ON THE STRONG LIFTING PROPERTY

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I. Introduction

Let X be a locally compact Hausdorff space, μ a positive Radon measure on X of support X. It is known that there is a strong lifting (definition below) for (X, μ) if X is metrizable [6, Ch. VIII, Theorem 3] or if X is a locally compact group and μ a Haar measure [5] or if X is a product of metrizable spaces and μ the product of measures on them [6, Ch. VIII, No. 2], or if μ is absolutely continuous with respect to any such measure [2].

Those are virtually the only cases for which the existence of a strong lifting is known, despite the interest a general existence theorem has for the disintegration theory of measures [3], [4], [6].

As it seems too difficult at present to decide whether or not there is always a strong lifting for (X, μ)—this might be an indecidable problem for all that is known—the next thing to do is to find measures admitting strong liftings. It is shown here in this context that one can generate new measures admitting strong liftings from known ones by taking sums, multiples, infima and suprema. The main result obtained can be summarized saying that the set $L(X)$ of Radon measures μ on X such that $|\mu|$ admits an almost strong lifting (definition below) forms a band in the complete vector lattice $M(X)$ of all Radon measures on X.

We note in passing that the desired result $M(X) = L(X)$ for arbitrary X is true if and only if it holds in case that X is any product of unit intervals [2]. Using the fact that $L(X)$ is a band in $M(X)$ one obtains the following reduction of the general existence problem.

THEOREM. *$L(X) = M(X)$ for all X if and only if for every product Π of unit intervals and every positive Radon measure ν on Π there is a Radon measure $\mu > 0$ on Π such that μ is absolutely continuous with respect to ν and admits an almost strong lifting.*

It is clear that the condition is necessary. If it is satisfied, on the other hand, then the band complementary to $L(\Pi)$ is zero for all Π, hence $L(\Pi) = M(\Pi)$ for all Π, hence $L(X) = M(X)$ for all X.

II. Notation and conventions

X is a set and $(X, \mathfrak{F}, \mu)$ a measure space on X with σ-algebra $\mathfrak{F}$ and positive measure μ on $\mathfrak{F}$, strictly localizable and equal to its Carathéodory extension. $\bar{\mu}$ is the associated essential integral, μ^*, $\bar{\mu}^*$ are the corresponding outer measures (upper integrals). A μ-null function or set is one which is locally negli-

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gible. $\mathfrak{L}^\infty(X, \mathfrak{F}, \mu)$ is the algebra of all bounded real-valued $\mathfrak{F}$-measurable functions. We identify the sets in $\mathfrak{F}$ with idempotents in $\mathfrak{L}^\infty(X, \mathfrak{F}, \mu)$ such that $\mathfrak{F} \subset \mathfrak{L}^\infty(X, \mathfrak{F}, \mu)$. We say A is *μ-essentially contained* in B, $A \dot{\subset} B$ (μ), if $B \backslash A$ is a μ-null set. The statements $A \doteq B$ (μ), $f \dot{\leqq} g$ (μ) etc. have the analogous meanings. $L^\infty(X, \mathfrak{F}, \mu)$ is the quotient of $\mathfrak{L}^\infty(X, \mathfrak{F}, \mu)$ by the null functions and $\dot{f}$ is the class of f in $L^\infty(X, \mathfrak{F}, \mu)$. If X is a locally compact space and μ a positive Radon measure on X then $\mathfrak{F}$ is always the σ-algebra of all μ-measurable sets in the sense of Bourbaki [1].

A lifting for $(X, \mathfrak{L}, \mu)$ is a map $T : \mathfrak{F} \to \mathfrak{F}$ with the properties

(L1) $T(A) \doteq A$ (μ)
(L2) $A \doteq B$ implies $T(A) = T(B)$
(L3) $T(\emptyset) = \emptyset$, $T(X) = X$
(L4) $T(A \cap B) = T(A) \cap T(B)$
(L5) $T(A \cup B) = T(A) \cup T(B)$

If $\tau \subset \mathfrak{F}$ is a topology on X then a lifting T is *strong* (with respect to τ) if

(LS) $T(U) \supset U$ for $U \in \tau$;

It is *almost strong with exceptional set* N if

(LAS) N is a null set and $T(U) \cup N \supset U$ for $U \in \tau$,

A lifting T has a unique linear and norm-continuous[1] extension to a positive algebra homomorphism $T : \mathfrak{L}^\infty(X, \mathfrak{F}, \mu) \to \mathfrak{L}^\infty(X, \mathfrak{F}, \mu)$. If τ is completely regular then T is almost strong with exceptional set N if and only if this extension satisfies $T\phi(x) = \phi(x)$ for $x \in \complement N$ and all τ-continuous functions ϕ [6, Ch. VIII].

An almost strong lifting T is strong if $\emptyset$ is an exceptional set.

Lastly, a *lower density* is a map $\theta : \mathfrak{F} \to \mathfrak{F}$ with properties (L1) through (L4). θ is almost strong (strong) if it satisfies LAS (LS) (cf. [6, Ch. VIII]). A standard reference is the monograph [6] by Ionescu-Tulcea.

III. Dense topologies and pre-densities

In this section we review the connection between liftings and the canonically associated topologies [6, Ch. V]. The upshot is that the topologies which can be used to generate liftings can be characterized, and that they in turn can be generated by predensities, which are maps $\theta : \mathfrak{F} \to \mathfrak{F}$ less restrictive than lower densities. The existence of an almost strong pre-density ensures the existence of an almost strong lifting, and this fact will be used in the later sections to establish the existence of strong liftings for sums and suprema of measures.

Definition. A topology $\mathfrak{I}$ on X is *compatible* (with $(\mathfrak{F}, \mu)$) if $\mathfrak{I} \subset \mathfrak{F}$ and if for all A in $\mathfrak{F}$ the set $A_{\mathfrak{I}} = \bigcup\{U \in \mathfrak{I} \mid U \dot{\subset} A\}$ is μ-essentially contained in A. That is to say $\mathfrak{I} \subset \mathfrak{F}$ is compatible if there is a maximal set $A_{\mathfrak{I}}$ in $\mathfrak{I}$ with $A_{\mathfrak{I}} \dot{\subset} A$, for all A in $\mathfrak{F}$.

[1] Essential supremums norm.

Remarks. (1) If X is a locally compact Hausdorff space under the topology τ and μ a positive Radon measure on (X, τ), then τ is compatible with $\mathfrak{F} = \mathfrak{F}(X, \mu)$. Let $A \in \mathfrak{F}$ and K a compact set in $\complement A$ such that supp $(\mu|_K) = K$. If U is open and $U \dot{\subset} A$ then $U \cap K \doteq \emptyset$, hence $U \cap K = \emptyset$. That is to say, $A_\tau \cap K = \emptyset$. But $\complement A$ is up to a locally negligible set the union of compacta K as above, hence $A_\tau \cap \complement A \doteq \emptyset$, $A_\tau \dot{\subset} A$.

(2) If $\mathfrak{I}$ is compatible then there is a maximal open null set $\phi_\mathfrak{I}$ and one can define the $\mathfrak{I}$-support of μ, $\text{supp}_\mathfrak{I}(\mu)$ as the complement of $\phi_\mathfrak{I}$.

(3) If $\mathfrak{I}$ is compatible then $(A \cap B)_\mathfrak{I} = A_\mathfrak{I} \cap B_\mathfrak{I}$. Monotony of $A \to A_\mathfrak{I}$ gives one inclusion. On the other hand, $A_\mathfrak{I} \cap B_\mathfrak{I}$ is an element of $\mathfrak{I}$ contained in both A and B essentially, hence $A_\mathfrak{I} \cap B_\mathfrak{I} \subset (A \cap B)_\mathfrak{I}$.

(4) $A \dot{\subset} B$ implies $A_\mathfrak{I} \subset B_\mathfrak{I}$.

From now on we assume that a fixed topology $\tau \subset \mathfrak{F}$ is given on X.

Definition. (i) A compatible topology $\mathfrak{I}$ on X is dense (in $\mathfrak{F}$) if

(D1) $\text{supp}_\mathfrak{I}(\mu) = X$ and

(D2) for every A in $\mathfrak{F}$ there is U in $\mathfrak{I}$ with $A \doteq U$ (μ).

(ii) If $\mathfrak{I}$ is dense, a set U in $\mathfrak{I}$ is maximal if $U = U_\mathfrak{I}$. If $\mathfrak{I}$ is generated by its maximal elements, it is a minimal dense topology.

(iii) A compatible topology $\mathfrak{I}$ is strong (with respect to τ) if $U_\mathfrak{I} \supset U$ for all U in τ.

(iv) A compatible topology $\mathfrak{I}$ is almost strong if there is a μ-null set N such that $U \subset U_\mathfrak{I} \cup N$ for all $U \in \tau$. Such an N is called an exceptional set for $\mathfrak{I}$.

(5) If $\mathfrak{I}$ is a dense topology for $(X, \mathfrak{F}, \mu)$ then $\theta_\mathfrak{I} : A \to A_\mathfrak{I}$ is a lower density for $(X, \mathfrak{F}, \mu)$: properties (L2) and (L4) follow from the definition of $A_\mathfrak{I}$ and Remark 3. $A_\mathfrak{I} \dot{\subset} A$ follows from compatibility and $A \dot{\subset} A_\mathfrak{I}$ from (D2), and this adds up to (L2). Finally, (L3) is a consequence of (D1) and $X \in \tau$. $\mathfrak{I}$ is (almost) strong if and only if $\theta_\mathfrak{I}$ is, with the same exceptional sets (iv).

One obtains from $\mathfrak{I}$ canonically two further topologies. $\mathfrak{I}_-$ is the topology generated by the maximal sets of $\mathfrak{I}$. It is a minimal dense topology and coincides with the first topology canonically [6, Ch. V, No. 3] associated with the lower density $\theta_\mathfrak{I}$. Then there is

$$\mathfrak{I}^- = \{U \backslash N \mid U \in \mathfrak{I}, N \text{ negligible}$$

which coincides with the second topology canonically associated with $\theta_\mathfrak{I}$ (ibidem). We have $\mathfrak{I}_- \subset \mathfrak{I} \subset \mathfrak{I}^-$ and all of these topologies are (almost) strong if one of them is. The map $\mathfrak{I} \to \theta_\mathfrak{I}$ maps the set of dense topologies onto the set of lower densities, preserving the property of being (almost) strong.

$$\theta \to \{\theta(A) \backslash N \mid A \in \mathfrak{F}, N \text{ null}\}$$

is an inverse.

(6) The topology $\mathfrak{I}^-$ has the property that a function on X is measurable

if and only if it is $\mathfrak{I}^-$-continuous on the complement of a null set. The set

$$J_x = \{\dot{f} \in L^\infty(X, \mathfrak{F}, \mu) \mid \text{there is } f \in \dot{f} \text{ such that } f \text{ is } \mathfrak{I}\text{-continuous and equal to zero at } x\}$$

is a proper ideal in $L^\infty(X, \mathfrak{F}, \mu)$. Selecting, for every x in X, a character χ_x of L^∞ which annihilates J_x one obtains a lifting T for $\mathfrak{L}^\infty(X, \mathfrak{F}, \mu)$ by setting

$$Tf(x) = \chi_x(\dot{f}), \quad x \in X.$$

For the proof of these statements see [6, Ch. V]. If $\mathfrak{I}$ is almost strong with exceptional set N then so is T. For if $U \in \tau$ then $U \doteq U_\mathfrak{I}$ and $U_\mathfrak{I}$ is $\mathfrak{I}^-$-continuous in $U_\mathfrak{I}$. Hence $\dot{1} - \dot{U}_\mathfrak{I} \in J_x$ for $x \in U_\mathfrak{I}$ and $T(U) = T(U_\mathfrak{I}) = 1$ on $U_\mathfrak{I}$. That is,

$$T(U) \cup N = T(U_\mathfrak{I}) \cup N \supset U_\mathfrak{I} \cup N \supset U.$$

Using Theorem 1 in [6, Ch. VIII] we find there is an almost strong lifting for $(X, \mathfrak{F}, \mu)$ (with respect to τ) if and only if there is a dense almost strong topology for $(X, \mathfrak{F}, \mu)$.

Definition. (i) A *pre-density* for $(X, \mathfrak{F}, \mu)$ is a map $\theta : \mathfrak{F} \to \mathfrak{F}$ with the properties:

(PD1) $\theta(A) \doteq A$ (μ) for all $A \in \mathfrak{F}$.

(PD2) $A \doteq B$ implies $\theta(A) = \theta(B)$.

(PD3) $A_1 \cap \cdots \cap A_k \doteq \emptyset$ implies $\theta(A_1) \cap \cdots \cap \theta(A_k) = \emptyset$ for all finite families $A_1, \cdots, A_k$ in $\mathfrak{F}$.

(ii) A pre-density θ is almost strong with exceptional set N if N is a null set such that $U \subset \theta(U) \cup N$ for all $U \in \tau$. An almost strong predensity θ is strong if $\emptyset$ is an exceptional set.

Proposition 1. *Let θ be a pre-density for $(X, \mathfrak{F}, \mu)$*

(i) *The topology $\mathfrak{I}^\theta$ spanned by the sets $\theta(A)$, $A \in \mathfrak{F}$, is dense.*

(ii) *If θ is (almost) strong, so is $\mathfrak{I}^\theta$. In particular, $(X, \mathfrak{F}, \mu)$ admits an (almost) strong lifting if and only if it admits a (almost) strong predensity.*

Proof. A set is in $\mathfrak{I}^\theta$ if it is the union of finite intersections of sets $\theta(A)$, $A \in \mathfrak{F}$. Condition (PD3) ensures that there is no non-empty null set in $\mathfrak{I}^\theta$, i.e., $\text{supp}_\mathfrak{I}\, \theta(\mu) = X$. If $A \in \mathfrak{F}$ and $\theta(A_1) \cap \cdots \cap \theta(A_k) \mathrel{\dot\subset} A$ then

$$\theta(A_1) \cap \cdots \cap \theta(A_k) \cap \theta(\complement A) = \emptyset,$$

hence $\theta(A_1) \cap \cdots \cap \theta(A_k) \subset \complement\theta(\complement A) \doteq A$. Hence $A_{\mathfrak{I}^\theta} \mathrel{\dot\subset} A$ and $\mathfrak{I}^\theta$ is compatible. Denseness is obvious. This proves (i). Let θ be almost strong and N an exceptional set. Then

$$U_{\mathfrak{I}^\theta} \cup N \supset \theta(U) \cup N \supset U$$

for U in τ and $\mathfrak{I}^\theta$ is almost strong with exceptional set N. If θ is strong, $\emptyset$ is an exceptional set for $\mathfrak{I}^\theta$ and thus $\mathfrak{I}^\theta$ is strong. This together with Remark (6) finishes the proof of (ii).

PROPOSITION 2. *Suppose* τ *is completely regular and compatible, and put* $K = \operatorname{supp}_\tau(\mu)$. *The following are equivalent.*

(i) $(X, \mathfrak{F}, \mu)$ *admits an almost strong lifting.*

(ii) $(X, \mathfrak{F}, \mu)$ *admits an almost strong lifting with exceptional set* N *such that* $N \cap K = \emptyset$.

(iii) $(K, \mathfrak{F} \cap K, \mu|_K)$ *admits a strong lifting* (*with respect to the induced topology* $K \cap \tau$).

Proof. Assume (i) and let S be an almost strong lifting with exceptional set M. The kernel $\mathcal{K}$ of the natural map of the τ-continuous bounded functions $C_b(X, \tau)$ into $L^\infty(X, \mathfrak{F}, \mu)$ consists of those functions ϕ which vanish on K. For $x \in K$ the character $\phi \to \phi(x)$ of $C_b(X, \tau)$ annihilates $\mathcal{K}$ and hence has an extension χ_x to $L^\infty(X, \mathfrak{F}, \mu)$. We select such an extension χ_x for every $x \in K \cap M$. The map $T : \mathfrak{L}^\infty \to \mathfrak{L}^\infty$ defined by

$$\begin{aligned} Tf(x) &= Sf(x) \quad \text{if } x \notin K \cap M \\ &= \chi_x(\bar{f}) \quad \text{if } x \in K \cap M \end{aligned}$$

is again an almost strong lifting [6, Ch. VII, No. 4, Remark 1]. If $U \in \tau$ then

$$TU(x) \geq \sup \{T\phi(x) \mid \phi \leq U\}$$

and obviously $TU(x) = 1$ if $x \in U \cap K$. Hence the complement of K is an exceptional set for T, whence (ii).

Assume (ii) and let T be an almost strong lifting with exceptional set N such that $N \cap K = \emptyset$. Define $T' : \mathfrak{F} \cap K \to \mathfrak{F} \cap K$ by $T'(A \cap K) = T(A \cap K) \cap K$. Then T' is easily checked to be a lifting for $\mu|_K$ and strong: if $U \cap K \in \tau \cap K$ then

$$T'(U \cap K) = T(U) \cap T(K) \cap K \supset U \backslash N \cap X \cap K \supset U \cap K.$$

That is (ii) implies (iii). The implication (iii) $\to$ (i) is contained in the proof of Proposition 12, [6, Ch. VIII, No. 4].

IV. Extension of almost strong liftings to smaller measures

For later use, we start proving a slight generalization of a result used in [2].

PROPOSITION 3. *Let* $(X, \mathfrak{F}, \mu)$, $(X, \mathfrak{G}, \nu)$ *be* (*complete, strictly localizable*[2]) *measure spaces such that* $\mathfrak{G} \subset \mathfrak{F}$ *and* $\mu \mid \mathfrak{G}$ *is absolutely continuous with respect to* ν.

(i) *There is a set* N *in* $\mathfrak{G}$ *with the following properties.*

(ia) $\mu(N) = 0$.

(ib) *If* $M \in \mathfrak{G}$, $\mu(M) = 0$ *then* $M \dot{\subset} N$ (ν).

(ic) *If* $A \in \mathfrak{F}$ *then* $A \backslash N \in \mathfrak{G}$.

(id) *If* $A, B \in \mathfrak{F}$, $A \doteq B$ (μ) *then* $A \backslash N \doteq B \backslash N$ (ν).

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[2] $\mu = \bar{\mu}$, $\nu = \bar{\nu}$.

(ii) *If* $N' \in \mathcal{G}$ *differs from* N *on a* ν*-null set then* N' *has again properties* (ia) *through* (id). *Any set* N' *with properties* (ia) *and* (ib) *differs from* N *on a* ν*-null set.*

Proof. Let $\dot{N}$ be the supremum in $L^\infty(X, \mathcal{G}, \nu)$ of classes $\dot{M}$ such that $\mu(M) = 0$ for $M \in \dot{M}$. Pick $N \in \dot{N}$. Then (ia) and (ib) are obvious. It follows from (ia) and (ib) that the restrictions of μ and ν to $\mathcal{G} \cap \complement N$ are equivalent. As both $\mathcal{G} \cap \complement N$ and $\mathfrak{F} \cap \complement N$ are their own Carathéodory completions (for $\nu \mid \complement N$, say,) they are equal. This proves (ic). If $A \doteq B$ (μ) then

$$A \,\Delta\, B \doteq \emptyset \ (\mu) \quad \text{and} \quad (A \,\Delta\, B)\backslash N \mathrel{\dot{\subset}} N \quad (\nu).$$

Hence $(A\backslash N) \,\Delta\, (B\backslash N) \doteq \emptyset$ (ν) which is just the claim of (id). (ii) is clear.

A set N with properties (ia) (ib) will be called a maximal ν-measurable μ-null set.

Corollary [2]. *Suppose* $\tau \subset \mathcal{G}$ *and* $(X, \mathcal{G}, \nu)$ *admits an almost strong lifting. Then so does* $(X, \mathfrak{F}, \mu)$.

Proof. Let S be an almost strong lifting with exceptional set M for $(\mathcal{G}, \nu)$ and let N be a maximal ν-measurable μ-null set. Put

$$T(A) = S(A\backslash N) \quad \text{for } A \in \mathfrak{F}.$$

Then T is a predensity for $(X, \mathfrak{F}, \mu)$. Indeed,

$$A \doteq A\backslash N \doteq S(A\backslash N) = T(A) \quad (\mu),$$

and if $A \doteq B$ (μ) then $A\backslash N \doteq B\backslash N$ (ν) and $T(A) = T(B)$. If $A_1 \cap \cdots \cap A_k \doteq \emptyset$ (μ) then

$$A_1\backslash N \cap \cdots \cap A_k\backslash N \doteq \emptyset \quad (\nu) \quad \text{and} \quad T(A_1) \cap \cdots \cap T(A_k) = S(\emptyset) = \emptyset.$$

Lastly, if $U \in \tau$ then

$$\begin{aligned} T(U) \cup M \cup S(N) &= S(U\backslash N) \cup M \cup S(N) \\ &= (S(U)\backslash S(N)) \cup M \cup S(N) \supset S(U) \cup M \supset U. \end{aligned}$$

Hence T is almost strong with exceptional set $M \cup S(N)$. Proposition 1, (ii) finishes the proof.

V. Extension of almost strong liftings to the sum of measures

For the sake of simplicity we assume henceforth that X is a locally compact Hausdorff space under the topology τ and that $\mu, \nu \cdots$ are positive Radon measures on X.

Theorem 1. *Let* μ_1, μ_2 *be positive Radon measures on* X *admitting almost strong liftings. Then their sum* $\mu = \mu_1 + \mu_2$ *also admits an almost strong lifting.*

Proof. Let S_i be an almost strong lifting for (X, μ_i), $i = 1, 2$. We may assume that $M_i = X\backslash\text{supp}\,(\mu_i)$ is an exceptional set for S_i (Proposition 2).

Let N_i' be a maximal μ-measurable μ_i-null set. Then $N_i = M_i \cup N_i'$ is at once a maximal μ-measurable μ_i-null set and an exceptional set for S_i. Furthermore, any open μ-null set is contained in the μ-null set $N = N_1 \cap N_2$.

We define an almost strong predensity θ for (X, μ) by

$$\theta(A) = (S_1(A)\backslash N_1) \cup (S_2(A \cap N_1) \cap N_1) \cup (A_\tau \backslash N),$$

where $A \in \mathfrak{F}(X, \mu)$ and A_τ is the maximal open set μ-essentially contained in A. It is clear that $A \doteq B\ (\mu)$ implies $\theta(A) = \theta(B)$. According to Proposition 3, (iid) we have

$$A\backslash N_1 = S_1(A)\backslash N_1\ (\mu).$$

For the same reason

$$\begin{aligned} A \cap N_1 &\doteq (A \cap N_1)\backslash N_2 \doteq S_2(A \cap N_1)\backslash N_2 \doteq S_2(A \cap N_1) \cap (N_1 \backslash N_2) \\ &\doteq S_2(A \cup N_1) \cap N_1 \quad (\mu) \end{aligned}$$

such that $A = (A\backslash N_1) \cup (A \cap N_1) \doteq \theta(A)\ (\mu)$. It is obvious from the definition that θ is almost strong with exceptional set N. It remains to be shown that property (PD3) for pre-densities holds. Let K be a finite set and A_k, $k \in K$, elements of $\mathfrak{F}(X, \mu)$ with $\bigcap\{A_k \mid k \in K\} = \emptyset\ (\mu)$. Then

$$\begin{aligned} \textstyle\bigcap_{k \in K} \theta(A_k) &= \textstyle\bigcap_{k \in K} \{(S_1(A_k)\backslash N_1) \cup (S_2(A_k \cap N_1) \cap N_1) \cup (A_{k\tau}\backslash N)\} \\ &= \textstyle\bigcup M(K', K'', K''') \end{aligned}$$

where the union is taken over all triples K', K'', K''' of mutually disjoint subsets of K whose union is K and where

$$\begin{aligned} &M(K', K'', K''') \\ &\quad = \textstyle\bigcap_{k \in K'} (S_1(A_k)\backslash N_1) \cap \bigcap_{k \in K''} (S_2(A_k \cap N_1) \cap N_1) \cap \bigcap_{k \in K'''} (A_{k\tau}\backslash N). \end{aligned}$$

It is clear that if both K' and K'' are non-void then $M(K', K'', K''') = \emptyset$. If $K' = K$ then

$$M\ (K', K'', K''') = \textstyle\bigcap_{k \in K} (S_1(A_k)\backslash N_1) = S_1(\bigcap_{k \in K} A_k)\backslash N_1 = \emptyset$$

as $\bigcap_{k \in K} A_k \doteq \emptyset\ (\mu_1)$. A similar argument shows that $M(K', K'', K''')$ is empty if $K'' = K$.
If $K''' = K$ then

$$M(K', K'', K''') = \textstyle\bigcap_{k \in K} (A_{k\tau}\backslash N) = (\bigcap_{k \in K} A_k)_\tau \backslash N = \emptyset$$

as $(\bigcap_{k \in K} A_k)_\tau$ is an open μ-null set and hence contained in N. If $K'' = \emptyset$ and K' and K''' are non-void then $M(K', K'', K''')$ is of the form

$$M(K', K'', K''') = (S_1(A)\backslash N_1) \cap (B_\tau \backslash N)$$

where $A = \bigcap_{k \in K'} A_k$, $B = \bigcap_{k \in K'''} A_k$ and $A \cap B \doteq \emptyset\ (\mu)$. We find

$$\begin{aligned} M(K', K'', K''') \subset S_1(A) \cap \complement N_1 \cap ((S_1(B_\tau) \cup N_1) \cap (\complement N_1 \cup \complement N_2) \\ \subset S_1(A \cap B_\tau) = \emptyset. \end{aligned}$$

The last case to consider is that $K' = \emptyset$, K'' and $K''' \neq \emptyset$. In this case $M(K', K'', K''')$ is of the form

$$M(K', K'', K''') = S_2(A \cap N_1) \cap N_1 \cap (B_\tau \backslash N)$$

where $A = \bigcap_{k \in K''} A_k$, $B = \bigcap_{k \in K'''} A_k$ and $A \cap B \doteq \emptyset \ (\mu)$. We find

$$M(K', K'', K''') \subset S_2(A \cap N_1) \cap N_1 \cap (S_2(B_\tau) \cup N_2) \cap (\complement N_1 \cup \complement N_2)$$
$$\subset S_2(A \cap B_\tau) = \emptyset.$$

This shows that θ is, indeed, an almost strong predensity and finishes the proof of Theorem 1.

As $\sup(\mu_1, \mu_2)$ is smaller than $\mu_1 + \mu_2$, the corollary to Proposition 3 yields the following:

COROLLARY. *Let μ_1, μ_2 be positive Radon measures on X admitting almost strong liftings. Then* sup (μ_1, μ_2) *also admits an almost strong lifting.*

VI. Extension of almost strong liftings to the supremum of measures

Let $(\mu_\alpha)_{\alpha \in A}$ be an increasingly directed family of positive Radon measures on X, bounded from above, and let μ be its supremum. If h is a lower semicontinuous function on X with $\mu^*(h) > \infty$ then given an $\varepsilon > 0$ one can find a $\phi \in C_{00}(X)$, set of all continuous functions on X with compact support, such that $\phi \leq h$ and $\mu(\phi) > \mu^*(h) - \varepsilon$. There is $\alpha_0 \in A$ such that $\alpha \geq \alpha_0$ implies $\mu_\alpha(\phi) > \mu(\phi) - \varepsilon$ and hence $\mu_\alpha^*(h) \geq \mu_\alpha(\phi) > \mu^*(h) - 2\varepsilon$. That is, $\mu^*(h) = \sup_\alpha \mu_\alpha^*(h)$ for μ-integrable lower semicontinuous functions h on X.

If now f is μ-integrable, we find a μ-integrable lower semicontinuous function $h \geq f$ such that

$$\mu(h - f) = \mu(h) - \mu(f) \leq \varepsilon$$

and see from $\mu_\alpha(h - f) \leq \mu(h - f) \leq \varepsilon$ that $\mu_\alpha(f) \geq \mu_\alpha(h) - \varepsilon$ and hence

$$\sup_\alpha \mu_\alpha(f) \geq \sup_\alpha \mu_\alpha(h) - \varepsilon = \mu(h) - \varepsilon \geq \mu(f) - \varepsilon$$

for sufficiently high $\alpha \in A$. That is, $\mu(f) = \sup\{\mu_\alpha(f) \mid \alpha \in A\}$ for μ-integrable functions f. Now let K be a compact set in X, ϕ a continuous function on K, ϕ' the function equal to ϕ on K and zero elsewhere. Then ϕ' is μ-integrable and we obtain for the restrictions of μ_α, μ to K the following:

$$\mu|_K(\phi) = \mu(\phi') = \sup_\alpha \mu_\alpha(\phi') = \sup_\alpha \mu_{\alpha|K}(\phi).$$

That is, $\mu|_K$ is the supremum of $(\mu_\alpha|_K)_{\alpha \in A}$

THEOREM 2. *If the μ_α, $\alpha \in A$ admit almost strong liftings, so does μ.*

Proof. Let $(K_i)_{i \in I}$ be a locally countable family of mutually disjoint compact sets whose union has μ-negligible complement. μ admits an almost strong lifting if $\mu|_{K_i}$ admits an almost strong lifting for all $i \in I$. If one replaces "almost strong" by "strong" in this statement, it is just Proposition 2 in [6, Ch. VIII, No. 1]. Our statement here can be obtained routinely using that result and Proposition 2. Now, $\mu|_{K_i}$ is the supremum of $(\mu_\alpha|_{K_i})_{\alpha\in A}$ and these measures admit almost strong liftings [corollary to Proposition 3). That is, the theorem is reduced to the case that X is compact.

Select now inductively an increasing sequence $(\mu_n)_{n=1,2,\cdots}$ of measures in $\{\mu_\alpha \mid \alpha \in A\}$ such that $\|\mu_n\| \geq \|\mu\| - 2^{-n}$. Then μ is the supremum of the increasing sequence of measures μ_n which admit almost strong liftings. Using Proposition 2, we select an almost strong lifting S_n for μ_n with $M_n = \complement\operatorname{supp}(\mu_n)$ as an exceptional set. The M_n form a decreasing sequence whose intersection contains every open μ-null set. Next let N_n'' be a maximal μ-measurable μ_n-null set. If $m < n$ then $N_n' \dot{\subset} N_m'$ (μ) such that

$$N_n' = \bigcap \{N_m'' \mid m \leq n\}$$

is again a maximal μ-measurable μ_n-null set. The sets $N_n = M_n \cup N_n'$, $n = 1, 2, \cdots$, are at once maximal μ-measurable μ_n-null sets and exceptional sets for S_n and form a decreasing sequence whose intersection N contains every open μ-null set and is a μ-null set itself. Augmenting this by $N_0 = X$, we define an almost strong pre-density θ for μ by

$$\theta(A) = \bigcup_{n=1,2,\cdots} (S_n(A \cap N_{n-1}) \cap N_{n-1} \cap \complement N_n) \cup (A_\tau \cap \complement N).$$

Here A is any μ-measurable set and A_τ is the maximal open set μ-essentially contained in A. θ is obviously almost strong with exceptional set N. It is also clear that if $A \doteq B$ (μ) then $\theta(A) = \theta(B)$. From

$$A \cap N_{n-1} \doteq S_n(A \cap N_{n-1}) \quad (\mu_n)$$

we get

$$(A \cap N_{n-1})\backslash N_n \doteq S_n(A \cap N_{n-1}) \cap \complement N_n \doteq S_n(A \cap N_{n-1}) \cap N_{n-1} \cap \complement N_n \quad (\mu)$$

and hence

$$\theta(A) \doteq \bigcup_{n=1,2\cdots} (A \cap N_{n-1})\backslash N_n \cup (A_\tau\backslash N) \doteq A\backslash N \cup (A_\tau\backslash N) \doteq A\backslash N \doteq A \quad (\mu)$$

There is only condition (PD3) to be checked.

Let $\bigcap_{k\in K} A_k \doteq \emptyset$ (μ) where K is finite and $A_k \in \mathfrak{F}(X, \mu)$, $k \in K$. Then

$$\begin{aligned}\bigcap_{k\in K} \theta(A_k) &= \bigcap_{k\in K} \bigcup_{n=1,2\cdots} (S_n(A_k \cap N_{n-1}) \cap (N_{n-1}\backslash N_n)) \cup (A_{k\tau}\backslash N)\\ &= \bigcup M(n^\sim)\end{aligned}$$

where

$$M(n^\sim)$$
$$= \bigcap_{n^\sim(k)\text{ finite}} (S_{n^\sim(k)}(A_k \cap N_{n^\sim(k)-1}) \cap (N_{n^\sim(k)-1}\backslash N_{n^\sim(k)}) \cap \bigcap_{n^\sim(k)=\infty} A_{k\tau}\backslash N$$

and the union is taken over all maps $n^\sim : K \to \{1, 2, \cdots, \infty\}$. It is to be shown that all the $M(n^\sim)$ are empty. Consider first the case that

$n^{\sim}(K) = \{\infty\}$. Then

$$M(n^{\sim}) = \bigcap_{k\in K} (A_{k\tau} \cap \complement N) = (\bigcap_{k\in K} A_k)_\tau \backslash N = \emptyset$$

as $(\bigcap_{k\in K} A_k)_\tau$ is an open μ-null set. If, on the other hand $n^{\sim}(k)$ is finite for all $k \in K$ then $M(n^{\sim})$ is obviously empty if the $n^{\sim}(k)$, $k \in K$, do not coincide. But even if they do, $n^{\sim}(K) = \{m\}$ say, then

$$M(n^{\sim}) \subset \bigcap_{k\in K} S_m(A_k) = S_m(\bigcap_{k\in K} A_k) = \emptyset.$$

In the general case the same argument applies: if the $n^{\sim}(k)$ which are finite do not coincide then $M(n^{\sim}) = \emptyset$. If they do, $n^{\sim}(k) = m$, say, if $n^{\sim}(k)$ is finite, then $M(n^{\sim})$ is of the form

$$M(n^{\sim}) = S_m(A \cap N_{m-1}) \cap (N_{m-1}\backslash N_m) \cap B_\tau \cap \complement N$$

where $A = \bigcap_{n^{\sim}(k)=m} A_k$, $B = \bigcap_{n(k)=\infty} A_k$ and hence $A \cap B \doteq \emptyset$ (μ). We find

$$M(n^{\sim}) \subset N_{m-1} \cap \complement N_m \cap S_m(A) \cap (S_m(B_\tau) \cup N_m) \cap \complement N \subset S_m(A) \cap S_m(B_\tau)$$
$$= S_m(A \cap B_\tau) = \emptyset.$$

That is, θ is indeed an almost strong pre-density, and Proposition 1, (ii) finishes the proof of the theorem.

Corollary. *The set $L(X)$ of Radon measures μ on X such that $|\mu|$ admits an almost strong lifting is a band in the completely reticulated vector space $M(X)$ of all Radon measures on X.*

Proof. If $|\mu|$ admits an almost strong lifting and λ is a real number then $|\lambda\mu| = |\lambda|\,|\mu|$ also admits an almost strong lifting. If $|\mu|$ does and $|\nu|$ is smaller than $|\mu|$ then so does $|\nu|$ (corollary to Proposition 3). If $|\mu|$ and $|\nu|$ do then so does $|\mu| + |\nu|$ which is bigger than $|\mu + \nu|$ (Theorem 1). Hence $L(X) + L(X) \subset L(X)$. Finally let $(\mu_\alpha)_{\alpha\in A}$ be a family in $L(X)$ with supremum μ. Then the positive part μ^+ of μ is the supremum of the family $(\mu_\alpha^+)_{\alpha\in A}$, whose members admit almost strong liftings (corollary to Proposition 3). Augmenting this family by the finite suprema of its elements, μ^+ becomes the supremum of a directed family of positive measures admitting almost strong liftings (corollary to theorem 1.). Hence μ^+ admits an almost strong lifting (Theorem 2). The negative part μ^- of μ is smaller than the negative part of any of the μ_α, $\alpha \in A$, and therefore also admits an almost strong lifting. Hence $|\mu| = \mu^+ + \mu^-$ admits an almost strong lifting, and $\mu \in L(X)$.

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