ILLINOIS JOURNAL OF MATHEMATICS
Volume 28, Number 2, Summer 1984

RESIDUAL NILPOTENCY OF FUCHSIAN GROUPS[1]

BY

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0. Introduction

If we have a group with a relatively complicated structure, but with many homomorphisms into groups of a simpler nature, the study of these homomorphisms may enable us to obtain information not easily available otherwise about the original group. If the family of groups which are targets of the homomorphisms is characterized by a property P, the information we obtain is in theory complete if the original group is "residually P" in Philip Hall's terminology; that is to say, if there are enough homomorphisms to groups with the property P to distinguish any one element of the group from any other, so that homomorphisms to groups with property P provide a sort of "coordinate system".

In the case of Fuchsian groups, there is an additional motivation for studying homomorphisms to different classes of group—the homomorphic images can always be realized as groups of automorphisms of Riemann surfaces. Homomorphisms of Fuchsian groups into finite groups, finite cyclic groups, finite abelian groups and finite soluble groups have been studied, with perhaps a disproportionate amount of attention, for which I must admit some personal responsibility, being paid to the very special and certainly fascinating class of Riemann surfaces for which Hurwitz's maximum of $84(g - 1)$ automorphisms is attained (see [1], [3], [4], [5], [8], [10], [16], [18], [19], [20], [21], [22], [24]).

All Fuchsian groups are residually finite, and many are residually finite-and-soluble (see [24]).

In his study of residual solubility, Sah was led to introduce the "p-periods" of a Fuchsian group, which can be regarded as the first step in the direction of the p-localization introduced in the present paper. Our object here is to close the obvious gap in the list of target groups for homomorphisms by studying maps into finite nilpotent groups. We restrict attention to co-compact Fuchsian groups, so that we can use the technique of Gruenberg [9] based on Hirsch's theorem [12] that a finitely generated nilpotent group is residually finite. Therefore a finitely generated group is

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[1] This material is based upon work supported by a grant from the National Science Foundation.

Received December 10, 1981.

residually nilpotent if and only if it is residually finite-of-prime-power-order. By considering the primes individually, we find easy access to the structure of the family of homomorphisms with nilpotent target.

We find that, while the existence of homomorphisms to nilpotent groups is the rule rather than the exception for Fuchsian groups, yet residual nilpotency is the exception rather than the rule. By localizing, we are able to prove that a Fuchsian group is residually nilpotent if all its periods are powers of a single prime, and only then. We are also able to characterize the intersection of all kernels of homomorphisms into nilpotent groups as a finite intersection of kernels of "p-local" homomorphisms. Essential to our argument is Baumslag's theorem that the fundamental group of a closed orientable surface is residually free [2].

Since the process of localization may change a perfectly respectable presentation of a Fuchsian group into a presentation which is in some way peculiar, it has seemed necessary to introduce some extra technical apparatus in order to express our results effectively. For this we apologize.

My thanks to Peter Neumann for drawing attention to Baumslag's theorem mentioned above.

Since writing the above, I have learned that R. S. Kulkarni has independently proved the equivalence of 8.1(i) and 8.1(ii) below, though he formulates 8.1(ii) slightly differently.

1. Signatures

A *signature* S is an ordered $(r + 1)$-tuple of integers

$$S = (g; m_1, \dots, m_r) \tag{1.1}$$

such that $r \geqslant 0$, $g \geqslant 0$, $m_i \geqslant 1$ $(i = 1, \dots, r)$.

The number g is called the genus, and those of the m_i which are not less than 2 are called the *periods*. If none of the m_i is equal to 1, S is said to be *reduced*. The signature $\overline{S}$ obtained from S by removing all those of the m_i which are equal to 1 is called the *reduced signature* or *reduced form* of S. With the signature S is associated a group presentation

$$\langle x_1, \dots, x_r, a_1, b_1, \dots, a_g, b_g \,|\, x_1^{m_1}, \dots, x_r^{m_r}, \\ x_1x_2 \dots x_r a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} \rangle. \tag{1.2}$$

The group defined by (1.2) is called the *group of the signature* and is denoted by $\Gamma(S)$. If $m_i = 1$, then x_i will denote the unit element and may be dropped from the generating system. Thus $\Gamma(S) \cong \Gamma(\overline{S})$, and for many purposes only reduced signatures need be considered. However, the operation of p-localization, which plays a key role in this paper, may lead from a reduced to a non-reduced signature, so non-reduced signatures must be considered. Unless the context makes it necessary, we shall not distinguish pedantically between a signature and its reduced form regarding two signatures S, T as equal if $\overline{S} = \overline{T}$.

The Euler characteristic of S is the rational number

$$\chi(S) = 2 - 2g + \sum_{\nu=1}^{r}\left(\frac{1}{m_\nu} - 1\right). \tag{1.3}$$

The signature S is *degenerate* if the associated reduced form satisfies one of the following:

(a) $g = 0, r = 1,$
(b) $g = 0, r = 2, m_1 \neq m_2.$

The group defined by a degenerate signature is cyclic of order ≥ 1 and can be defined by another signature which is non-degenerate. Like signatures which are not reduced, degenerate signatures may be ignored for many purposes, but in this context they must be considered because they too can arise as a result of p-localization.

2. Automorphisms of Simply Connected Riemann Surfaces [6], [8], [11], [15], [26]

Let S be a non-degenerate signature. Then $\Gamma(S)$ is a cocompact discrete group of biholomorphic mappings of a simply connected Riemann surface $\tilde{X}$. There are three simply connected Riemann surfaces, yielding three classes of group. If S is the signature (1.1), then there are r distinct orbits of points with nontrivial stabilizer, the orders of the stabilizers being $m_1, \ldots, m_r$.

(a) If $\chi(S) > 0$, $\Gamma(S)$ is finite of order $2/\chi(S)$ and acts on the Riemann sphere. The only reduced non-degenerate signatures with $\chi(S) > 0$ are $(0;m,m)$, $(0;2,2,m)$, $(0;2,3,3)$, $(0;2,3,4)$, $(0;2,3,5)$ and those obtained from them by permuting the periods.

(b) If $\chi(S) = 0$, then $\Gamma(S)$ is infinite and soluble, being either free abelian of rank 2 or having a free abelian group of rank 2 as a normal subgroup of finite index, the factor group being cyclic. The signatures S of Euler characteristic zero are $(1;)$, $(0;2,2,2,2)$, $(0;2,4,4)$, $(0;2,3,6)$, $(0;3,3,3)$ and those obtained from them by permuting the periods. In this case $\Gamma(S)$ acts on the complex plane $\mathbf{C}$.

(c) If $\chi(S) < 0$, $\Gamma(S)$ can be realized as a *Fuchsian group,* that is a group of linear fractional transformations of the complex upper half-plane, $\{z \in \mathbf{C}: 2i(z - \bar{z}) < 0\}$.

If Γ_1 is a subgroup of finite index in the group $\Gamma(S)$, where S is non-degenerate, then there is a signature S_1 such that $\Gamma_1 \cong \Gamma(S_1)$, and we have

$$\chi(S_1) = (\Gamma : \Gamma_1)\chi(S). \tag{2.1}$$

In fact, if $\chi(S) \leq 0$, then $\chi(S)$ coincides with the Euler characteristic of $\Gamma(S)$ in the sense recently introduced into group theory. However, if $\chi(S) > 0$, it is twice the group-theoretic Euler characteristic. This unfortunate fact is related to the topological properties of the spaces on which the groups act. The sphere is not contractible, while the other two spaces are. The equation (2.1) is classical, and is known as the *Riemann-Hurwitz relation.*

3. Smooth Homomorphisms

Let G_1, G_2 be groups, Σ a family of subgroups of G_1. A homomorphism $f\colon G_1 \to G_2$ is defined to be *Σ-injective* (or, injective on Σ) if, for every $H \in \Sigma$, the restriction $f|_H$ is injective. If $f|_H$ is injective and $K \subset H$, then $f|_K$ is also injective; and, since $\operatorname{Ker} f \cap H = t(\operatorname{Ker} f \cap t^{-1}Ht)t^{-1}$, $f|_{t^{-1}Ht}$ is also injective. Thus:

(3.1) *If Σ is a family of groups and Σ_1 is a subfamily such that every element of Σ is a subgroup of a conjugate of an element of Σ_1, then f is Σ-injective if and only if f is Σ_1-injective.*

In particular, if Σ denotes the family of all finite subgroups of G_1, we shall use the term *finite-injective* instead of Σ-injective. If p is a prime number, we shall use the term *p-injective* for a homomorphism which is injective on the family of all subgroups which are finite p-groups. Clearly:

(3.2) *The homomorphism f is finite-injective if and only if f is p-injective for every prime number p.*

When $G_1 = \Gamma(S)$ is the group of the signature (1.1), a homomorphism $f\colon G_1 \to G_2$ is called *smooth* if the order of $f(x_i)$ is precisely m_i for each $i = 1, \ldots, r$. If p is a prime number, then f is called *p-smooth* if the order of $f(x_i)$ is divisible by the highest power p^{α_i} of p which divides m_i. Clearly again:

(3.3) *f is smooth if and only if f is p-smooth for every prime factor p of the product $m_1m_2 \ldots m_r$.*

We require the following well-known property of groups defined by signatures. It is straightforward if $\chi(S) > 0$, since only the groups listed in 2(a), (b) need be considered. If $\chi(S) \leq 0$, it was initially proved by studying the geometry of the group action, but in the past ten years or so elegant algebraic proofs have appeared. See [13], [14], [17]. For classical proofs, see [7], [15], [26].

(3.4) THEOREM. *If S is a non-degenerate signature, then every element of finite order in $\Gamma(S)$ is conjugate to a power of some x_i. Moreover, the order of x_i is precisely m_i. If $\chi(S) \leq 0$, every finite subgroup of $\Gamma(S)$ is cyclic.*

(3.5) COROLLARY. *The identity* id: $\Gamma(S) \to \Gamma(S)$ *is smooth if and only if S is non-degenerate.*

Proof. If S is non-degenerate, this is simply the second assertion of

Theorem 3.4. If S is degenerate of the form $(0;m)$, then the order of x_1 is $1 \neq m$. If S is of the form $(0;m,n)$, then $\Gamma(S)$ is cyclic of order $d = (m,n)$, so the order of both x_1 and x_2 is d. This is equal to both m and n if and only if $m = n$, i.e., if S is non-degenerate.

(3.6) THEOREM. *Let S be non-degenerate. Then f: $\Gamma(S) \to G$ is smooth if and only if f is finite-injective.*

Proof. Let Σ be the family of groups $\{\langle x_1\rangle, \ldots, \langle x_r\rangle\}$. By Theorem 3.4, f is smooth if and only if f is Σ-injective. Again by (3.4), every finite subgroup of $\Gamma(S)$ is conjugate to a subgroup of an element of Σ if $\chi(S) \leq 0$. This proves the theorem if $\chi(S) \leq 0$ using (3.1). If $\chi(S) > 0$, the groups can be dealt with individually, and it can easily be checked that the only smooth homomorphisms are isomorphisms. Essentially the same argument shows

(3.7) THEOREM. *If S is non-degenerate, f: $\Gamma(S) \to G$ is p-smooth if and only if f is p-injective.*

4. Automorphisms of Compact Riemann Surfaces

If X is a compact Riemann surface and G is a finite group of automorphisms, i.e., biholomorphic self-mappings, of X, then there is a group $\tilde{G}$ of automorphisms of the universal covering space $\tilde{X}$ of X obtained by taking all liftings of all elements of G. See [8], [15], [18], [23], [24].

We shall say that $\tilde{G}$ *covers* the Riemann surface automorphism group G. In that case there is a homomorphism ϕ of the covering group $\tilde{G}$ onto G such that the kernel of ϕ is the fundamental group of the surface X, and such that, if p denotes the covering map and the horizontal arrows denote the group actions, the diagram below commutes:

$$\begin{array}{ccccc} \tilde{G} & \times & \tilde{X} & \to & \tilde{X} \\ \phi\downarrow & & \downarrow p & & \downarrow p. \\ G & \times & X & \to & X \end{array} \tag{4.1}$$

In this case $\tilde{X}$ will be one of the three simply connected Riemann surfaces (§2) and $\tilde{G}$ will be the group of a signature S. The kernel $\pi_1(X)$ will be a group of signature $(g;)$, where g is the genus of X, and, by (2.1),

$$\chi(S)|G| = 2 - 2g. \tag{4.2}$$

Thus $\tilde{G}$ is a Fuchsian group if and only if $g \geq 2$. Since $\pi_1(X)$ has no elements of finite order, ϕ is smooth. Conversely, it can easily be shown that every smooth homomorphism ϕ: $\Gamma(S) \to G$ induces a group action of G as a group of automorphisms of the Riemann surface $\tilde{X}/\ker \phi$. Therefore:

(4.3) *We obtain all Riemann surface automorphism groups (G, X) with G finite and X compact by finding all smooth homomorphisms of groups $\Gamma(S)$ onto finite groups G.*

5. Localization

Suppose that p is a prime number and that S is the signature (1.1). For $i = 1, \ldots, r$, let α_i be the largest number such that p^{α_i} is a divisor of m_i. The signature

$$S_p = (g; p^{\alpha_1}, \ldots, p^{\alpha_r}) \tag{5.1}$$

is called the *p-localization* of S. If every period of S is already a power of p, so that $S = S_p$, then the signature S is said to be *p-local*.
Now the group $\Gamma(S_p)$ has the presentation

$$\langle x_1', \ldots, x_r', a_1', b_1', \ldots, a_g', b_g' \mid x_1'^{p^{\alpha_1}}, \ldots, x_r'^{p^{\alpha_r}}, x_1'x_2' \ldots x_r'a_1'b_1'a_1'^{-1}b_1'^{-1} \ldots a_g'b_g'a_g'^{-1}b_g'^{-1}\rangle. \tag{5.2}$$

Since $p^{\alpha_i} \mid m_i$, we have $(x_i')^{m_i} = 1$, so the function defined on the generating set by

$$x_i \to x_i', \; a_j \to a_j', \; b_k \to b_k' \quad (i = 1, \ldots, r; j, k = 1, \ldots, g)$$

can be extended to a homomorphism l_p: $\Gamma(S) \to \Gamma(S_p)$. We shall call l_p the *p-localization homomorphism.*

(5.3) THEOREM. *If G_p is a finite p-group and ϕ: $\Gamma(S) \to G_p$ is a homomorphism, then there is a unique homomorphism ϕ_p: $\Gamma(S_p) \to G_p$ such that $\phi = \phi_p \circ l_p$.*

Proof. Use the presentations (1.2), (5.2), for $\Gamma(S)$, $\Gamma(S_p)$ respectively. If ϕ_p exists as claimed, it must satisfy

$$\phi_p(x_i') = \phi_p(l_p(x_i)) = \phi(x_i), \; \phi_p(a_j') = \phi(a_j), \; \phi_p(b_k') = \phi(b_k) \tag{5.4}$$

and, since the x', a', b' generate $\Gamma(S_p)$, this will determine ϕ_p uniquely. To show that this does define a homomorphism, we must check that it is compatible with the defining relators of (5.2). Now it is certainly compatible with the long relator, because there is an exact match of the long relators in (1.2), (5.2). We only need check that

$$\phi_p(x_i')^{p^{\alpha_i}} = 1,$$

i.e., that $\phi(x_i)^{p^{\alpha_i}} = 1$. Since $\phi(x_i^{m_i}) = 1$ and the order of $\phi(x_i)$, being an element of a p-group, is a power of p, it follows that $\phi(x_i)^{p^{\alpha_i}} = 1$. Thus ϕ_p extends to a homomorphism as claimed. The usefulness of Theorem 5.3 in studying Riemann surface automorphism groups is clear from the following theorem.

(5.4) THEOREM. *Let G be a finite nilpotent group and, for each prime p, let G_p be its p-Sylow subgroup. For formal simplicity, let $G_p = \{1\}$ if p is not a factor of the order of G. Let ϕ: $\Gamma(S) \to G$ be a homomorphism and let λ_p: $G \to G_p$ be the projection of G (as a product of its Sylow* *subgroups) onto G_p. Then ϕ is smooth if and only if $(\lambda_p \circ \phi)_p$ is smooth for each $p \mid m_1 m_2 \ldots m_r$.*

Proof. Using (3.3) we need only show that $(\lambda_p \circ \phi)_p$ is smooth if and only if ϕ is p-smooth.

1. If $(\lambda_p \circ \phi)_p$ is smooth, then $\lambda_p(\phi(x_i))$ has order precisely p^{α_i}, so, since λ_p is a homomorphism, $\phi(x_i)$ has order divisible by $p_i^{\alpha_i}$ and ϕ is p-smooth.
2. If the order of $\phi(x_i)$ is divisible by p^{α_i}, then so is the order of $\lambda_p(\phi(x_i))$, since the order of the kernel of λ_p is relatively prime to p. But

$$\lambda_p(\phi(x_i)) = (\lambda_p \circ \phi)_p(x_i'),$$

so $(\lambda_p \circ \phi)_p$ is smooth.

Theorem 5.4 reduces the study of nilpotent Riemann surface automorphism groups to the study of smooth homomorphisms of p-local groups onto finite p-groups. The set, which we shall denote by $\Pi(S)$, of prime factors of $m_1 m_2 \ldots m_r$, plays a critical role, because, if $p \notin \Pi(S)$, then S_p has no periods and every homomorphism from $\Gamma(S_p)$ to a finite group is smooth. Let $p_1, \ldots, p_k$ be any set of prime numbers including $\Pi(S)$. By Theorem 5.4, each smooth homomorphism ϕ from $\Gamma(S)$ to the finite nilpotent group

$$G = G_{p_1} \times \ldots \times G_{p_k}$$

determines a set of smooth homomorphisms

$$\psi_{p_i}: \Gamma(S_{p_i}) \to G_{p_i} \quad (i = 1, \ldots, k)$$

such that, if $\gamma \in \Gamma(S)$ and $g_i = \psi_{p_i}(l_{p_i}(\gamma))$,

then $$\phi(\gamma) = g_1 g_2 \ldots g_k.$$

Thus one may obtain all smooth homomorphisms from $\Gamma(S)$ to G by taking all possible smooth homomorphisms from $\Gamma(S_p)$ to the Sylow p-subgroups of G.

6. The p-Frattini Series

Let G be any group, p a prime number. Let G^p be the subgroup generated by all commutators and all pth powers of elements of G. Clearly G^p is a characteristic subgroup and, if G is finitely generated, G/G^p is an elementary abelian p-group. It is also the intersection of all normal subgroups of index p, and is a generalization of the Frattini subgroup of a finite p-group. If G has a presentation

$$\langle a_1, \ldots, a_m \mid R_1(a), \ldots, R_k(a) \rangle$$

then the presentation of G/G^p is obtained by adding the extra relators

$$a_i a_j a_i^{-1} a_j^{-1} \text{ and } a_i^p \quad (i, j = 1, \ldots, m).$$

We write $(G^p)^p = G_2^p$, and inductively define $G_{r+1}^p = (G_r^p)^p$. Then G_r^p

is characteristic in G and G/G_r^p is a finite p-group. Conversely, if N is any normal subgroup of G such that G/N is a finite p-group, then $N \supset G_i^p$ for some i, because a non-trivial finite p-group always has a normal subgroup of index p, so that the series

$$G/N \rhd G^p/N \rhd G_2^p/N \rhd \ldots$$

must reach the unit element after a finite number of steps. Hence we have:

(6.1) THEOREM. *G is residually a finite p-group if and only if* $\bigcap_n G_n^p = \{1\}$.

It is known that a free group is residually a finite p-group, and G. Baumslag has shown that the fundamental group of an orientable surface is residually free (see [2]). We deduce:

(6.2) THEOREM. *A surface group (that is, a group isomorphic to $\Gamma(g;)$ for some g) is residually a finite p-group, for every prime p.*

Let us now consider the p-Frattini series of $\Gamma = \Gamma(S_p)$, where

$$S_p = (g; p^{\alpha_1}, \ldots, p^{\alpha_r})$$

is a reduced p-local signature. Let $N = \max(\alpha_1, \ldots, \alpha_r)$ and assume that $\chi(S_p) \leqslant 0$, so that, if $g = 0$, $r \geqslant 2$.

(6.3) LEMMA. *If $r \geqslant 2$, the maximum period of $\Gamma(S_p)^p$ is p^{N-1}.*

Proof. It is easily verified that Γ/Γ^p is elementary abelian of rank $2g + r - 1$, and that none of the elements x_i or their conjugates belong to Γ^p. Thus the maximum period of Γ^p is p^{N-1}.

(6.4) LEMMA. *If $r = 1$, the number of periods of Γ^p is $\geqslant 2$.*

Proof. If $S = (g; p^N)$, then the long relation shows that x_1 is in the derived group, therefore in Γ^p. In this case, since we are assuming that $\chi(S) \leqslant 0$, we must have $g \geqslant 1$, Γ/Γ^p has order p^{2g}. By Singerman's Theorem [25] the number of periods of Γ^p is $p^{2g} \geqslant 2$.

Combining 6.3 and 6.4 we deduce:

The maximum period of Γ_2^p is strictly less than the maximum period of Γ.

We immediately have:

(6.5) THEOREM. *If S_p is a p-local signature with $\chi(S_p) \leqslant 0$, then Γ_i^p has no periods if i is sufficiently large.*

Since the natural homomorphism of Γ on Γ/Γ_i^p is smooth if and only if Γ_i^p has no periods, we deduce:

(6.6) COROLLARY. *If $\chi(S_p) \leq 0$, then $\Gamma(S_p)$ covers infinitely many Riemann surface automorphism groups which are finite p-groups.*

(The number is infinite because each Γ_i^p has finite index in the one before, and $\Gamma(S_p)$ is an infinite group. Thus all the terms of the series Γ_i^p are distinct).

(6.7) COROLLARY. *If S_p is a p-local signature, then $\Gamma(S_p)$ is residually a finite p-group.*

Proof. Suppose k is such that Γ_k^p is a surface group. Then Γ_k^p is residually a finite p-group by (6.2), so, by (6.1) the intersection $\cap_{i=1}^{\infty} (\Gamma_k^p)_i^p$ is trivial, and Γ is residually a finite p-group, again by (6.1). This proves the result if $\chi(S_p) \leq 0$. If $\chi(S_p) > 0$, then $\Gamma(S_p)$ *is* a finite p-group, being either a cyclic group of prime power order or a dihedral 2-group.

7. Relationship Between the Lower Central Series and Localization

Let S be the signature (1) and let $\Pi(S)$ denote, as before, the set of prime divisors of the periods of S. Let $\Gamma_f(S)$ denote the characteristic subgroup of Γ generated by all the elements of $\Gamma(S)$ of finite order, so that, by (3.4)

$$\Gamma_f(S) = \text{normal closure } (x_1, \ldots, x_r).$$

7.1 LEMMA. *For all primes p, $\ker l_p \subset \Gamma_f(S)$, and $\ker l_p = \Gamma_f(S)$ if and only if $p \notin \Pi(S)$.*

Proof. $\Gamma(S_p)$ is obtained from $\Gamma(S)$ by adjoining the relators

$$x_1^{p^{\alpha_1}}, \ldots, x_r^{p^{\alpha_r}},$$

so

$$\ker l_p = \text{normal closure } (x_1^{p^{\alpha_1}}, \ldots, x_r^{p^{\alpha_r}}) \subset \Gamma_f(S).$$

If, in particular, p is not a divisor of any of the periods, then $\alpha_1 = \alpha_2 = \ldots = \alpha_r = 0$ and $\ker l_p = \Gamma_f(S)$. Now consider the lower central chain

$$\Gamma = \gamma_1(\Gamma) \rhd \gamma_2(\Gamma) \rhd \ldots \quad \text{where } \gamma_{i+1}(\Gamma) = [\Gamma, \gamma_i(\Gamma)].$$

Let $\gamma_\infty(\Gamma) = \cap_{n=1}^{\infty} \gamma_n(\Gamma)$. The group $\gamma_\infty(\Gamma)$ is referred to as the *nilpotent residual.*

(7.2) LEMMA. *$\gamma_\infty(\Gamma)$ consists of those elements of Γ that are mapped on the unit element by all homomorphisms of Γ onto nilpotent groups. In particular, Γ is residually nilpotent if and only if $\gamma_\infty(\Gamma) = \{1\}$.*

Proof. If $x \notin \gamma_\infty(\Gamma)$ then $x \notin \gamma_k(\Gamma)$ for some k. The canonical homomorphism of Γ onto the nilpotent group $\Gamma/\gamma_k(\Gamma)$ does not map x on 1. Conversely, if $\phi: \Gamma \to G$, $\phi(x) \neq 1$, G nilpotent of class k, then $1 = \gamma_{k+1}(G) = \phi(\gamma_{k+1}(\Gamma))$, so $x \notin \gamma_{k+1}(\Gamma)$.

(7.3) THEOREM. *If l_p is the p-localization map and $\Pi(S)$ denotes the set of prime factors of $m_1 m_2 \ldots m_r$, then $\gamma_\infty(\Gamma(S)) = \cap_{p \in \Pi(S)} \ker l_p$.*

Proof. We prove that $\gamma_\infty(\Gamma(S)) = \cap_p \ker l_p$, where p ranges over *all* primes. This will do, since, by (7.1), $\ker l_p \subset \ker l_q$ if $q \notin \Pi(S)$.

1. Suppose that $x \notin \gamma_\infty(\Gamma(S))$. Then there is a nilpotent group G_1 and a homomorphism $\phi : \Gamma \to G_1$ such that $\phi(x) \neq 1$, by (7.2). Since Γ is finitely generated, so is $\phi(\Gamma)$, so by a theorem of Gruenberg [9], there is a further homomorphism $\psi: \phi(\Gamma) \to G_2$, where G_2 is a finite nilpotent group, such that $\psi(\phi(x)) \neq 1$. Since G_2 is a product of p-groups, there is a homomorphism $\omega: G_2 \to G_p$ of G_2 onto one of its Sylow subgroups G_p, such that $\omega(\psi(\phi(x))) \neq 1$. By Theorem 5.3, $x \notin \ker l_p$.

Conversely, suppose that $x \notin \ker l_p$, for some p. Then $l_p(x) \in \Gamma(S_p)$ and $l_p(x) \neq 1$. Since $\Gamma(S_p)$ is residually a finite p-group, by (6.7), there is a homomorphism $\psi: \Gamma(S_p) \to G_p$, where G_p is a finite p-group, such that $\psi(l_p(x)) \neq 1$. Since G_p is nilpotent it follows from (7.2) that $x \notin \gamma_\infty(\Gamma)$.

(7.4) THEOREM. *The group $\Gamma(S)$ is residually nilpotent if and only if S is p-local for some p.*

Proof. If $\Gamma(S)$ contains elements u of order p and v of order q, where p and q are distinct primes, then, on replacing v by a conjugate if necessary, we may assume that u and v do not commute. (In $\Gamma(S)$ the centralizer of an element of finite order is finite cyclic, so there are plenty of conjugates of v which do not commute with u). If ϕ maps $\Gamma(S)$ onto a finite nilpotent group G, then, since G is the product of its Sylow groups, $\phi(u)$ commutes with $\phi(v)$. This being true of all homomorphisms of $\Gamma(S)$ onto nilpotent groups, we conclude that $uvu^{-1}v^{-1}$ does not belong to $\gamma_\infty(\Gamma(S))$, so $\Gamma(S)$ is not residually nilpotent.

Conversely, if S is p-local, then $\Gamma(S)$ is residually a finite p-group by (6.7), so $\Gamma(S)$ is residually nilpotent.

8. Covering Groups of Nilpotent Riemann Surface Automorphism Groups

In this section we characterize precisely those signatures S for which $\Gamma(S)$ can cover a nilpotent group of automorphisms of Riemann surface. Since the finite groups $\Gamma(S)$ with $\chi(S) > 0$ cover only themselves, we shall assume that $\chi(S) \leqslant 0$.

(8.1) THEOREM. *The following are equivalent.*

(i) $\Gamma(S)$ *covers at least one nilpotent Riemann surface automorphism group.*
(ii) S_p *is non-degenerate for all* $p \in \Pi(S)$.
(iii) *The intersection* $\gamma_\infty(\Gamma(S))$ *of the lower central series is torsion-free.*

Proof. (ii) *implies* (i). If $\chi(S_p) \leqslant 0$, there exists a finite p-group G_p and a smooth homomorphism ψ_p: $\Gamma(S_p) \to G_p$ by (6.6). If $\chi(S_p) > 0$ and S_p is non-degenerate, then $\Gamma(S_p) = G_p$ is itself a finite p-group and the identity map ψ_p is a smooth homomorphism, by (3.5). Thus for all $p \in \Pi(S)$ we have smooth homomorphisms ψ_p: $\Gamma(S_p) \to G_p$ and we can combine these as at the end of §5 into a smooth homomorphism onto the product of the groups G_p.

(i) *implies* (iii). If $\Gamma(S)$ covers the nilpotent automorphism group G, then there is a smooth homomorphism ϕ: $\Gamma(S) \to G$. If G is nilpotent of class k, then

$$\phi(\gamma_{k+1}(\Gamma(S))) = \gamma_{k+1}(G) = \{1\}$$

so $\gamma_\infty(\Gamma(S)) \subset \gamma_{k+1}(\Gamma(S)) \subset \ker \phi$. Since $\ker \phi$ is torsion-free, $\gamma_\infty(\Gamma(S))$ is torsion-free.

(iii) *implies* (ii). Suppose S_p is degenerate. Then the order of one of the x_i is less than p^{α_i} by (3.5). Then $x_i^{m_i/p} \in \ker l_p$. Also if q is a prime different from p, then m_i/p is divisible by any power of q that divides m_i, so $x_i^{m_i/p} \in \ker l_q$. Thus $x_i^{m_i/p}$ is an element of order p which belongs to all the kernels of localization homomorphisms, therefore to $\gamma_\infty(\Gamma(S))$ by (7.3). Thus $\gamma_\infty(\Gamma(S))$ is not torsion-free.

In view of Theorem 8.1 we call S *nilpotent-admissible* if every S_p is non-degenerate. If S is nilpotent-admissible, it may happen that $\Gamma(S)$ covers only one nilpotent Riemann surface automorphism group. For instance, the only group covered by the signature

$$(0;\ 2,\ 2g + 1,\ 2(2g + 1))$$

is the cyclic group of order $2(2g + 1)$, discovered by Wiman and Harvey [10] to be the largest cyclic group of automorphisms of a surface of genus $g \geqslant 2$. Our next result shows that the number of nilpotent automorphism groups covered by a nilpotent-admissible signature is either 1 or countably infinite.

(8.2) THEOREM. *Suppose that* S *is a nilpotent-admissible signature. Then one of the following holds.*

(i) $\chi(S_p) > 0$ *for all* p, *and there is only one nilpotent Riemann surface automorphism group covered by* $\Gamma(S)$. *In this case the lower central series becomes constant after a finite number of steps. All the terms of the series have finite index, only the constant one being torsion-free.*

(ii) $\chi(S_p) \leqslant 0$ *for at least one p, there are infinitely many distinct Riemann surface automorphism groups covered by* $\Gamma(S)$ *and all the terms of the lower central series are distinct.*

Proof. If $\chi(S_p) > 0$, then either $S_p = (0;p,p)$ or $p = 2$ and $S_p = (0;2,2,2^n)$. The only smooth homomorphisms in these cases are the identity maps. Therefore if S is such that $\chi(S_p) > 0$ for all $p \in \Pi(S)$, there is only one smooth homomorphism ψ_p: $\Gamma(S_p) \rightarrow G_p$ for each p, therefore only one smooth homomorphism ϕ of $\Gamma(S)$ to a finite nilpotent group, which must be the product of the $\Gamma(S_p)$.

All the localizations being finite, and only finitely many of them non-trivial (since g must be 0), it follows that ker $l_p = \gamma_\infty(\Gamma(S))$ has finite index. Therefore there are only finitely many terms in the lower central series. Only the last one is torsion-free, otherwise we should have more than one finite nilpotent group covered by $\Gamma(S)$.

If, on the other hand, $\chi(S_p) \leqslant 0$ for some p, then there are infinitely many smooth homomorphisms from $\Gamma(S_p)$ to different finite nilpotent p-groups (Corollary 6.6). Combining these as at the end of §5 with at least one homomorphism ψ_q: $\Gamma(S_q) \rightarrow G_q$ for $q \neq p$, $q \in \Gamma(S)$, we find infinitely many nilpotent Riemann surface automorphism groups covered by $\Gamma(S)$.

In this case the lower central series of $\Gamma(S_p)$ does not become constant, for if it did, the group, being residually nilpotent, would be actually nilpotent. This cannot be, because it is known to have trivial centre. It follows that the lower central chain of $\Gamma(S)$ does not become constant. For if $\gamma_i(\Gamma(S)) = \gamma_{i+1}(\Gamma(S))$, then, applying the localizaton homomorphism, we should have $\gamma_i(\Gamma(S_p)) = \gamma_{i+1}(\Gamma(S_p)$, which we have just seen is not so.

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