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EULER CHARACTERISTICS OVER UNRAMIFIED REGULAR LOCAL RINGS

BY

MELVIN HOCHSTER[1]

Let M, N be finitely generated modules over a local ring (R, m) (all rings are assumed commutative, with identity; (R, m) is "local" means that R is Noetherian with maximal ideal m). If $Tor_j^R(M, N)$ has finite length for $j \geqslant i$, i a nonnegative integer, and vanishes for all sufficiently large j, we define

$$\chi_i^R(M, N) = \sum_{j \geqslant i} (-1)^{j-i} l(Tor_j^R(M, N)),$$

where l denotes length. The main result here is the following:

THEOREM. *Let R be an unramified regular local ring and let M, N be finitely generated R-modules such that $Tor_i^R(M, N)$ has finite length, $i \geqslant 1$. If $\chi_i(M, N) = 0$, then*

$$Tor_j^R(M, N) = 0 \quad \text{for } j \geqslant i.$$

It was already known (see [1], [2], [3]) that if R is regular and $Tor_i^R(M, N)$ is 0 (respectively, has finite length) then

$$Tor_j^R(M, N) = 0$$

(respectively, has finite length) for $j \geqslant i$. Moreover, in [3] it is shown that if R is an unramified regular local ring and $Tor_i^R(M, N)$ has finite length, $i \geqslant 1$, then $\chi_i^R(M, N) \geqslant 0$, and that if $i \geqslant 2$ or M or N is torsion-free, then $\chi_i^R(M, N) = 0$ if and only if $Tor_j^R(M, N) = 0, j \geqslant i$. Thus, the theorem is new only in the case $i = 1$.

As usual, we may reduce at once to the case where R is complete and then assume $R = V[[x_2, \ldots, x_n]]$, where $n = \dim R$ and V is a complete discrete valuation ring with maximal ideal x_1V. We abbreviate $x = x_1$.

We write $M \hat{\otimes}_V N$ and $\hat{Tor}_i^V(M, N)$ for the complete tensor product and complete Tor_i, respectively, of M and N over V (see [4, p. V-6].) Let $S = R \hat{\otimes}_V R$. S is regular and if we map $S \to R$ by

$$a \hat{\otimes} b \mapsto ab,$$

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we have $R \simeq S/(z_2, \ldots, z_n)$, where $z_i = x_i \hat{\otimes} 1 - 1 \hat{\otimes} x_i$, $2 \leq i \leq n$. Here $z_2, \ldots, z_n$ is a regular sequence in S.

Following [4], [3], we note that there is a spectral sequence

$$Tor_p^S(\hat{Tor}_q^V(M, N), R) \Rightarrow Tor_{p+q}^R(M, N).$$

When V is a discrete valuation ring, $\hat{Tor}_q^V(M, N) = 0$ for $q \geq 2$, and this spectral sequence yields a long exact sequence

$$\to Tor_{j-1}^S(\hat{Tor}_1^V(M, N), R) \to Tor_j^R(M, N)$$
$$\to Tor_j^S(M \hat{\otimes}_V N, R) \to \ldots \to Tor_2^S(M \hat{\otimes}_V N, R)$$
$$\to \hat{Tor}_1^V(M, N) \otimes_S R \to Tor_1^R(M, N) \to Tor_1^S(M \hat{\otimes}_V N, R) \to 0$$

It was already shown in [3] that when $Tor_1^R(M, N)$ has finite length, so do all terms in the above sequence, and so we have

$$\chi_1^R(M, N) = \chi_1^S(M \hat{\otimes}_V N, R) + \chi_0^S(\hat{Tor}_1^V(M, N), R).$$

Because $R = S/(z_2, \ldots, z_n)$, both terms on the right are known to be nonnegative (see [3, Theorem 1 and Lemma 1]), and so $\chi_1^R(M, N) = 0$ implies that both $\chi_1^S(M \hat{\otimes}_V N, R) = 0$ and $\chi_0^S(\hat{Tor}_1^V(M, N), R) = 0$. From [3, Theorem 1 and Lemma 1], we then know that

(a) $\mathrm{Tor}_j^S(M \hat{\otimes}_V N, R) = 0, j \geq 1$, and
(b) $\dim \hat{Tor}_1^V(M, N) < n - 1$.

We shall use this information to show that $\hat{Tor}_1^V(M, N) = 0$. Then, as seen in [3], we have $Tor_j^S(M \hat{\otimes}_V N, R) \simeq Tor_j^R(M, N)$ and the result follows from [3, Theorem 1].

Let $M_0 = \cup_t \mathrm{Ann}_M x^t$ and $N_0 = \cup_t \mathrm{Ann}_N x^t$. To complete the proof, we shall establish the following facts:

(1) If $M_0 \neq 0$ and $N_0 \neq 0$, then $W = \mathrm{Im}(M_0 \hat{\otimes}_V N_0 \to M \hat{\otimes}_V N)$ is nonzero.

(2) $\dim M_0 \hat{\otimes}_V N_0 = \dim \hat{Tor}_1^V(M_0, N_0) = \dim \hat{Tor}_1^V(M, N)$ ($= \dim M_0 + \dim N_0$).

Assume (1) and (2) for the moment. From (a) above, $z_2, \ldots, z_n$ is a regular sequence on $M \hat{\otimes}_V N$, so that depth $M \hat{\otimes}_V N \geq n - 1$. From (1), if $M_0 \neq 0$, $N_0 \neq 0$, we have $W \subset M \hat{\otimes}_R N$ and

$$\dim W \leq \dim M_0 \hat{\otimes}_V N_0 = \dim \hat{Tor}_1^V(M, N) \quad \text{(from (2))} \leq n - 2.$$

But over any local ring, a module of depth d cannot have a nonzero submodule of dimension less than d; see [4, Prop. 7, p. IV-16] (the same fact is used in the proof of Theorem 1 in [3]). This shows that either $M_0 = 0$ or $N_0 = 0$, i.e., that x is a nonzero divisor on at least one of the modules M, N, which is known [4, Propriété (g), p. V-9] to imply that $\hat{Tor}_1^V(M, N) = 0$, as required.

Thus, to complete the proof of the theorem, it suffices to establish the assertions (1) and (2) listed above.

To prove (1), we first note that if $M_0 \neq 0$ (respectively, $N_0 \neq 0$) then $M_0 \not\subset xM$ (respectively, $N_0 \not\subset xN$). For if $M_0 \subset xM$, given $u \in M_0$ we have $u = xv$, and since $x^t u = 0$ for some t, $x^{t+1} v = 0$ and $v \in M_0$. But then $M_0 \subset xM_0$ and so $M_0 = 0$ by Nakayama's lemma.

Hence, if $M_0 \neq 0$, it has a nonzero image G_0 in M/xM and, similarly, if $N_0 \neq 0$, N_0 has nonzero image H_0 in N/xM. Let $K = R/m$. Then we have

$$M \mathbin{\hat{\otimes}_V} N \twoheadrightarrow (M/xM) \mathbin{\hat{\otimes}_V} (N/xN) \simeq (M/xM) \mathbin{\hat{\otimes}_K} (N/xN).$$

Since $\hat{\otimes}_K$ is faithfully exact,

$$0 \neq G_0 \mathbin{\hat{\otimes}_K} H_0 \hookrightarrow (M/xM) \mathbin{\hat{\otimes}_K} (N/xN).$$

Since the image of $M_0 \mathbin{\hat{\otimes}_V} N_0$ in $(M/xM) \mathbin{\hat{\otimes}_K} (N/xN)$ is nonzero, its image in $M \otimes_V N$ must have been nonzero, as required.

It remains to establish (2). First note that since x is a nonzerodivisor on M/M_0, $\hat{Tor}_1^V(M/M_0, N) = 0$. This fact and the long exact sequence for $\hat{Tor}$ arising from $0 \to M_0 \to M \to M/N_0 \to 0$ show that

$$\hat{Tor}_1^V(M_0, N) \simeq \hat{Tor}_1^V(M, N)$$

while

$$\hat{Tor}_1^V(M_0, N_0) \simeq Tor_1^V(M_0, N)$$

because x is a nonzero divisor on N/N_0.

The remaining assertions in (2) then follow from the lemma below applied in the case $A = M_0$, $B = N_0$.

LEMMA. *Let V be a complete discrete valuation ring with maximal ideal xV and let $R = V[[x_2, \ldots, x_n]]$. Let A, B be nonzero finitely generated R-modules each of which is killed by a power of x. Then:*

(a) $\dim A = \dim A/xA = \dim \mathrm{Ann}_A x$ *and* $\dim B = \dim B/xB = \dim \mathrm{Ann}_B x$.

(b) *If* $xA = xB = 0$, $\hat{Tor}_1^V(A, B) \simeq A \mathbin{\hat{\otimes}_V} B \simeq A \mathbin{\hat{\otimes}_K} B$, *where* $K = V/xV$, *and each has dimension* $\dim A + \dim B$.

(c) *More generally,* $\dim A \mathbin{\hat{\otimes}_V} B = \dim \hat{Tor}_1^V(A, B) = \dim A + \dim B$.

Proof. Since x is nilpotent on A,

$$\dim A = \dim A/xA.$$

If $P \in \mathrm{Ass}\, A$, $R/P \hookrightarrow A$ implies $x \in P$ and then $R/P \hookrightarrow \mathrm{Ann}_A x$. Thus,

$$\dim A = \dim \mathrm{Ann}_A x.$$

This establishes (a).

To prove (b), consider a resolution $F_\bullet$ of A over R. $\widehat{Tor}_\bullet^V(A, B)$ is the homology of $F_\bullet \mathbin{\widehat{\otimes}}_V B$. But we may identify

$$(\widehat{\otimes}_V B) \simeq (((\otimes_V (V/xV)) \mathbin{\widehat{\otimes}}_K B),$$

since $xB = 0$. When we apply $\otimes_V V/xV$ to $F_\bullet$, we get a complex whose first homology module is $Tor_1^V(A, V/xV)$ (this is *ordinary Tor*, *not* complete *Tor*) which, since x kills A, is $\simeq A$. But then since $\widehat{\otimes}_K B$ is faithfully exact, we find

$$\widehat{Tor}_1^V(A, B) \simeq A \mathbin{\widehat{\otimes}}_K B \simeq A \mathbin{\widehat{\otimes}}_V B \simeq A \mathbin{\widehat{\otimes}}_K B,$$

as required. The dimension statement now follows from [4, pp. V-10, V-11, case (a)]. This proves (b).

It remains to prove (c). Given an exact sequence

$$0 \to A_1 \to A \to A_2 \to 0$$

we have the long exact sequence

$$\to \widehat{Tor}_j^V(A_1, B) \to \widehat{Tor}_j^V(A, B) \to \widehat{Tor}_j^V(A_2, B) \to$$

whence $\dim \widehat{Tor}_t^V(A, B) \leq \max_i \dim \widehat{Tor}_t^V(A_i, B)$. Using this fact repeatedly, by a straightforward induction argument, one shows that given finite filtrations of A, B with factors A_i, B_j respectively,

$$\dim \widehat{Tor}_t^V(A, B) \leq \max_{i,j} \dim \widehat{Tor}_t^V(A_i, B_j).$$

Applying this fact for $t = 0, 1$ with the filtrations with factors

$$A_i = x^iA/x^{x+1}A, \quad B_j = x^jB/x^{j+1}B$$

we have

$$\begin{aligned}\dim \widehat{Tor}_t^V(A, B) &\leq \max_{i,j} \dim \widehat{Tor}_t^V(x^iA/x^{i+1}A, x^jB/x^{j+1}B)\\ &= \max_{i,j}(\dim x^iA/x^{i+1}A + \dim x^jB/x^{j+1}B)\\ &\leq \dim A + \dim B.\end{aligned}$$

On the other hand, since $A \mathbin{\widehat{\otimes}}_V B$ (respectively, $\widehat{Tor}_1^V(A, B)$) has a quotient (respectively, submodule)

$$(A/xA) \mathbin{\widehat{\otimes}}_V (B/xB)$$

(respectively, $\widehat{Tor}_1^V(\mathrm{Ann}_A x, \mathrm{Ann}_B x)$), which has dimension $\dim A + \dim B$, we have equality, Q.E.D.

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UNIVERSITY OF MICHIGAN
ANN ARBOR, MICHIGAN