COMPOSITION SERIES FOR SIMPLEX SPACES

BY

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A general theory of composition series was given in [3]. It was there applied to the case of separable simplex spaces. The authors characterized the separable GC-spaces and partially characterized the separable GM-spaces. We shall, here, generalize and extend those results.

The notations and definitions are those used in [1], [2], [3], [4]. V will always denote a simplex space. For a set $A \subseteq P_1(V)$, we let $\bar{A}$ or A^- be the weak* closure of A and $A^+ = A - \{0\}$, with the exception that $E^+ = EP_1(V)^+$. For $q \in P_1(V)$, we shall denote by π_q the unique maximal probability measure with resultant q. If V is separable, then π_q is supported by $EP_1(V)$.

Let X be any topological space and p any topological property. If a subset $G \subseteq X$ has property p, we write $G \subseteq_p X$ and say that G is a *p-subset* of X. (For a full account, see [3, §3].) A property p is *inductive* if for each nonempty closed set F and each open G in X we have: $G \subseteq_p X$ implies that $G \cap F \subseteq_p F$. We say a property p is *strongly inductive* if (1) p is inductive and, given G_1, G_2 open, F closed in X, we also have:

(2) $G_1 \subseteq G_2 \subseteq X$ and $G_1 \subseteq_p X$ imply $G_1 \subseteq_p G_2$.
(3) $G_1 \subseteq G_2 \subseteq_p X$ implies $G_1 \subseteq_p X$.
(4) $G_1 \subseteq_p F \subseteq X$ implies $G_1 \subseteq_p X$.

For $X = \max V$, we shall consider the following properties:

(C) $G \subseteq_C \max V$ means that elements of V restrict to continuous functions on G.

(M) $G \subseteq_M \max V$ if each net in G which converges to a point of G converges to no other point of $\max V$.

(n) (for $n \geq 2$) $G \subseteq_n \max V$ if each sequence in G which converges to a point of G converges to at most n points in $\max V$.

PROPOSITION 1. *The properties* (C), (M) *and* (n) *are strongly inductive.*

Proof. That (C) and (M) are strongly inductive is shown in [3, Prop' 4.3]. That (n) is strongly inductive is obvious.

If J is a closed ideal in V, we let $P_1(J)$ be the positivestates of J and $EP_1(J)$ be the pure states of J when we consider J as a simplex space in its own right.

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The restriction map $\rho : V^* \to J^*$ restricts to a continuous affine map of $P_1(V)$ onto $P_1(J)$ [1, Thm. 4.4]. This, in turn, restricts to a continuous one-to-one map of $EP_1(V) - J^\perp$ onto $EP_1(J)^+$. With these maps, we may consider $P_1(J)$ and $EP_1(J)^+$ to be subsets of $P_1(V)$ and $EP_1(V) - J^\perp$, respectively. As such $EP_1(J)$ considered as a subset of $EP_1(V)$ with the structure topology induced from $EP_1(V)$ is homeomorphic to max (J) [1, Thm. 4.4]. It is, therefore, structurally open in $EP_1(V)$. Furthermore, we may consider $EP_1(V)$ to be the union of $EP_1(J)$ with $EP_1(V) \cap J^\perp$.

In order to find structure closed sets, we use the following Proposition, cf. [4, Prop. 1.1]. First, a set $D \subseteq P_1(V)$ is *dilated* if for each $q \in D$, we have supp $\pi_q \subseteq D$.

Proposition 2. (A) *Let $D \subseteq P_1(V)$ be dilated and weak* closed. Then the weak* closed convex hull of $D \cup \{0\}$ is a face of $P_1(V)$ and $D \cap E^+$ is structurally closed.*

(B) *Let $D \subseteq E^+$. Then the following are equivalent:*

1. *D is structure closed.*
2. *D is weak* closed in E^+ and $\bar{D} \cup \{0\}$ is dilated.*

(C) *Let $q \in Z$ and suppose π_q is supported by E^+. Then*

$$\text{supp } \pi_q = (\text{supp } \pi_q \cap E^+)^-.$$

Proof. (A) The first conclusion is [2, Thm. 3.3] while the second follows easily from the Milman Theorem [5, p. 9].

(B) (1) $\to$ (2). Obviously D is weak* closed in E^+. Let K be the closed face containing zero such that $K \cap E^+ = D$. Let $q \in \bar{D} \cup \{0\}$. Then $q \in K$ and so supp $\pi_q \subseteq K$. Hence supp $\pi_q \subseteq (K \cap EP_1(V))^-$ [5, p. 30.]. As the latter set is $\bar{D} \cup \{0\}$, the implication is clear.

(2) $\to$ (1) follows easily from (A).

(C) is [4, Prop. 1.1 (A)].

We let $R_nP_1(V)$ be the lateral n-skeleton of $P_1(V)$, i.e.

$$R_0P_1(V) = EP_1(V),$$

$$R_nP_1(V) = \{\textstyle\sum_{i=1}^n \lambda_i p_i \mid p_i \in E^+,\ \lambda_i \geq 0,\ \sum_{i=1}^n \lambda_i \leq 1\} \quad \text{for} \quad n > 0.$$

In terms of $R_nP_1(V)$, we may recast properties (C), (M), and (n).

Proposition 3. *Let J be a closed ideal in V. Then:*

(1) max $J \subseteq_C$ max V *if and only if*

$$Z \subseteq EP_1(V) \cup J^\perp = R_0P_1(V) \cup J^\perp.$$

(2) max $J \subseteq_M$ max V *if and only if*

$$Z \subseteq R_1P_1(V) \cup J^\perp.$$

(3) *Suppose V is separable. Then* max $J \subseteq_n$ max V *if and only if*

$$Z \subseteq R_nP_1(V) \cup J^\perp.$$

Proof. (1) and (2) are contained in [3, Thm. 2.2] and [3, Thm. 2.5], respectively.

(3) Suppose $\max J \subseteq_n \max V$. Let $q \in Z - E^+$. Then there is a sequence $\{p_k\} \subseteq E^+$ such that $p_k \to q$. Let F be the structure closure of supp $\pi_q \cap E^+$. Then $\{p_k\}$ converges structurally to each element of F and to no others [4, Cor. 1.5]. Suppose there is a $z \in EP_1(J) \cap F$. As $EP_1(J)$ is structurally open, $p_k \in EP_1(J)$ eventually. Therefore cardinality $(F) \leq n$ and so $q \in R_nP_1(V)$. Suppose, on the other hand, that $EP_1(J) \cap F = \emptyset$. Then $F \subseteq E^+ \cap J^\perp$ and so Prop. 2(C) yields supp $\pi_q \subseteq J^\perp$. Thus $q \in J^\perp$. Therefore

$$Z - E^+ \subseteq R_nP_1(V) \cup J^\perp.$$

Since $E^+ = EP_1(J)^+ \cup (E^+ \cap J^\perp)$, we have $Z \subseteq R_nP_1(V) \cup J^\perp$.

Conversely, suppose $Z \subseteq R_nP_1(V) \cup J^\perp$ and let $\{p_k\} \subseteq EP_1(J)$, $p \in EP_1(J)$ be such that $\{p_k\}$ converges structurally to p. Since Z is a compact metric space, going to a subsequence and re-indexing, there is a point $q \in Z$ such that $p_k \to q$. Let F be the structure closure of supp $\pi_q \cap E^+$. Then $\{p_k\}$ converges to each point of F and to no others [4, Cor. 1.5]. Hence, $p \in F$. Suppose $q \in J^\perp$. Then supp $\pi_q \cap E^+ \subseteq J^\perp \cap E^+$. As the latter is structure closed, $p \in F \subseteq J^\perp \cap E^+$. This contradicts $p \in EP_1(J)$ and so $q \in R_nP_1(V)$. Hence F has at most n points and the proposition has been proven.

Let J be a closed ideal in V. We define the following simplex properties (for a full account of such properties see [3, §4]):

J is a C-*ideal* (or a 0-*ideal*) if $\max J$ has property (C).
J is an M-*ideal* (or a 1-*ideal*) if $\max J$ has property (M).
J is an n-*ideal* if V is separable and $\max J$ has property (n).

For the simplex properties $n \geq 0$, a closed ideal $J \subseteq V$ is *Gn in* V if for all closed ideals I, either $J \subseteq I$ or $(J + I)/I$ contains a non-zero closed n-ideal in V/I. If V is Gn in V we say that V is a *Gn-space*. We say that V is an *Nn-space* if it contains no non-zero n-ideal. With this terminology we have the following theorem [3, Lemma 4.1 and Proposition 4.2].

THEOREM 4. *Let V be a simplex space, separable if $n \geq 2$. Then there is a largest Gn-ideal J. If $V \neq J$, then V/J is an Nn-space. There is a collection of distinct closed ideals J_γ indexed by ordinals $0 \leq \gamma \leq \gamma_0$ such that:*

(1) $J_0 = \{0\}$, $J_{\gamma_0} = J$.

(2) *If $\gamma < \gamma_0$ is a successor ordinal, then J_γ is a proper subset of $J_{\gamma+1}$ and $J_{\gamma+1}/J_\gamma$ is an n-ideal in V/J_γ.*

(3) *If $\gamma \leq \gamma_0$ is a limit ordinal, then $J_\gamma = (\cup_{\beta<\gamma} J_\beta)^-$.*

Further, ideals and quotients of Gn-spaces are again Gn-spaces.

Such a sequence of closed ideals is called an *n-composition series* for V.

To ease the notation, we let

$$Z_n = Z \cap R_n P_1(V)^+.$$

Hence

$$Z_n = \{z \in Z \mid z = \textstyle\sum_{i=1}^n \lambda_i p_i,\ 0 \le \lambda_i \le 1,\ p_i \in E^+,\ z \ne 0\},\ n > 0,$$

$$Z_0 = E^+.$$

For any $L \subseteq E^+$, let

$$k_n(L) = (\cup \{\text{supp}\ \pi_q \mid q \in \bar{L} - Z_n\})^-.$$

We let

$$e_n(L) = \text{structure closure}\ (k_n(L) \cap E^+).$$

We note the following.

PROPOSITION 5. *Let $L \subseteq E^+$ be structurally closed. Then:*
(1) $e_n(L) \subseteq L$.
(2) *$k_0(L)$ is closed and dilated.*
(3) $k_0(L) \cap E^+ = e_0(L)$.

Proof. If $q \in \bar{L}$, then supp $\pi_q \subseteq \bar{L} \cup \{0\}$ by Prop. 2(B). Hence

$$k_n(L) \subseteq \bar{L} \cup \{0\}.$$

So $k_n(L) \cap E^+ \subseteq (\bar{L} \cup \{0\}) \cap E^+ = L$. Thus $e_n(L) \subseteq L$ which is (1). In particular, $k_0(L) \subseteq \bar{L} \cup \{0\}$. Let $q \in k_0(L) \cup \{0\}$. If $q \in EP_1(V)$, then supp $\pi_q = \{q\} \subseteq k_0(L) \cup \{0\}$. If $q \notin EP_1(V)$, then $q \in \bar{L} - L$. Hence supp $\pi_q \subseteq k_0(L)$ and so (2) holds. Therefore $k_0(L) \cap E^+$ is already structure closed by Prop. 2(A) which yields (3).

With these concepts we may now attack the problem of characterizing the Gn spaces for $n \ge 0$.

PROPOSITION 6. *Let V be a simplex space. We assume π_q is supported by $EP_1(V)$ for each $q \in Z$ if $n = 1$ and that V is separable if $n \ge 2$. Let $F \subseteq E^+$ be a non-empty structure closed set. Let I be the closed ideal satisfying $I^\perp \cap E^+ = F$. Then the following are equivalent:*

(1) *There is a closed non-trival ideal J such that $(J + I)/I$ is an n-ideal in V/I.*

(2) *$U = F - e_n(F)$ is non-empty.*

In fact, there is a one-to-one correspondence between closed, non-trivial ideals J such that $(J + I)/I$ is an n-ideal in V/I and non-empty sets $W \subseteq U$ which are structure-open relative to F.

Proof. Since $I^\perp$ is a closed face, the structure and weak* topologies for $(V/I)^*$ coincide with the restrictions to $I^\perp$ of the structure and weak* topologies of V, respectively [1, Thm. 3.4]. Thus, it suffices to consider the case that $F = E^+$ and $I = \{0\}$.

(1) → (2). Let J be a closed, non-trivial n-ideal. We know that

$$Z \subseteq J^{\perp} \cup R_n P_1(V)$$

and so $Z \subseteq J^{\perp} \cup Z_n$. If $q \in Z - Z_n$. then $q \in J^{\perp}$. Since $J^{\perp}$ is a closed face containing zero, supp $\pi_q \subseteq J^{\perp}$. Hence

$$k_n(E^+) \cap E^+ \subseteq J^{\perp} \cap E^+.$$

Because $J^{\perp} \cap E^+$ is structurally closed, $e_n(E^+) \subseteq J^{\perp} \cap E^+$. Letting $W = E^+ - J^{\perp}$, we have $W \subseteq E^+ - e_n(E^+)$ and W is a non-empty structurally open set.

(2) → (1). Let W be a non-empty structurally open set such that

$$W \subseteq E^+ - e_n(E^+).$$

Let J be the closed ideal satisfying $J^{\perp} \cap E^+ = E^+ - W$. Suppose $q \in Z - Z_n$. Thus supp $\pi_q \subseteq k_n(E^+)$. If $n = 0$, then $k_0(E^+) \cup \{0\}$ is closed and dilated so its closed convex hull F is a face by Prop. 2(A). As

$$F \cap E^+ = k_0(E^+) \cap E^+ = e_0(E^+) \subseteq E^+ - W = J^{\perp} \cap E^+,$$

we must have $F \subseteq J^{\perp}$. Hence supp $\pi_q \subseteq J^{\perp}$. If $n \geq 1$, then π_q is supported by $EP_1(V)$. So

$$\text{supp } \pi_q = (\text{supp } \pi_q \cap E^+)^- \subseteq (k_n(E^+) \cap E^+)^- \subseteq (e_n(E^+))^- \subseteq (J^{\perp} \cap E^+)^- \subseteq J^{\perp}.$$

In either case, supp $\pi_q \subseteq J^{\perp}$. Thus, $q \in J^{\perp}$. Therefore

$$Z \subseteq J^{\perp} \cup Z_n \subseteq J^{\perp} \cup R_n P_1(V)$$

and, consequently, J is a non-trivial n-ideal.

COROLLARY 7. *Let J be the closed ideal in V satisfying*

$$J^{\perp} \cap E^+ = (\bigcup\{\text{supp } \pi_q | q \in Z - E\})^- \cap E^+.$$

Then J is a C-ideal and it contains every other C-ideal.

Using Proposition 6 we easily get the following main result.

THEOREM 8. *Suppose V is a simplex space. We assume that π_q is supported by $EP_1(V)$ for each $q \in Z$ if we are considering the property* (M); *we assume that V is separable if we are considering property* (n), $n \geq 2$. *Then V is a GC-, GM-, or Gn-space if and only if, for each non-empty structure closed set F, we have $F \neq e_0(F)$, $F \neq e_1(F)$, or $F \neq e_n(F)$, respectively.*

COROLLARY 9. *Suppose V is a simplex space satisfying the hypothesis o. Theorem* 8. *If*

$$\text{cardinality } (\{z \mid z \in \text{supp } \pi_q \text{ for some } q \in Z - Z_m\}) < \infty,$$

then V is a GC-, GM-, or Gn-space for $m = 0$, $m = 1$, *or* $m = n \geq 2$, *respectively.*

Proof. $e_m(E^+)$ is a finite set so Theorem 8 applies trivially.

COROLLARY 10. *Suppose there is a* $q_0 \in Z$ *such that* $q_0 \in \text{supp } \pi_{q_0}$. *If* π_{q_0} *is supported by* $EP_1(V)$, *then V is not a GC-space or a GM-space. Further, if V is separable, then V is not a Gn-space for any* $n \geq 2$.

Proof. We take F to be the structure closure of supp $\pi_{q_0} \cap E^+$. Then F is a non-empty structure closed set which satisfies $F = e_n(F)$ for each n.

REFERENCES

1. E. EFFROS, *Structure in simplexes*, Acta Math., vol. 117 (1967), pp. 103–121.
2. ———, *Structure in simplexes II*, J. Functional Anal., vol. 1 (1967), pp. 379–391.
3. E. EFFROS AND A. GLEIT, *Structure in simplexes III*, Trans. Amer. Math. Soc., vol. 142 (1969), pp. 355–379.
4. A. GLEIT, *On the structure topology of simplex spaces*, Pacific J. Math., vol. 34 (1970), pp. 389-405.
5. R. PHELPS, *Lectures on Choquet's Theorem*, van Nostrand, Princeton, 1966.

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