}{ccc} A \otimes B \otimes C & \xrightarrow{\nabla(A,B) \otimes C} & A \times B \otimes C \\ \Big\downarrow{\scriptstyle A \otimes \nabla(B,C)} & & \Big\downarrow{\scriptstyle \nabla(A \times B, C)} \\ A \otimes B \times C & \xrightarrow[\nabla(A, B \times C)]{} & A \times B \times C \end{array}$$

$$\begin{tikzcd}
A \times B \times C \arrow[r, "f(A \times B, C)"] \arrow[d, "f(A, B \times C)"] & A \times B \otimes C \arrow[d, "f(A, B) \otimes C"] \\
A \otimes B \times C \arrow[r, "A \otimes f(B, C)"'] & A \otimes B \otimes C
\end{tikzcd}$$

$$\begin{tikzcd}
A \times B \otimes C \arrow[r, "\nabla(A \times B, C)"] \arrow[d, "f(A, B) \otimes C"] & A \times B \times C \arrow[d, "f(A, B \times C)"] \\
A \otimes B \otimes C \arrow[r, "A \otimes \nabla(B, C)"'] & A \otimes B \times C
\end{tikzcd}$$

$$\begin{tikzcd}
A \times B \times C \arrow[r, "f(A, B \times C)"] \arrow[d, "\Phi(A \times B, C)"] & A \otimes B \times C \arrow[d, "A \otimes \Phi(B, C)"] \\
A \times B \times C \arrow[r, "f(A, B \times C)"'] & A \otimes B \times C
\end{tikzcd}$$

$$\begin{tikzcd}
A \times B \otimes C \arrow[r, "\nabla(A \times B, C)"] \arrow[d, "\Phi(A, B) \otimes C"] & A \times B \times C \arrow[d, "\Phi(A, B \times C)"] \\
A \times B \otimes C \arrow[r, "\nabla(A \times B, C)"'] & A \times B, C
\end{tikzcd}$$

A Serre-fibration $\pi : E \to B$ with fibre $F$ can always be replaced by a "twisted Cartesian product" $E = B \times_\xi F$ which is defined as follows, cf. [2], [7]:

4.1 DEFINITION. Let $B$, $F$ be simplicial sets and $A$ a simplicial group acting on $F$. By $B \times_\xi F$, where $\xi : B_n \to A_{n-1}$ is a given "twisting function" we denote the simplicial set defined as follows:

(i) As a set, $(B \times_\xi F)_n$ is the Cartesian product $B_n \times F_n$

(ii) The degeneracy operators are given by

$$s_i(b, f) = (s_i b, s_i f).$$

(iii) The face operators are given by

$$\begin{aligned} \partial_i(b, f) &= (\partial_i b, \partial_i f), \quad i < n \\ &= (\partial_n b, \xi(b) \cdot \partial_n f), \quad i = n \end{aligned}$$

if $b \in B_n$, $f \in F_n$. Here $\xi$ has to satisfy two identities, namely

(iv) $\xi(\partial_n b) \cdot \partial_{n-1} \xi(b) = \xi(\partial_{n-1} b)$, $b \in B_n$

(v) $\xi(s_n b) =$ the unit of $A_n$ if $b \in B_n$.

*Remark.* Here we give preferred treatment to the last face operator, as is done in [8], and not to the first, as in [7]. This turns out to be the appropriate thing to do if we want to represent our fibration as Base $\times$ Fibre rather than Fibre $\times$ Base. In order to use the theory of Chapter 3, we shall denote the differential of $B \times_\xi F$ by $d_\xi$ and that of $B \times F$ (untwisted) by $d$. Then, writing $t = d_\xi - d$,

$$t(b, f) = (\partial_n(b), (\xi(b) - 1)\cdot\partial_n f)), \quad b \in B_n, f \in F_n.$$

We now introduce the usual "Serre-filtrations" namely $(b, f) \in B \times F$ has filtration $\leq p$ if $b$ is the degeneration of an element of $B_p$; and

$$F_p(B \otimes F) = \textstyle\sum_{i \leq p} B_i \otimes F.$$

4.2 LEMMA.

(i) $\nabla(B, F), f(B, F), \Phi(B, F)$ *are filtration-preserving.*
(ii) $tF_p(B \times F) \subset F_{p-1}(B \times F)$

Property (i) is well known (and trivial). Properly (ii) follows easily from (v) in 4.1. Hence, using the notation of Chapter 3, an operator belonging to $I_n$ will reduce filtration at least by $n$; and therefore will be zero when applied to an element of grading $< n$. The operators $D_n$, $\nabla_n$, $f_n$, $\Phi_n$ will therefore, by 3.2, converge as $n \to \infty$, to operators $D_\xi$, $\nabla_\xi$, $f_\xi$, $\Phi_\xi$. It will be convenient to denote $B \otimes F$ with the differential $D_\xi$ by $B \otimes_\xi F \in C^+$. We summarise our result:

4.3 LEMMA.

$$B \otimes_\xi F \underset{f_\xi}{\overset{\nabla_\xi}{\rightleftarrows}} B \times_\xi F$$

*are chain-maps satisfying*

$$f_\xi \nabla_\xi = B \otimes F, \quad \nabla_\xi f_\xi = B \times F + d_\xi \Phi_\xi + \Phi_\xi d_\xi$$

*where* $\Phi_\xi$ *is a chain-homotopy satisfying*

$$\Phi_\xi f_\xi = 0, \quad f_\xi \Phi_\xi = 0, \quad \Phi_\xi \Phi_\xi = 0.$$

Now, let $\beta : B' \to B$ be a map of simplicial sets; the twisted Cartesian product $B \times_\xi F$ induces the twisted Cartesian product $B' \times_{\xi'} F$ where $\xi' = \xi\beta$; this takes the place of the induced fibration:

$$\begin{tikzcd} B' \times_{\xi'} F \arrow[r, "\beta \times F"] \arrow[d, "\pi'"] & B \times_\xi F \arrow[d, "\pi"] \\ B' \arrow[r, "\beta"] & B \end{tikzcd}$$

where $\pi(b, f) = b$, $\pi'(b', f) = b'$. Note that the map $\beta \times F$ is "untwisted". It is evident that the operators of 4.3 are natural in relation to this construction. We shall adhere to these notations for the rest of this chapter.

With the simplicial set $B$ we associate the diagonal map

$$B \xrightarrow{\delta_B} B \times B, \quad b \mapsto (b, b).$$

This turns $B \in C^+$ into a coalgebra, using the usual augmentation and

$$\psi_B : B \to B \otimes B$$

being defined as the composition

$$B \xrightarrow{\delta_B} B \times B \xrightarrow{f(B, B)} B \otimes B.$$

More generally, we have the simplicial maps

$$\sigma' = (B' \times \beta)\delta_{B'} : B' \to B' \times B$$

and

$$\sigma' \times F : B' \times_\xi F \to B' \times (B \times_\xi F)$$

where, again, we note that the last map is "untwisted". From these maps we form the compositions

$$\sigma = f(B', B)\sigma' : B' \to B' \otimes B$$

which gives $B'$ the structure of a right $B$-comodule and

$$\lambda_1 = f(B', B \times F)(\sigma' \times F) : B' \times_\xi F \to B' \otimes (B \times_\xi F).$$

This is a chain map because $f(B', B \times F) = f(B', B \times_{\xi'} F)$; which is true because $f(X, Y)$ depends on the operators $\partial_0$ in $Y$ only—and they are the same for $B \times F$ and $B \times_\xi F$.

In the light of 4.3 we want to compare $\lambda_1$ with the map

$$\lambda_2 = \sigma \otimes F : B' \otimes_{\xi'} F \to B' \otimes (B \otimes_\xi F),$$

The first question which must be answered is whether $\lambda_2$ is a chain-map at all:

4.4* Lemma. $(d_{B'} \otimes B \otimes F + B' \otimes D_\xi)\lambda_2 = \lambda_2 D_{\xi'}$.

Next, to show the "equivalence" of $\lambda_1$ and $\lambda_2$ we must prove

4.5* Lemma. *The following diagrams are commutative:*

$$\begin{array}{ccc}
B' \otimes_{\xi'} F & \xrightarrow{\nabla_{\xi'}} & B' \times_{\xi'} F \\
\Big\downarrow{\scriptstyle \lambda_2} & & \Big\downarrow{\scriptstyle \lambda_1} \\
B' \otimes (B \otimes_\xi F) & \xrightarrow[B' \otimes \nabla_\xi]{} & B' \otimes B \times_\xi F
\end{array}$$

$$\begin{tikzcd}
B' \otimes_{\xi'} F \arrow[d, "\lambda_2"] & B' \times_{\xi'} F \arrow[l, "f_{\xi'}"'] \arrow[d, "\lambda_1"] \\
B' \otimes (B \otimes_{\xi} F) & B' \otimes B \times_{\xi} F \arrow[l, "B' \otimes f_{\xi}"]
\end{tikzcd}$$

$$\begin{tikzcd}
B' \times_{\xi'} F \arrow[r, "\Phi_{\xi'}"] \arrow[d, "\lambda_1"] & B' \times_{\xi'} F \arrow[d, "\lambda_1"] \\
B' \otimes B \times_{\xi} F \arrow[r, "B' \otimes \Phi_{\xi}"'] & B' \otimes B \times_{\xi} F
\end{tikzcd}$$

*Proof of* 4.4*. In the untwisted case—when $\xi(b)$ is always the unit and $B \times_{\xi} F = B \times F$, $D = d_{B \otimes F}$—the statement is true. Thus, writing

$$T = D_{\xi} - d_{B \otimes F} \quad \text{and} \quad T' = D_{\xi'} - d_{B' \otimes F'}$$

we must prove $(B' \otimes T)\lambda_2 = \lambda_2 T'$. Now, since in the notation of chapter 3, $T = \sum_{n=1}^{\infty} y_n$ where $y_n = f(B, F)t_n \nabla (B, F)$, and an analogous notation for $T'$, it suffices to prove

$$(B' \otimes y_n)\lambda_2 = \lambda_2 y_n'$$

for all $n$. Consider the diagram

$$\begin{tikzcd}
B' \otimes F \arrow[rr, "\sigma' \times F"] \arrow[dr, "\nabla(B', F)"] \arrow[ddd, "y_n'"'] \arrow[drr, phantom, "(1)"] & & B' \times B \otimes F \arrow[rr, "f(B', B) \otimes F"] \arrow[d, "\nabla(B' \times B, F)"] \arrow[drr, phantom, "(2)"] & & B' \otimes B \otimes F \arrow[dl, "B' \otimes \nabla(B, F)"'] \arrow[ddd, "B' \otimes y_n"] \\
{} \arrow[d, phantom, "(5)"] & B' \times F \arrow[r, "\sigma' \times F"] \arrow[d, "t_n'"] & B' \times B \times F \arrow[r, "f(B', B \times F)"] & B' \otimes B \times F \arrow[d, "B' \otimes t_n"'] & {} \arrow[d, phantom, "(6)"] \\
{} & B' \times F \arrow[r, "\sigma' \times F"'] \arrow[dl, "f(B', F)"] \arrow[dr, phantom, "(3)"] & B' \times B \times F \arrow[r, "f(B', B \times F)"'] \arrow[d, "f(B' \times B, F)"] \arrow[drr, phantom, "(4)"] & B' \otimes B \times F \arrow[dr, "B' \otimes f(B, f)"] & {} \\
B' \otimes F \arrow[rr, "\sigma' \otimes F"'] & & B' \otimes B \otimes F \arrow[rr, "f(B', B) \otimes F"'] & & B' \otimes B \otimes F
\end{tikzcd}$$

In this diagram, (1) and (3) are commutative due to naturality; (2) and (4) by 4.0; and (5) and (6) by definition. Hence the outer rectangle will commute—and that is the required result—if the inner one does. Now, since

$$t_{n+1} = t\Phi(B, F)t_n, \quad t_{n+1}' = t'\Phi(B', F, t')t_n',$$

it remains to prove that the inner rectangle will commute if $t_n'$, $t_n$ are replaced either by $t'$, $t$ or by $\Phi(B', F)$, $\Phi(B, F)$. Now, the first of these assertions is $(B' \otimes t)\lambda_1 = \lambda_1 t'$ and this follows—cf. the remarks after the definition of $\lambda_1$—

because

$$(B' \otimes d)\lambda_1 = \lambda_1 d \quad \text{and} \quad (B' \otimes d_\xi)\lambda_1 = \lambda_1 d_{\xi'}$$

are *both* true. For the second assertion, consider the diagram

$$\begin{array}{ccccc} B' \times F & \xrightarrow{\sigma' \times F} & B' \times B \times F & \xrightarrow{f(B', B \times F)} & B' \otimes B \times F \\ \downarrow{\scriptstyle \Phi(B',F)} & (7) & \downarrow{\scriptstyle \Phi(B' \times B, F)} & (8) & \downarrow{\scriptstyle B' \otimes \Phi(B,F)} \\ B' \times F & \xrightarrow[\sigma' \times F]{} & B' \times B \times F & \xrightarrow[f(B', B \times F)]{} & B' \otimes B \times F \end{array}$$

where (7) commutes due to naturality, and (8) by 4.0. This completes the proof of 4.4*.

The proofs of 4.5* are analogous.

If, in Definition 4.1 we replace $F$ by the simplicial group $A$—operating on itself by the group-action—we obtain the definition of the associated *principal* twisted Cartesian product, cf. [7], denoted by $B \times_\xi A$. We now consider the simplicial maps

$$\tau' : A \times F \to F$$

i.e. the action of $A$ on $F$, and

$$B \times \tau' : (B \times_\xi A) \times F \to B \times_\xi F.$$

From these maps we form the compositions

$$\tau = \tau' \nabla (A, F) : A \otimes F \to F$$

which gives $F$ the structure of a left $A$-module and

$$\mu_1 = (B \times \tau') \nabla (B \times A, F) : (B \times_\xi A) \otimes F \to B \times_\xi F.$$

This is a chain map, because $\nabla(B \times A, F) = \nabla(B \times_\xi A, F)$; which is true because $\nabla(X, Y)$ depends on degeneracy operators only—and they are the same for $B \times F$ and $B \times_\xi F$.

In the light of 4.3 we want to compare $\mu_1$ with the map

$$\mu_2 = (B \otimes \tau) : (B \otimes_\xi A) \otimes F \to B \otimes_\xi F.$$

It is now clear that the situation is dual to that which leads to Lemmas 4.4* 4.5*.

4.4$_*$ LEMMA. *$\mu_2(D_\xi \otimes F + B \otimes A \otimes d_F) = D_\xi \mu_2$ where $D_\xi$ denotes the differentials of $B \otimes_\xi A$ and $B \otimes_\xi F$.*

4.5$_*$ LEMMA. *The following diagrams are commutative:*

$$\begin{array}{ccc} (B \otimes_\xi A) \otimes F & \xrightarrow{\nabla_\xi \otimes F} & B \times_\xi A \otimes F \\ \downarrow{\scriptstyle \mu_2} & & \downarrow{\scriptstyle \mu_1} \\ B \otimes_\xi F & \xrightarrow[\nabla_\xi]{} & B \times_\xi F \end{array}$$

$$\begin{tikzcd}
(B \otimes_\xi A) \otimes F \arrow[d, "\mu_2"] & B \times_\xi A \otimes F \arrow[l, "f_\xi \otimes F"'] \arrow[d, "\mu_1"] \\
B \otimes_\xi F & B \times_\xi F \arrow[l, "f_\xi"]
\end{tikzcd}$$

$$\begin{tikzcd}
B \times_\xi A \otimes F \arrow[r, "\Phi_\xi \otimes F"] \arrow[d, "\mu_1"] & B \times_\xi A \otimes F \arrow[d, "\mu_1"] \\
B \times_\xi F \arrow[r, "\Phi_\xi"'] & B \times_\xi F
\end{tikzcd}$$

The proofs are dual to those of $4.4^*$, $4.5^*$.

Let us now substitute $F = A$ and $\tau =$ the multiplication of $A$ in $4.4_*$ ; and $B' = B, \beta = B, \xi' = \xi$ in $4.4^*$. The lemmas then assert, cf. 2.1, in particular $1_*$ and $1^*$ in the proof, that $B \otimes_\xi A$ is a principal twisted object. It follows, from 2.2, 2.3 that $D_\xi = d_{B \otimes A} + x \cap$ where $x = x(\xi) = j_A D_\xi i_B$. In the notation of 2.3 this can be written succinctly as $B \otimes_\xi A = Bx(\xi)A$.

From this result, $2.4_*$ and $4.4_*$ we deduce $B \otimes_\xi F = Bx(\xi)F$ which is the theorem of E. H. Brown, cf. [1], [2].

Finally, from $4.4^*$, $2.4^*$ and $2.6^*$ we deduce that the chain-complex of the induced fibration is given by the cotensor-product over $B$:

$4.6^*$ PROPOSITION. $B' \otimes_{\xi'} F = B' \otimes^B (B \otimes_\xi F)$.

Dually, we have a proposition relating the chain-complex of a twisted Cartesian product with that of the associated principal one:

$4.6_*$ PROPOSITION. $B \otimes_\xi F = (B \otimes_\xi A) \otimes_A F$.

## 5. Cotor

5.1. An element of ${}^B\mathfrak{M}$ is called *injective* if it is a direct summand—in ${}^B\mathfrak{M}$—of an "extended comodule" $B \otimes F$, $F \in C^+$.

5.2. By $C^-(B)$ we denote the category of negative complexes

$$X : X^0 \to X^1 \to \cdots \to X^n \to X^{n+1} \to \cdots$$

consisting of objects and morphisms in ${}^B\mathfrak{M}$. They are called "negative" because of the usual notation $X^i = X_{-i}$, cf. [4]. We embed ${}^B\mathfrak{M}$ in $C^-(B)$ by regarding $M \in {}^B\mathfrak{M}$ as $M \to 0 \to 0 \to \cdots$. A map $\varepsilon : M \to X$ of $C^-(B)$ is called a *resolution* of $M$ if

(i) $0 \to M \to X^0 \to X^1 \to \cdots$ is split exact in $C^+$
(ii) $X^i$ $(i \geq 0)$ is injective, cf. 5.1 above.

The two notions employed—namely what sort of sequences are considered as exact and what are the injectives—are not independent; on the contrary, each determines the other, cf. [6].

5.3. The canonical resolution $M \to U_B(M)$, cf. [4], [6], is defined as follows.

$$0 \to M \xrightarrow{\varepsilon} B \otimes M = U_B(M)^0 \quad \text{with } \varepsilon = \sigma_M .$$

We let $\bar{U}_B(M)^1 = \operatorname{coker} \varepsilon$—note that ${}^B\mathfrak{M}$ has cokernels—and define

$$U_B(M)^1 = B \otimes \bar{U}_B(M)^1.$$

Inductively, $\bar{U}_B(M)^{n+1}$ is defined by the exact sequence

$$0 \to \bar{U}_B(M)^n \xrightarrow{\sigma} B \otimes \bar{U}_B(M)^n \to \bar{U}_B(M)^{n+1} \to 0,$$

with $U_B(M)^{n+1} = B \otimes \bar{U}_B(M)^{n+1}$. Since the above sequence splits in $C^+$, we get a resolution. Note that, $\square$ again denoting the functor which forgets the differentials of ${}^B\mathfrak{M}$, if $M, N \in {}^B\mathfrak{M}$ and $\square M = \square N$, then $\square U_B(M) = \square U_B(N)$.

5.4. Let $M \in {}^B\mathfrak{M}$. A filtration $F_p M$ on $M$ in $C^+$ is called a $B$-filtration provided $F_{-1} M = 0$, $M = \bigcup_{0 \leq p} F_p M$ and $\sigma_M : M \to B \otimes M$ is filtration-preserving, where $B \otimes M$ has the tensor product filtration and $B$ the skeleton filtration; cf. 6.3 in [6].

5.5. It is clear that a $B$-filtration on $M$ can be extended in a canonical way to a filtration on $U_B(M)$, i.e. a filtration on each $U_B(M)^i$ so that

$$U_B(M)^i \to U_B(M)^{i+1}$$

is filtration-preserving.

5.6. By $C$ we denote the category of complexes

$$\cdots \to K_{n+1} \to K_n \to K_{n-1} \to \cdots$$

over $R$. Let $L \in \mathfrak{M}^B$, $M \in {}^B\mathfrak{M}$ and $M \to X$ be a resolution of $M$. We define $\hat{W}(L, X) \in C$ as follows:

$$\hat{W}(L, X)_n = \textstyle\prod_{i=0}^{\infty} (L \otimes^B X^i)_{n+i}$$

where $\prod$ denotes the product and not, beware, the coproduct (sum).

$$d_n \mid (L \otimes^B X^i)_{n+i}$$

is defined by $d = d' + d''$ where

$$d' : (L \otimes^B X^i)_{n+i} \to (L \otimes^B X^i)_{n+i-1}$$

is induced by the differentials in $L$, $X^i$, and

$$d'' : (L \otimes^B X^i)_{n+i} \to (L \otimes^B X^{i+1})_{n+i}$$

is defined by taking the map

$$(-1)^p L_p \otimes \{X^i \to X^{i+1}\} \quad \text{on } L_p \otimes X^i.$$

Denoting the homology-functor by $H$ we define

$$\operatorname{Cotor}^B (L, M) = H\hat{W}(L, X).$$

For the usual invariance proofs, as well as the fact that we could just as well have resolved $L$, or both $L$ and $M$, see [4], [6]. Cotor is bigraded in the sense that

$$\operatorname{Cotor}_n^B (L, M) = H_n \hat{W}(L, X) = \textstyle\prod_{p+q=n} \operatorname{Cotor}_{p,q}^B (L, M)$$

where $\operatorname{Cotor}_{p,q}^B (L, M)$ consists of those elements represented by cycles in $(L \otimes X^{-p})_q$. Thus $\operatorname{Cotor}_{p,q}^B (L, M) = 0$ unless $p \leq 0$, $q \geq 0$.

5.7. $B$ will be said to be connected if $\varepsilon : B_0 \to R$ is an isomorphism; in that case it is easy to deduce—from the canonical resolution—that $\operatorname{Cotor}_n^B (L, M) = 0$ if $n < 0$; cf. [4, 6.2].

5.8. If $M$ is injective, we can use $0 \to M \to M \to 0$ as the resolution of $M$, and we get $\operatorname{Cotor}^B (L, M) = H\hat{W}(L, M) = H(L \otimes^B M)$.

5.9. Let $\phi : B_1 \to B_2$ be a morphism of coalgebras, let $L_i \in \mathfrak{M}^{B_i}$, $M_i \in {}^{B_i}\mathfrak{M}$ $(i = 1, 2)$ and let $f : L_1 \to L_2$, $g : M_1 \to M_2$ be $\phi$-morphisms; assume further that $B_1, B_2, M_1, M_2$ are $R$-flat. If

$$H(\phi) : H(B_1) \to H(B_2), \quad H(f) : H(L_1) \to H(L_2), \quad H(g) : H(M_1) \to H(M_2)$$

are isomorphisms, then so is the induced morphism.

$$\operatorname{Cotor}^\phi (f, g) : \operatorname{Cotor}^{B_1} (L_1, M_1) \to \operatorname{Cotor}^{B_2} (L_2, M_2),$$

Cf. Theorem 7.1 in [4].

## 6. 1-trivial modules

6.1 Definition. $M \in {}^B\mathfrak{M}$ will be called $n$-trivial if:

(i) $\square M = \square (B \otimes F)$ with $F \in C^+$;

(ii) If $F_p$ denotes the filtration of $B \otimes F$ by $B$-degree (i.e. $F_p(B \otimes F) = \sum_{i=0}^{p} B_i \otimes F$), then $(d_M - d_{B\otimes F})F_p \subset F_{p-n-1}$.

We shall prove that if $n \geq 1$, $n$-trivial modules behave, to some extent, like injective ones, cf. 5.8:

6.2 Theorem. *Let $M \in {}^B\mathfrak{M}$ be 1-trivial and $L \in \mathfrak{M}^B$. Then*

$$H(L \otimes {}^B M) = \operatorname{Cotor}^B (L, M).$$

*Proof.* $M$ is filtered as in 6.1; we filter $L$ by

$$F_p L = \sigma_L^{-1}(L \otimes F_p B).$$

We filter $L \otimes {}^B M$ by $L$-degree, noting $\square (L \otimes {}^B M) \subset \square (L \otimes F)$. The filtration on $M$ is a $B$-filtration and can be extended to $U_B(M)$, cf. 5.5. The filtrations can now, writing

$$F_p \hat{W} = \textstyle\prod_{q+r-i \leq p} F_q L \otimes^B F_r U_B(M)^i,$$

be extended to $\hat{W}(L, U_B(M))$. Now, for brevity, let us write $U_B(M) = U$, $U_B(B \otimes F) = U_\otimes$; and let us call the differentials of $\hat{W}(L, U)$, $\hat{W}(L, U_\otimes)$

$D$, $D_{\otimes}$ respectively. Then, cf. 5.3, $\square U = \square U_{\otimes}$ and from

$$(d_M - d_{B\otimes F})F_p \subset F_{p-2}$$

we easily deduce

$$(*) \qquad (D - D_{\otimes})F_p \subset F_{p-2}.$$

Now, consider the diagram

$$\begin{tikzcd} \hat{W}(L, B \otimes F) \arrow[r, "i"] \arrow[d, "u"] & \hat{W}(L, M) \arrow[d, "v"] \\ \hat{W}(L, U_{\otimes}) \arrow[r, "j"] & \hat{W}(L, U) \end{tikzcd}$$

where $i$, $j$ are the identities and $u$, $v$ are induced by the resolutions. $i$, $j$ do *not* commute with the differentials. But due to $(*)$ above, in the spectral sequences induced by our filtrations, $E^2(i)$, $E^2(j)$ exist and are isomorphisms. Since $\hat{W}(L, B \otimes F) = L \otimes F$ we see that the spectral sequences on the left collapse and $E^2(u)$ is an isomorphism; indeed both terms are $H(L \otimes F)$ Hence $E^2(v)$ is an isomorphism.

But, the filtrations we have introduced are bicomplete, cf. [5], hence $H(v)$ is an isomorphism; since $\hat{W}(L, M) = L \otimes^B M$, our theorem is proved.

6.3 *Application.* Suppose $\pi : E \to B$ is a Serre-filtration in which the action of the fundamental group of $B$ on the homotopy of the fibre is trivial; then $E$ can be replaced by $B \times_{\xi} F$ where $\xi : B_1 \to A_0$ is trivial. It follows, cf. Chapter 4 above, that $x(\xi) : B_1 \to A_0$ is trivial. Hence, as is easily seen (cf. [2, 10.3])

$$(D^{x(\xi)} - d_{B\otimes F})F_p \subset F_{p-2}$$

in Proposition 4.6*. Applying 6.2 we obtain the theorem of Eilenberg, Moore and Husemoller [4], [6], that in this case the homology of the fibration $E'$ induced by $\beta : B' \to B$ is given by

$$\operatorname{Cotor}^B (B', B \otimes_{\xi} F) = \operatorname{Cotor}^B (B', E).$$

The last identity follows from 4.5* and 5.9.

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