A NORMAL HEREDITARILY SEPARABLE NON-LINDELÖF SPACE

BY
MARY ELLEN RUDIN

A. Hajnal and I. Juhasz have defined a Hausdorff hereditarily σ-separable non-σ-Lindelöf space. R. Countryman has raised the question of the existence of a regular, hereditarily separable, non-Lindelöf space. The purpose of this paper is to show that the existence of a Souslin tree of cardinality $\aleph_1$ (which is consistent with the usual axioms for set theory) implies the existence of such a space which is also normal.

A partially ordered set $(T, \leq)$ is a Souslin tree provided:

1. $(T, \leq)$ is a tree ($t \in T$ implies $\{s \in T \mid s \leq t\}$ is well ordered).
2. T is uncountable.
3. Every chain (totally ordered set) is countable.
4. Every antichain (pairwise unordered set) is countable.

Suppose $(T, \leq)$ is a Souslin tree.

For $t \in T$, define $p(t) = \{s \in T \mid s \leq t\}$ and $f(t) = \{s \in T \mid t \leq s\}$; if $X \subset T$, define $p(X) = \bigcup_{x \in X} p(x)$ and $f(X) = \bigcup_{x \in X} f(x)$.

For each countable ordinal α, let T_α be the α^{th} level of T: that is

$$T_\alpha = \{t \in T \mid p(t) \text{ is order isomorphic to } \alpha\}.$$

Clearly $T = \bigcup_{\alpha < \omega_1} T_\alpha$. Without loss of generality we assume that $t \in T_\alpha$ and $\alpha < \beta$ implies $f(t) \cap T_\beta$ is infinite.

I. Preliminary definitions

Let $\mathfrak{A} = \{(n, \alpha, t) \in \omega \times \omega_1 \times T \mid \alpha \text{ is a limit ordinal and } t \in T_\gamma \text{ for some } \gamma > \alpha\}$.

For each limit ordinal α, select $\alpha^0 < \alpha^1 < \cdots$ having α as a limit.

For $(n, \alpha, t) \in \mathfrak{A}$, let $\mathcal{Z}(n, \alpha, t)$ be the set of all nonempty chains Z such that:

(a) $p(t) \cap T_{\alpha^n} \in p(Z)$ but $p(t) \cap T_{\alpha^{n+1}} \notin p(Z)$.
(b) $Z \cap f(T_\alpha) = \emptyset$.
(c) If $z \in Z \cap T_\beta$ and $\beta < \gamma < \alpha$, then $Z \cap T_\gamma \neq \emptyset$.
(d) If $r \in T_\alpha$, $Z \not\subset p(r)$.

For $Z \subset T$, define $Z^* = \{Y \subset T \mid \text{for some finite } F \subset T, Y = Z - p(F)\}$. Observe that $Z \in \mathcal{Z}(A)$ implies $Z^* \subset \mathcal{Z}(A)$.

In Section III we choose for each $\gamma < \omega_1$ and $A \in \mathfrak{A}$, a subset $R_\gamma(A)$ of T. If $A = (n, \alpha, t)$ and $t \in T_\beta$, define $Z(A) = R_{\beta+1}(A)$. The following properties hold.

Received June 4, 1970.

(1) $Z(A) \in \mathcal{Z}(A)$.

(2) For all $\gamma < \omega_1$, there is a term of $Z(A)^*$ contained in $R_\gamma(A)$.

(3) $A = (n, \alpha, s)$, $B = (m, \beta, r)$, $\alpha \leq \beta < \gamma$, $r \neq s$ and

$$p(s) \cap p(r) \cap T_\gamma = \emptyset \quad \text{implies} \quad R_\gamma(A) \cap R_\gamma(B) = \emptyset.$$

II. A topological space Σ which is normal, Hausdorff, hereditarily separable and not Lindelöf

Assume $R_\gamma(A)$ and $Z(A)$ as in the last paragraph of I.

The terms of T will be the points of Σ. Let U be open in Σ if and only if, for each $t \in U$ there is an $m \in \omega$ such that, for $n > m$ and $(n, \alpha, t) \in \mathfrak{A}$, there is a $Y \in Z(n, \alpha, t)^*$ such that $Y \subset U$.

For each $\alpha < \omega_1$, $p(T_\alpha)$ is countable and open; hence Σ is not Lindelöf. The complement of a point t is also obviously open since for each $A \in \mathfrak{A}$ one can pick $Y \in Z(A)^*$ avoiding t. Hence Σ is normal implies Σ is Hausdorff.

1. *Proof that Σ is hereditarily separable.* Suppose $X \subset T$. Let

$$V = \{t \in T \mid f(t) \cap X = \emptyset\}$$

and let

$$W = \{t \in V \mid p(t) \cap V = \{t\}\}.$$

Since W is an antichain there is an upper bound β on $\{\delta \mid W \cap T_\delta \neq \emptyset\}$. Since $p(T_\beta)$ is countable, it will suffice to show that, for each $r \in T_\beta - V$, there is a countable dense subset of $X \cap f(r)$.

Suppose $r \in T_\beta - V$. Define $\alpha_0 = \beta$. Then, for each $n \in \omega$, define $\alpha_n < \omega_1$ and $W_n \subset X \cap f(r)$ by induction as follows. If α_n has been defined, let

$$W_n = \{t \in X \cap f(r) \mid p(t) \cap X \cap f(T_{\alpha_n}) = \{t\}\}.$$

Clearly W_n is an antichain. Let α_{n+1} be greater than some upper bound on $\{\delta \mid W_n \cap T_\delta \neq \emptyset\}$. Let α be the limit of $\{\alpha_n\}_{n \in \omega}$.

I claim $f(r) \cap f(T_{\alpha+1})$ is a subset of the closure of $\bigcup_{n\in\omega} W_n$. Suppose $t \in T_\gamma \cap f(r)$ and $\gamma > \alpha$ and U is open and $t \in U$. We show $U \cap X \neq \emptyset$. Since U is open, there is an n such that $\beta < \alpha^n$ and a $Y \in Z(n, \alpha, t)^*$ such that $Y \subset U$. Select $y \in Y$; for some i, $y \in p(T_{\alpha_i})$. Let $z = Y \cap T_{\alpha_{i+1}}$. There is an $x \in X \cap f(z)$ since, by the definition of β, $f(r) \cap V = \emptyset$. There is a first term x' of $p(x) \cap X \cap f(T_{\alpha_i})$ and $x' \in W_i$ by definition. Since $y < x' < z$, $x' \in U \cap X$.

2. *Proof that Σ is normal.* Suppose H is closed. By the proof given in 1, there is an $\alpha < \omega_1$ such that $t \in T_{\alpha+1}$ implies $f(t) \subset H$ or $f(t) \cap H = \emptyset$.

Suppose H and K are disjoint and closed. There is clearly a nonlimit ordinal $\gamma < \omega_1$ such that $t \in T_\gamma$ implies $f(t) \subset H$ or $f(t) \subset K$ or $f(t) \cap (H \cup K) = \emptyset$. Without loss of generality we assume that $f(T_\gamma) \subset H \cap K$, for K and $f(T_\gamma) - K$ are closed and disjoint.

For $t \in T$ there is an m_t such that, for every $(n, \mu, t) \in \mathfrak{A}$ and $n > m_t$, there is a $Y \in Z(n, \mu, t)^*$ such that $Y \subset X - H$ if $t \notin H$ and $Y \subset X - K$ if $t \notin K$.

Thus, if $A = (n, \mu, t) \in \mathcal{A}$, $n > m_t$, and $u < \gamma$, I(2) allows us to pick $Y(A) \in Z(A)^*$ such that:

(a*) $Y(A) \subset R_\gamma(A)$, and
(b*) $Y(A)$ intersects H only if $t \in H$ and $Y(A)$ intersects K only if $t \in K$.

Define $U_0 = H$ and $V_0 = K$. If subsets U_{k-1} and V_{k-1} of T have been defined, define

$$U_k = \bigcup \{Y(n, \mu, t) \mid t \in U_{k-1}, n > m_t, \mu < \gamma \text{ and } (n, \mu, t) \in \mathcal{A}\}$$

and similarly

$$V_k = \bigcup \{Y(n, \mu, t) \mid t \in V_{k-1}, n > m_t, \mu < \gamma \text{ and } (n, \mu, t) \in \mathcal{A}\}.$$

Clearly $U = \bigcup_{k\in\omega} U_k$ and $V = \bigcup_{k\in\omega} V_k$ are open and $U \supset H$ and $V \supset K$. Also $U \cap V = \emptyset$. Suppose on the contrary that i is the smallest integer such that $U_i \cap V \neq \emptyset$. Select $x \in U_i \cap V_j$. Since H and K are disjoint, by (b*), $i > 0$ and $j > 0$. Hence for some $\mu < \gamma$ and $\eta < \gamma$ and $s \in U_{i-1}$ and $r \in V_{j-1}$,

$$x \in Y(n, \mu, s) \cap Y(m, \eta, r).$$

By (a*), $R_\gamma(n, \mu, s) \cap R_\gamma(m, \eta, r) \neq \emptyset$. The minimality of i implies $r \neq s$. So property I(3) guarantees some $t \in p(s) \cap p(r) \cap T_\gamma$. But our definition of γ then implies r and s are either both in H or both in K which is a contradiction.

Since $p(T_\gamma)$ is countable, a slightly more complicated construction of U and V would yield a cover of T. Hence Σ also has the property that any two disjoint closed sets are contained in the union of disjoint open and closed sets.

III. The construction of $R_\gamma(A)$

1. *Some definitions and lemmas.* If $S \subset T$ and $s \in S$, define

$$S(s) = \{t \in f(s) \mid \text{for all } s \leq r \leq t, r \in S\}.$$

If $A = (n, \alpha, t) \in \mathcal{A}$, define $\mathcal{S}(A)$ to be the set of all nonempty $S \subset \bigcup \mathcal{Z}(A)$ such that $s \in S \cap T_\beta$ and $\beta < \gamma < \alpha$ implies there exists δ with $\gamma < \delta < \alpha$ where $S(s) \cap T_\delta$ has at least two terms.

If R, S belong to $\mathcal{S}(A)$ define $R < S$ if

(i) for each $s \in S$ there is an $r \in R$ such that $R(r) \subset S(s)$, and
(ii) for each $r \in R$ there is a $V \in R(r)^*$ such that $V \subset S$.

LEMMA 1. *Suppose $\{A_n\}_{n\in\omega}$ and $\{B_n\}_{n\in\omega}$ are disjoint countable subsets of $\mathcal{A}$ and, for each $n \in \omega$, $S_n \in \mathcal{S}(A_n)$ and $Y_n \in \mathcal{Z}(B_n)$, and $n \neq m$ implies $p(Y_n) \neq p(Y_m)$. Then, for each $n \in \omega$, there exist $R_n \in \mathcal{S}(A_n)$ and $X_n \in Y_n^*$ such that $R_n < S_n$ and the terms of $\{R_n\}_{n\in\omega} \cup \{X_n\}_{n\in\omega}$ are disjoint.*

Proof. Define $\{C_n\}_{0<n<\omega} = \{B_n\}_{n\in\omega} \cup \{A_i, j, k\}_{i,j,k\in\omega}$; assume $n \neq m$ implies $C_n \neq C_m$. Index $S_i = \{s_{ij}\}_{j\in\omega}$.

For $n \in \omega$, we define by induction a function $g_n : (0, 1, \cdots, n) \to$ the set of all subsets of T.

Define $g_n(0) = \emptyset$ for all $n \in \omega$.

Fix $n > 0$ in order to define g_n.

Assume $g_{n-1}(m)$ has been defined for all $m < n$. Let $W = \bigcup_{m<n} g_{n-1}(m)$. Also assume:

(a) $0 < m < n$ and $C_m = B_i$ implies $g_{n-1}(m) \in Y_i^*$.
(b) $0 < m < n$ and $C_m = (A_i, j, k)$ implies there is an $s \in S_i$ and a finite set E_m of branch points of $S_i(s)$ such that:

(b. 1) $g_{n-1}(m) = f(s) \cap p(E_m)$,
(b. 2) $e \in E_m$ implies $f(e) \cap (W - \{e\}) = \emptyset$,
(b. 3) $q < n$ and $E_q \cap E_m \neq \emptyset$ implies $g_{n-1}(q) = g_{n-1}(m)$ and the first two terms of C_m are the first two terms of C_q.

Note that W and E_m are functions of n.

We now define $g_n(n)$. Observe that W is the union of finitely many chains.

Case 1. Suppose $C_n = B_i$. Choose $g_n(n) \in Y_i^*$ such that $g_n(n) \cap W = \emptyset$.

Case 2. Suppose $C_n = (A_i, j, k)$ and for no $m < n$ is $C_m = (A_i, j, h)$ for any h. Choose a branch point t of $S_i(s_{ij})$ such that $f(t) \cap W = \emptyset$. Then define $g_n(n) = \{t\}$.

Case 3. Suppose $C_n = (A_i, j, k)$ and $m < n$ and m is the smallest integer such that $C_m = (A_i, j, h)$ for some $h \in \omega$. For each $x \in E_m$ choose distinct branch points x_1 and x_2 of $S_i(x)$ belonging to T_β for some $\beta > \alpha^k$ where α is the second term of A_i. Then define

$$g_n(n) = g_{n-1}(m) \cup \bigcup_{x \in E_m} (f(x) \cap p(x_1, x_2)).$$

Suppose $m < n$. Define $g_n(m) = g_{n-1}(m)$ unless there is a point $e \in E_m$ such that $g_n(n) \cap f(e) \neq \emptyset$. If $e \in g_n(n) \cap E_m$, define $g_n(m) = g_n(n)$.

Suppose there *is* a point e such that for some $q < n$, $e \in E_q - g_n(n)$ and $g_n(n) \cap f(e) \neq \emptyset$. Let $M = \{m < n \mid$ the first two terms of C_m are the first two terms of $C_q\}$; let A_i be the first term of C_q. Since $e \in S_i$ and $S_i \in \mathcal{S}(A_i)$, there are unordered branch points e_1 and e_2 of $S_i(e)$. Since $e \notin g_n(n)$, Case 1 or 2 holds and $g_n(n)$ is contained in a single chain. Hence (b. 2) implies that, for some $h = 1$ or 2, $f(e_h) \cap g_n(n) = \emptyset$. Define

$$g_n(m) = g_{n-1}(m) \cup (f(e) \cap p(e_h)) \quad \text{for all} \quad m \in M.$$

The induction hypotheses are again satisfied.

If $B_i = C_n$ define $X_i = g_n(n)$. And define

$$R_i = \bigcup_{n,m,j,k \in \omega} \{g_n(m) \mid C_m = (A_i, j, k)\}.$$

Then $X_i \in Y_i^*$, $R_i \in \mathcal{S}(A_i)$, $R_i < S_i$, and the terms of $\{R_i\}_{i \in \omega} \cup \{X_i\}_{i \in \omega}$ are disjoint.

LEMMA 2. *Suppose* $A \in \mathfrak{A}$ *and* γ *is a countable limit ordinal and* $\{X_\beta\}_{\beta<\gamma} \subset \mathcal{S}(A)$ *and* $\alpha < \beta < \gamma$ *implies* $X_\beta < X_\alpha$. *Then there is an* $X \in \mathcal{S}(A)$ *such that. for all* $\beta < \gamma$, $X < X_\beta$.

Proof. Index $\{(x_n, \beta_n)\}_{n\in\omega} = \{(x, \beta) \mid \beta < \gamma \text{ and } x \in X_\beta\}$. Let D be the set of all finite sequences of 0's and 1's.

For each $n \in \omega$ we define R_n as follows. Define $R_0 = \emptyset$. Suppose $R_n \subset \mathcal{T}$ has been defined for all $m < n$. If, for all $s \in X_{\beta_n}(x_n)$, $X_{\beta_n}(s) \cap \bigcup_{m<n} R_m \neq \emptyset$, then define $R_n = \emptyset$.

Suppose there is an $s \in X_{\beta_n}(x_n)$ such that $X_{\beta_n}(s) \cap \bigcup_{m<n} R_m = \emptyset$; we define R_n in this case after the following inductive construction. Choose $s' \in X_{\beta_n}(s) - \{s\}$. There is $k \in \omega$ such that $\gamma^k > \beta_n$. Choose unordered r_0 and r_1 belonging to X_{γ^k} such that $X_{\gamma^k}(r_i) \subset X_{\beta_n}(s')$ for $i = 0, 1$. Suppose $d = d_0, d_1, \cdots, d_j \in D$ and $r_d \in X_{\gamma^{k+j}}$ has been chosen. Choose unordered $r_{d_0,\cdots,d_j,0}$ and $r_{d_0,\cdots,d_j,1}$ belonging to $X_{\gamma^{k+j+1}}$ such that, for $i = 0, 1$, $X_{\gamma^{k+j}}(r_d) \supset X_{\gamma^{k+j+1}}(r_{d_0,\cdots,d_j,i})$. Having thus chosen r_d for all $d \in D$, define $R_n = f(s') \cap \bigcup_{d\in D} p(r_d)$.

Then define $X = \bigcup_{n\in\omega} R_n$.

Clearly $X \in \mathcal{S}(A)$. Observe that $r \in R_n$ implies $X(r) = R_n(r)$. Suppose $\beta < \gamma$ and let us indicate why $X < X_\beta$.

To test (ii), assume $r \in X$. Then $X(r) \subset R_n$ for some n. By the construction of R_n, there is $k \in \omega$ with $\beta < \gamma^k$ and a finite subset F of $R_n \cap X_{\gamma^k}$ such that $R_n - p(F) \subset X_{\gamma^k}$. Since $\beta < \gamma^k$, for each $v = F$, there is $V_v \in X_{\gamma^k}(v)^*$ such that $V_v \subset X_\beta$. Since $V_v \cap R_n \in R_n(v)^*$, $\bigcup_{v\in F} V_v \supset V \in R_n^*$. Thus $V \cap R_n(r) \in R_n(r)^* = X(r)^*$ and $V \subset X_\beta$ so (ii) is satisfied.

To test (i) assume $s \in X_\beta$; then $(s, \beta) = (x_n, \beta_n)$ for some n. We need to find $r \in X$ such that $X(r) \subset X_\beta(s)$. This is obvious if $R_n \neq \emptyset$. So assume $R_n = \emptyset$. Choose $t \in X_\beta(s) \cap R_m$ for some $m < n$. By the preceding paragraph there is $V \in X(t)^*$ such that $V \subset X_\beta$; thus $V \subset X_\beta(t) \subset X_\beta(s)$. Choose $r \in V$; then $X(r) \subset V \subset X_\beta(s)$ and (i) is satisfied.

2. We now use Lemmas 1 and 2 to define for each $\gamma < \omega_1$ and $A \in \mathfrak{A}$, a set $R_\gamma(A)$ so that conditions (1), (2), and (3) of I are satisfied. We need further definitions.

If $\gamma < \omega_1$, define $\mathfrak{A}^\gamma = \{(n, \alpha, t) \in \mathfrak{A} \mid t \in T_\gamma\}$.

If $A = (n, \alpha, t) \in \mathfrak{A}^\gamma$ and $\alpha < \beta \leq \gamma$, let $A_\beta = (n, \alpha, s)$ where $\{s\} = T_\beta \cap p(t)$.

Let $\mathfrak{A}' = \{(n, \alpha, r) \in \mathfrak{A} \mid r \in T_{\alpha+1}\}$; if $A = (n, \alpha, t) \in \mathfrak{A}$, define $A' = A_{\alpha+1}$. For each $A \in \mathfrak{A}'$ choose arbitrarily an $R(A) \in \mathcal{S}(A)$.

If $A = (n, \alpha, t) \in \mathfrak{A}$ and $\gamma \leq \alpha$, define $R_\gamma(A) = R(A')$.

Suppose $\gamma < \omega_1$ and for all $A \in \mathfrak{A}$ and $\beta < \gamma$, $R_\beta(A)$ has been defined satisfying:

(a) $A \in \mathfrak{A}^\delta$ and $\delta < \beta$ implies $R_\beta(A) \in \mathcal{Z}(A)$; $A \in \mathfrak{A}^\delta$ and $\beta \leq \delta$ implies $R_\beta(A) \in \mathcal{S}(A)$.

(b) $A = (n, \alpha, t) \in \mathfrak{A}^{\delta}$ and $\alpha < \eta \leq \beta \leq \delta$ implies $R_\beta(A) = R_\beta(A_\beta) < R_\eta(A_\eta)$.

Before we complete the definition of R_γ, we define $X(A)$ for all $A \in \bigcup_{\beta \leq \gamma} \mathfrak{A}^\beta$.

Suppose γ is a limit ordinal. By Lemma 2 and (b), if $A = (n, \alpha, t) \in \mathfrak{A}^\gamma$, we can choose $X(A) \in \mathcal{S}(A)$ such that $X(A) < R_\beta(A)$ for all $\alpha < \beta < \gamma$. If $\beta < \gamma$ and $A \in \mathfrak{A}^\beta$, define $X(A) = R_{\beta+1}(A)$.

Suppose γ is not a limit ordinal. If $A \in \mathfrak{A}^\gamma$, define $X(A) = R_{\gamma-1}(A)$. If $A \in \mathfrak{A}^{\gamma-1}$, choose $X(A) \in \mathcal{Z}(A)$ such that $X(A) \subset R_{\gamma-1}(A)$. And if $\beta < \gamma - 1$ and $A \in \mathfrak{A}^\beta$, define $X(A) = R_\beta(A)$.

Observe that $\bigcup_{\beta<\gamma} \mathfrak{A}^\beta$ is countable and $A \in \bigcup_{\beta \leq \gamma} \mathfrak{A}^\beta$ implies $X(A) \in \mathcal{Z}(A)$ and $\mathfrak{A}^\gamma$ is countable and $A \in \mathfrak{A}^\gamma$ implies $X(A) \in \mathcal{S}(A)$. So we can apply Lemma 1 and find disjoint $R_\gamma(A)$ for the $A \in \bigcup_{\beta \leqq \gamma} \mathfrak{A}^\beta$ such that $R_\gamma(A) \in X(A)^*$ for $A \in \bigcup_{\beta<\gamma} \mathfrak{A}^\beta$ and $R_\gamma(A) \in \mathcal{S}(A)$ and $R_\gamma(A) < X(A)$ for $A \in \mathfrak{A}^\gamma$.

If $A = (n, \alpha, t) \in \mathfrak{A}^\beta$ for some $\beta > \gamma$, define $R_\gamma(A) = R_\gamma(A_\gamma)$ if $\alpha < \gamma$; we have already defined $R_\gamma(A) = R(A')$ if $\gamma \leq \alpha$.

Our induction hypotheses (a) and (b) are clearly again satisfied. We need only check (1), (2), and (3) of I.

If $A \in \mathfrak{A}^\beta$, then $R_{\beta+1}(A) \in \mathcal{Z}(A)$ so (1) is satisfied.

Suppose $A = (n, \alpha, t) \in \mathfrak{A}^\beta$. If $\gamma \leq \alpha$, we defined $R_\gamma(A) = R(A')$. We chose $R_{\alpha+1}(A) = R_{\alpha+1}(A') < R(A')$ and if $\alpha < \delta \leq \gamma \leq \beta$ we chose $R_\gamma(A) < R_\delta(A) \in \mathcal{S}(A)$. And $R_{\beta+1}(A) \subset R_\beta(A)$ and, for $\beta + 1 < \gamma$, $R_\gamma(A) \in R_{\beta+1}(A)^*$. Thus for all $\gamma < \omega_1$ there is a term of $R_{\beta+1}(A)^*$ contained in $R_\gamma(A)$ and (2) is satisfied.

Suppose $A = (n, \alpha, s)$, $B = (m, \beta, r)$, $\alpha \leq \beta < \gamma$, $r \neq s$, and $p(s) \cap p(r) \cap T_\gamma = \emptyset$. If $s \in T_\delta$, define $\hat{A} = A$ if $\delta \leq \gamma$ and $\hat{A} = A^\gamma$ if $\delta > \gamma$; define $\hat{B}$ similarly. By our assumption $\hat{A} \neq \hat{B}$. Thus we chose $R_\gamma(\hat{A})$ and $R_\gamma(\hat{B})$ disjoint. And, since $R_\gamma(A) = R_\gamma(\hat{A})$ and $R_\gamma(B) = R_\gamma(\hat{B})$, condition (3) is satisfied and we have the desired construction.

Bibliography

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University of Wisconsin
Madison, Wisconsin