ILLINOIS JOURNAL OF MATHEMATICS
Volume 28, Number 2, Summer 1984

A GENERALIZED NONCOMMUTATIVE KOROVKIN THEOREM AND $*$-CLOSEDNESS OF CERTAIN SETS OF CONVERGENCE

BY

B. V. LIMAYE AND M. N. N. NAMBOODIRI

Introduction

Let A be a complex C^*-algebra with identity 1_A, and for $n = 1, 2, \ldots$, let $\phi_n : A \to A$ be a Schwarz map, i.e., a *linear map such that

$$\phi_n(a)^* \phi_n(a) \leq \phi_n(a^*a)$$

for all $a \in A$. Robertson [4] has proved that the set

$$C = \{a \in A : \|\phi_n(a) - a\| \to 0, \|\phi_n(a^*a) - a^*a\| \to 0, \|\phi_n(aa^*) - aa^*\| \to 0\}$$

is a C^*-subalgebra of A. This is a noncommutative analogue of a classical theorem of Korovkin which states that if $A = C([a, b])$, the set of all continuous functions on $[a, b]$, and $\phi_n : A \to A$ is a positive map for $n = 1, 2, \ldots$, then

$$C = \{f \in A : \phi_n(f) \to f, \phi_n(|f|^2) \to |f|^2 \text{ uniformly on } [a, b]\}$$

is a norm-closed and conjugate closed subalgebra of A; in particular, if, 1, t and t^2 belong to C, then by the Stone-Weierstrass theorem, $C = C([a, b])$.

Let B be another C^*-algebra with identity 1_B, $\phi : A \to B$ a *homomorphism, and, for $n = 1, 2, \ldots, \phi_n : A \to B$ a Schwarz map. Note that each ϕ_n is a positive map with $\phi_n(1_A) \leq 1_B$. Consider the set

$$D = \{a \in A : \phi_n(a) \to \phi(a), \phi_n(a^*a) \to \phi(a^*a)\},$$

where the convergence is in the norm topology or in the weak topology. In Section 1, we show that the set D is a norm-closed (but not necessarily $*$-closed) subalgebra of A (Theorem 1.2). By considering $D \cap D^*$, we obtain a straightforward generalization of Robertson's result (Corollary 1.4).

In case A is commutative, the set D is clearly $*$-closed. The purpose of this paper is to investigate the $*$-closedness of the set D in case A is noncommutative. Let $B = A$ and ϕ be the identity map. Robertson has asked whether the $*$-closedness of the set D for all choices of Schwarz maps ϕ_n characterizes the commutativity of A. We answer this question in the negative by using the theorem proved in Section 1. We show that if $A = M_2$, the noncommutative C^*-algebra consisting of all 2×2 complex matrices, then the set D is $*$-closed

Received December 2, 1981.

for all choices of Schwarz maps ϕ_n (Theorem 2.3). Further, we show that M_2 is the only finite dimensional noncommutative C^*-algebra for which this result holds (Theorem 2.6).

As for infinite dimensional algebras, let H be a Hilbert space of infinite dimension and let $\beta(H)$ (respectively, $\varkappa(H)$) denote the C^*-algebra of all bounded (respectively, compact) operators on H. We show that if $A = \beta(H)$ or if A is an infinite dimensional noncommutative C^*-subalgebra of $\varkappa(H)$, then there is a Schwarz map $\phi : A \to A$ and there exists $T \in A$ such that

$$\phi(T) = T, \quad \phi(T^*T) = T^*T, \quad \text{but} \quad \phi(TT^*) \neq TT^*,$$

so that the set

$$D_\phi = \{T \in A : \phi(T) = T, \phi(T^*T) = T^*T\}$$

is not $*$-closed. The question whether this can be done for any infinite dimensional noncommutative C^*-subalgebra of $\beta(H)$ remains open.

1. A generalization of Robertson's theorem

We begin with a convergence result for the C^*-algebra $\beta(H)$ of all bounded operators on a complex Hilbert space H.

LEMMA 1.1. *Let (R_n), (S_n) and (U_n) be sequences in $\beta(H)$ and $R, U \in \beta(H)$.*

(a) *For all n and for all real numbers t, let*

$$t^2R_n + tS_n + U_n \geq 0,$$

and for all n and for some positive real number α, let $R_n \leq \alpha I$, where I denotes the identity operator in $\beta(H)$. Let "$\to$" denote either norm or weak convergence in $\beta(H)$. Then $U_n \to 0$ implies $S_n \to 0$.

(b) *For all n, let*

$$R_n^*R_n \leq S_n,$$

and let (R_n) and (S_n) converge weakly to R and R^*R respectively. Then, in fact, (R_n) converges strongly to R. If, in addition, (U_n) converges to U weakly, then $(U_nR_n^*)$ converges weakly to UR.

Proof. (a) For all natural numbers n and all real numbers t, we have

$$-tS_n \leq t^2R_n + U_n \leq \alpha t^2 I + U_n.$$

Let $\|U_n\| \to 0$. For a fixed $t \neq 0$ and all large enough n, $U_n \leq \alpha t^2 I$. Hence

$$-tS_n \leq 2\alpha t^2 I.$$

Changing t to $-t$, we have $tS_n \leq 2\alpha t^2 I$. Thus, for any given $t > 0$, and all large enough n,

$$-2\alpha tI \leq S_n \leq 2\alpha tI.$$

Hence $\|R_n\| \to 0$.

Next, let $U_n \to 0$ weakly. Fix $x \in H$. Then, the above procedure shows that for any given $t > 0$ and all large enough n,

$$-2\alpha t <x,x> \leq <S_n(x),x> \leq 2\alpha t <x,x>.$$

Hence $S_n \to 0$ weakly.

(b) Let $x \in H$. Then

$$\begin{aligned} \|R_n(x) - R(x)\|^2 &= <R_n^*R_n(x),x> + <R^*R(x),x> - 2Re<R_n(x),R(x)> \\ &\leq <S_n(x),x> + <R^*R(x),x> - 2Re<R_n(x),R(x)>. \end{aligned}$$

Since $R_n \to R$ and $S_n \to R^*R$ weakly, we see that the right side of the above inequality tends to zero. Since $x \in H$ is arbitrary, $R_n \to R$ strongly.

Let $U_n \to U$ weakly. For $x, y \in H$,

$$<(U_nR_n - UR)(x),y> \leq |<(U_nR_n - U_nR)(x),y>| + |<(U_nR - UR)(x),y|.$$

Now, $(U_n^*(x))$ converges weakly in H and hence it is bounded. Also,

$$\|R_n(x) - R(x)\| \to 0$$

by the above. Hence $<U_nR_n(x),y>$ tends to $<UR(x),y>$. Since $x, y \in H$ are arbitrary, $U_nR_n \to UR$ weakly, Q.E.D.

We now prove a generalization of Robertson's result [4]. Our proof, like Robertson's, uses an idea of Palmer [3].

THEOREM 1.2. *Let A and B be complex C^*-algebras with identities 1_A and 1_B, respectively. Let (ϕ_n) be a sequence of Schwarz maps and ϕ a *homomorphism from A to B. Then the set*

$$D = \{a \in A : \phi_n(a) \to \phi(a), \ \phi_n(a^*a) \to \phi(a^*a)\}$$

is a norm-closed subalgebra of A, where "$\to$" denotes either norm or weak convergence.

Proof. It is easy to see that D is norm-closed. To see that D is a subalgebra, it is enough to prove that if $a \in D$ and $\phi_n(b) \to \phi(b)$, then $\phi_n(ba) \to \phi(ba)$. Now, for any real number t,

$$\begin{aligned} &t\{\phi_n(b)\phi_n(a) + \phi_n(a)^*\phi_n(b)^*\} \\ &= \phi_n(tb^* + a)^*\phi_n(tb^* + a) - t^2\phi_n(b)\phi_n(b)^* - \phi_n(a)^*\phi_n(a) \\ &\leq \phi_n((tb^* + a)^*(tb^* + a)) - t^2\phi_n(bb^*) - \phi_n(a)^*\phi_n(a) + t^2(\phi_n(bb^*) - \phi_n(b)\phi_n(b)^*) \\ &= t\phi_n(ba + a^*b^*) + \phi_n(a^*a) - \phi_n(a)^*\phi_n(a) + t^2(\phi_n(bb^*) - \phi_n(b)\phi_n(b)^*). \end{aligned}$$

Hence, if we let

$$\begin{aligned} R_n &= \phi_n(bb^*) - \phi_n(b)\phi_n(b^*), \\ S_n &= \phi_n(ba + a^*b^*) - \phi_n(b)\phi_n(a) - \phi_n(a)^*\phi_n(b)^*, \\ U_n &= \phi_n(a^*a) - \phi_n(a)^*\phi_n(a), \end{aligned}$$

we see that for all real numbers t,

$$t^2R_n + tS_n + U_n \geq 0.$$

Since $\phi_n(a) \to \phi(a)$, where $\to$ denotes either norm or weak convergence, we have $\phi_n(a)^* \to \phi(a)^*$. If "$\to$" denotes norm convergence, then clearly $\phi_n(a)^*\phi_n(a) \to \phi(a)^*\phi(a)$, and if "$\to$" denotes weak convergence, then Lemma 1.1(b) shows that

$$\phi_n(a)^*\phi_n(a) \to \phi(a)^*\phi(a).$$

Since $\phi_n(a^*a) \to \phi(a^*a) = \phi(a)^*\phi(a)$, it follows that in both the cases,

$$U_n = \phi_n(a^*a) - \phi_n(a)^*\phi_n(a) \to 0.$$

Hence, by Lemma 1.1(a),

(1) $$S_n = \phi_n(ba + a^*b^*) - \phi_n(b)\phi_n(a) - \phi_n(a)^*\phi_n(b)^* \to 0.$$

Since $\phi_n(b) \to \phi(b)$, we see, by using Lemma 1.1(b) in the case of weak convergence, that

(2) $$\phi_n(b)\phi_n(a) \to \phi(b)\phi(a) = \phi(ba).$$

Taking adjoints in (2), we have

(3) $$\phi_n(a)^*\phi_n(b)^* \to \phi(a^*b^*).$$

From (1), (2) and (3), we obtain

(4) $$\phi_n(ba + a^*b^*) \to \phi(ba + a^*b^*).$$

Replacing b by ib in (4), we have

(5) $$\phi_n(ba - a^*b^*) \to \phi(ba - a^*b^*).$$

Adding (4) and (5), we obtain $\phi_n(ba) \to \phi(ba)$, as desired, Q.E.D.

Remark 1.3. We have not been able to settle the question of whether D is a subalgebra when "$\to$" denotes strong convergence. The difficulty lies in the fact that the adjoint operation is not continuous in the strong topology. Although the multiplication operation is not continuous in the weak topology, this problem is taken care of by Lemma 1.1(b). We cannot apply the same procedure for strong convergence, as the following example shows. Let H be a separable infinite dimensional Hilbert space and T denote a unilateral left shift operator on H. Let $R_n = T^n$ for $n = 1, 2, \ldots$. Then (R_n) and $(R_n^*R_n)$ converge to the zero operator strongly, but (R_n^*) does not converge to the zero operator strongly.

However, it is interesting to note that the following corollary is just as valid for strong convergence as it is for norm or weak convergence.

COROLLARY 1.4. *Under the assumptions of Theorem* 1.2, *the set*

$$C = \{a \in A : \phi_n(a) \to \phi(a),\ \phi_n(a^*a) \to \phi(a^*a),\ \phi_n(aa^*) \to \phi(aa^*)\}$$

is a C^*-subalgebra of A, where "$\to$" denotes norm, weak or strong convergence.

Proof. For norm or weak convergence, the result follows immediately from Theorem 1.2 since $C = D \cap D^*$, where $D^* = \{d^* : d \in D\}$. Now, consider

$$C_w = \{a \in A : \phi_n(a) \to \phi(a), \phi_n(a^*a) \to \phi(a^*a), \phi_n(aa^*) \to (aa^*) \text{ weakly}\},$$

and

$$C_{st} = \{a \in A : \phi_n(a) \to \phi(a), \phi_n(a^*a) \to \phi(a^*a), \phi_n(aa^*) \to (aa^*) \text{ strongly}\}.$$

Then $C_{st} \subset C_w$. We show that, in fact, $C_{st} = C_w$, which establishes the desired result.

Let $a \in C_w$. Let

$$R_n = \phi_n(a), \quad R = \phi(a), \quad S_n = \phi_n(a^*a).$$

Since ϕ_n is a Schwarz map, we have $R_n^*R_n \leq S_n$. Now, $R_n \to R$ and $S_n \to R^*R$ weakly. Hence by Lemma 1.1(b), $\phi_n(a) \to \phi(a)$ strongly.

Next, by Theorem 1.2, the set

$$D_w = \{a \in A : \phi_n(a) \to \phi(a), \phi_n(a^*a) \to \phi(a^*a) \text{ weakly}\}$$

is an algebra. Since $a, a^* \in D_w$, we see that $(a^*a)^2 \in D_w$. Again, letting

$$R_n = \phi_n(a^*a), R = \phi(a^*a), S_n = \phi_n((a^*a)^2)$$

in Lemma 1.1(b), we see that $\phi_n(a^*a) \to \phi(a^*a)$ strongly. Similarly, it can be shown that $\phi_n(aa^*) \to \phi(aa^*)$ strongly, so that $a \in C_{st}$, Q.E.D.

2. $*$-closedness of the set D

Let A be a C^*-algebra with identity 1_A, and, for $n = 1, 2, \ldots$, let $\phi_n : A \to A$ be a Schwarz map. If A is commutative, the set

$$D = \{a \in A : \|\phi_n(a) - a\| \to 0, \|\phi_n(a^*a) - a^*a\| \to 0\}$$

is clearly $*$-closed. In general, this need not be the case as the following example shows.

Example 2.1. Let H be a separable infinite dimensional Hilbert space and S a unilateral right shift operator on H. For $T \in \beta(H)$, the C^*-algebra of all bounded operators on H, let

$$\phi(T) = \phi_n(T) = S^*TS, \quad n = 1, 2, \ldots.$$

Then

$$D = \{T \in \beta(H) : S^*TS = T, S^*T^*TS = T^*T\}.$$

Now, since S^*S is the identity operator on H, we see that $S \in D$; but $S^* \notin D$, since SS^* is not the identity operator on H. In fact, $T \in \beta(H)$ belongs to D if and only if T commutes with S, since for all $x \in H$,

$$\| TS(x) - ST(x) \|^2 = \langle (S^*T^* - T^*S^*)(TS - ST)(x), x \rangle .$$

It is proved in [5] that the commutant of S is the strong closure of the set of polynomials in S. Thus, $T \in D$ if and only if T can be strongly approximated by a sequence of polynomials in S.

Next, we show that A need not be commutative for the set D to be *-closed for every choice of Schwarz maps ϕ_n. Let M_k denote the C^*-algebra of all $k \times k$ matrices with complex entries, and $I \in M_k$ denote the identity matrix.

LEMMA 1.2. *Let $\phi : M_2 \to M_2$ be a positive linear map with $\phi(I) \leq I$. Let $T \in M_2$ be such that T^*T is a one-dimensional projection and $\phi(T) = T$, $\phi(T^*T) = T^*T$. Then either $\phi(I) = I$ or T is a normal matrix.*

Proof. Let

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

so that

$$T^*T = \begin{bmatrix} |a|^2 + |c|^2 & \bar{a}b + \bar{c}d \\ a\bar{b} + c\bar{d} & |b|^2 + |d|^2 \end{bmatrix}.$$

Suppose, first, that

$$T^*T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then, $b = d = 0$, so that

$$T = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

If $c = 0$, then T is a normal matrix. Now, let $c \neq 0$. Since $T = \phi(T)$ and

$$\phi\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \phi(T^*T) = T^*T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

we have

$$\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = a\phi\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) + c\phi\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)$$
$$= a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c\phi\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right),$$

i.e.,

$$c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = c\phi(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}).$$

Since $c \neq 0$, we have

$$\phi(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Now, ϕ is a positive linear map on the C^*-algebra M_2 and $\phi(I) \leq I$. Hence, by Kadison's Schwarz inequality [2],

$$\phi(S^*)\phi(S) + \phi(S)\phi(S^*) \leq \phi(S^*S + SS^*)$$

for all $S \in M_2$. Putting

$$S = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

we have

$$\begin{aligned} I &= S^*S + SS^* \\ &= \phi(S^*)\phi(S) + \phi(S)\phi(S^*) \\ &\leq \phi(S^*S + SS^*) \\ &= \phi(I) \\ &\leq I. \end{aligned}$$

This shows that $\phi(I) = I$.

Thus, we have proved the lemma in the case where

$$T^*T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

In the general case, since T^*T is a one-dimensional projection, let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be pairs of complex numbers such that

$$|x_1|^2 + |x_2|^2 = 1 = |y_1|^2 + |y_2|^2, \quad x_1\overline{y_1} + x_2\overline{y_2} = 0,$$

and

$$T^*T(x) = x, \ T^*T(y) = 0.$$

Let

$$U = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}.$$

Then U is a unitary matrix, and

$$T^*T = U\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}U^*.$$

Define $\psi : M_2 \to M_2$ by $\psi(R) = U^*\phi(URU^*)U$, $R \in M_2$. Then it is easy to see that ψ is a positive linear map on M_2 and $\psi(I) \leq I$. If we let $S = U^*TU$, then

$$S^*S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and $\psi(S) = S$, $\psi(S^*S) = S^*S$. Hence, by the particular case considered above, we see that either $\psi(I) = I$, or S is normal. This, in turn, shows that either $\phi(I) = I$ or T is normal.

THEOREM 2.3. *For $n = 1, 2, \ldots$, let $\phi_n : M_2 \to M_2$ be a Schwarz map. Then the set*

$$D = \{T \in M_2 : \|\phi_n(T) - T\| \to 0, \|\phi_n(T^*T) - T^*T\| \to 0\}$$

is $*$-closed, and hence is a C^*-subalgebra of M_2.

We prove the following:

(1) Let $\phi : M_2 \to M_2$ be a Schwarz map. Then the set

$$D_\phi = \{T \in M_2 : \phi(T) = T, \ \phi(T^*T) = T^*T\}$$

is a C^*-subalgebra of M_2.

(2) $D = \cap D_\phi$, where the intersection is taken over all cluster points ϕ of the sequence (ϕ_n) in

$$P(M_2, M_2) = \{\psi : M_2 \to M_2 : \psi \text{ positive linear}, \ \|\psi\| \leq 1\}.$$

Proof of (1). It follows from the Cayley-Hamilton theorem that for any $T \in M_2$,

$$(*) \qquad T^2 = \operatorname{tr}(T)T - \det(T)I,$$

where $\operatorname{tr}(T)$ is the trace of T and $\det(T)$ is the determinant of T.

Let $T \in D_\phi$, i.e., $\phi(T) = T$ and $\phi(T^*T) = T^*T$. Assume that $\|T\| = 1$.

Case (i). $I \notin D_\phi$, i.e., $\phi(I) \neq I$. Since D_ϕ is a subalgebra of M_2 by Theorem 1.2, we see that $T^2 \in D_\phi$. Now by $(*)$,

$$T^2 = \operatorname{tr}(T)T - \det(T)I,$$

so that

$$\phi(T^2) = \operatorname{tr}(T)\phi(T) - \det(T)\phi(I).$$

But $\phi(T) = T$ and $\phi(T^2) = T^2$. Hence

$$\det(T)I = \det(T)\phi(I).$$

Since $I \neq \phi(I)$, it follows that $\det(T) = 0$. Consequently,

$$\det(T^*T) = |\det(T)|^2 = 0.$$

Then, by (*),

$$(T^*T)^2 = \operatorname{tr}(T^*T)\,T^*T.$$

Since $\|T\| = 1$, we have $\|T^*T\| = 1 = \|(T^*T)^2\|$. This shows that $\operatorname{tr}(T^*T) = 1$. Hence $(T^*T)^2 = T^*T$. Moreover, $T \neq 0$ and $\det(T^*T) = 0$. Thus, T^*T is a one-dimensional projection. Now, Lemma 2.2 shows that T is normal, i.e., $T^*T = TT^*$. Hence $\phi(TT^*) = \phi(T^*T) = T^*T = TT^*$. Since $\phi(T^*) = T^*$ always, we see that $T^* \in D_\phi$.

Case (ii). $I \in D_\phi$, i.e., $\phi(I) = I$. Let $T = R + iS$ with $R^* = R$ and $S^* = S$. Then, by (*),

$$2(T^*T + TT^*) = R^2 + S^2 = \operatorname{tr}(R)R + \operatorname{tr}(S)S - (\det(R) + \det(S))I.$$

Since $\phi(T) = T$, we see that $\phi(R) = R$ and $\phi(S) = S$. Also, $\phi(I) = I$. Hence

$$2\phi(T^*T + TT^*) = \operatorname{tr}(R)R + \operatorname{tr}(S)S - (\det(R) + \det(S))I.$$

This shows that

$$T^*T + TT^* = \phi(T^*T + TT^*).$$

But since $T \in D_\phi$, we have $T^*T = \phi(T^*T)$. Hence $TT^* = \phi(TT^*)$ and again, $T \in D_\phi$.

Thus D_ϕ is *-closed. By Theorem 1.2, it is a norm-closed subalgebra of M_2. Hence D_ϕ is a C^*-subalgebra of M_2.

Proof of (2). Let Φ denote the set of all cluster points of the sequence (ϕ_n) in $P(M_2, M_2)$. For $\phi \in \Phi$, clearly $D \subset D_\phi$. Let

$$E = \bigcap \{D_\phi : \phi \in \Phi\}, \quad \text{and} \quad \psi_n = \phi_{n|E},\ n = 1, 2, \ldots.$$

Then, by (1), E is C^*-subalgebra of M_2; it contains D and

$$\psi_n \in P(E, M_2) = \{\psi : E \to M_2,\ \psi \text{ positive linear},\ \|\psi\| \leq 1\}.$$

To show that E is contained in D, we argue as follows. We claim that $\psi_n(T) \to T$ for all $T \in E$. Suppose this is not the case. Then, by the compactness of $P(E, M_2)$, there is $\psi \in P(E, M_2)$ and a subnet (ψ_α) of (ψ_n) such that

$$\psi_\alpha(T) \to \psi(T) \text{ for all } T \in E, \quad \text{and} \quad \psi(T_0) \neq T_0 \text{ for some } T_0 \in E.$$

Let (ϕ_α) be the corresponding subnet of (ϕ_n), so that $\phi_{\alpha|E} = \psi_\alpha$. Now,

$$\phi_\alpha(T) = \psi_\alpha(T) \to \psi(T) \quad \text{for all } T \in E.$$

Let ϕ be a cluster point of (ϕ_α) in $P(M_2, M_2)$. Then $\phi(T) = \psi(T)$ for all $T \in E$. But ϕ is also a cluster point of (ϕ_n), i.e., $\phi \in \Phi$, while

$$\phi(T_0) = \psi(T_0) \neq T_0.$$

This contradicts the fact that $T_0 \in E$. Hence $\psi_n(T) \to T$ for all $T \in E$. Now, let $T \in E$. Since E is a *subalgebra of M_2, we have $T^*T \in E$, so that

$$\phi_n(T) = \psi_n(T) \to T \quad \text{and} \quad \phi_n(T^*T) = \psi_n(T^*T) \to T^*T,$$

i.e., $T \in D$. Thus, $D = E$ as desired, Q.E.D.

Remark 2.4. If $\phi : M_2 \to M_2$ is merely a positive linear map with $\phi(I) \leq I$, then the set D_ϕ may not be closed even under addition, as the following example shows. Let

$$\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix},$$

and

$$R = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}, S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then R and S belong to D_ϕ, but

$$R + S = \begin{bmatrix} i & 1 \\ 1 & 0 \end{bmatrix}$$

does not. Also D_ϕ is not closed under the Jordan product:
Let

$$T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, U = \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}.$$

Then T and U belong to D_ϕ, but

$$\frac{1}{2}(TU + UT) = \begin{bmatrix} i+1 & i+(1/2) \\ i+(1/2) & 1 \end{bmatrix}$$

does not. For this particular map ϕ, D_ϕ is closed under the squares. Examples of positive linear maps $\phi : M_2 \to M_2$ with $\phi(I) \leq I$ for which D_ϕ is not closed under the squares and/or D_ϕ is not $*$-closed are lacking.

The following example shows that M_2 cannot be replaced by any M_k, $k \geq 3$, in Theorem 2.3.

Example 2.5. Let $k \geq 3$ be an integer. Define $\phi^{(k)} : M_k \to M_k$ by

$$\phi^{(k)}\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1k} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2k} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3k} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & a_{k3} & \dots & a_{kk} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ 0 & 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Then $\phi^{(k)}$ is a Schwarz map on M_k. This can be proved as follows. Let $T = (a_{ij})$, $i,j = 1,\dots,n$. Then

$$\phi^{(k)}(T)^*\phi^{(k)}(T) = \begin{bmatrix} |a_{11}|^2 + |a_{21}|^2 & a_{11}\overline{a}_{12} + \overline{a}_{21}a_{22} & 0 & \dots 0 \\ \overline{a}_{11}a_{12} + a_{21}\overline{a}_{22} & |a_{12}|^2 + |a_{22}|^2 & 0 & \dots 0 \\ 0 & 0 & |a_{22}|^2 & \dots 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots 0 \end{bmatrix}$$

and

$$\phi^{(k)}(T^*T) = \begin{bmatrix} \sum_{j=1}^{k} |a_{j1}|^2 & \sum_{j=1}^{k} \overline{a}_{j1}a_{j2} & 0 & \dots 0 \\ \sum_{j=1}^{k} a_{j1}\overline{a}_{j2} & \sum_{j=1}^{k} |a_{j2}|^2 & 0 & \dots 0 \\ 0 & 0 & \sum_{j=1}^{k} |a_{j2}|^2 & \dots 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots 0 \end{bmatrix}$$

Hence

$$\phi^{(k)}(T^*T) - \phi^{(k)}(T^*)\phi^{(k)}(T) = \begin{bmatrix} \sum_{j=3}^{k} |a_{j1}|^2 & \sum_{j=3}^{k} \overline{a}_{j1}a_{j2} & 0 & \dots 0 \\ \sum_{j=3}^{k} a_{j1}\overline{a}_{j2} & \sum_{j=3}^{k} |a_{j2}|^2 & 0 & \dots 0 \\ 0 & 0 & \sum_{j=3}^{k} |a_{j2}|^2 & \dots 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots 0 \end{bmatrix}$$

which is clearly a positive matrix.

Let

$$D = \{T \in M_k : \phi^{(k)}(T) = T,\ \phi^{(k)}(T^*T) = T^*T\},$$

and

$$T^{(k)} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Then it can be easily seen that $T^{(k)}$ belongs to D, but its conjugate transpose does not. Thus, D is not $*$-closed.

We now prove a kind of converse of Theorem 2.3.

THEOREM 2.6. *Let A be a finite dimensional noncommutative C^*-algebra. Assume that for every Schwarz map $\phi : A \to A$, the set*

$$D_\phi = \{a \in A : \phi(a) = a, \ \phi(a^*a) = a^*a\}$$

is $*$-closed. Then A is isometrically $*$-isomorphic to M_2.

Proof. By Theorem 11.2, p. 50 of [6], A is isometrically $*$-isomorphic to a direct sum of matrix algebras. Hence we can assume that

$$A = M_{n_1} \oplus \ldots \oplus M_{n_m}$$

for some non-negative integers $n_1, \ldots, n_m$. Let d denote the vector space dimension of A. Then $d \geq 4$, since A is noncommutative. If $d = 4$, then $A = M_2$ and we are done. Let, now, $d \geq 5$. Again, since A is noncommutative, we consider the following mutually exclusive and exhaustive cases. For $T \in A$, let $T = T^{n_1} \oplus \ldots \oplus T_{n_m}$.

Case (i). At least one n_j, say, n_k, is at least 3. For $T \in A$, let

$$\phi(T) = T_{n_1} \oplus \ldots \oplus \phi^{(n_k)}(T_{n_k}) \oplus \ldots \oplus T_{n_m},$$

where $\phi^{(n_k)} : M_{n_k} \to M_{n_k}$ is the map considered in Example 2.5. Then it follows that $\phi : A \to A$ is a Schwarz map and

$$T = 0 \oplus \ldots \oplus T^{(n_k)} \oplus \ldots \oplus 0$$

belongs to D_ϕ, but T^* does not.

Case (ii). $m \geq 2$ and all n_j's equal 2. An element of $M_2 \oplus M_2$ can be regarded as a 4×4 matrix of the following form:

$$T_2 \oplus T_2 = \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ a_3 & a_4 & 0 & 0 \\ 0 & 0 & b_1 & b_2 \\ 0 & 0 & b_3 & b_4 \end{bmatrix}.$$

Define

$$\psi(T_2 \oplus T_2) = \begin{bmatrix} a_1 & a_2 & 0 & 0 \\ a_3 & a_4 & 0 & 0 \\ 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then

$$\psi : M_2 \oplus M_2 \to M_2 \oplus M_2$$

is the restriction of the Schwarz map $\phi^{(4)} : M_4 \to M_4$ of Example 2.5 to the

C^*-subalgebra formed of all elements of the type $T_2 \oplus T_2$. Define $\phi : A \to A$ by

$$\phi(T_2 \oplus T_2 \oplus \ldots \oplus T_2) = \psi(T_2 \oplus T_2) \oplus T_2 \oplus \ldots \oplus T_2.$$

Then ϕ is a Schwarz map and

$$T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \oplus \ldots \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

belongs to D_ϕ, but T^* does not.

Case (iii). $A = M_2 \oplus M_1 \oplus M_{n_3} \oplus \ldots \oplus M_{n_m}$, where $n_3, \ldots, n_m$ are either 2 or 1. An element of $M_2 \oplus M_1$ can be regarded as a 3×3 matrix of the following form:

$$T_2 \oplus T_1 = \begin{bmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \\ 0 & 0 & a_5 \end{bmatrix}.$$

Define

$$\psi(T_2 \oplus T_1) = \begin{bmatrix} a_1 & a_2 & 0 \\ a_3 & a_4 & 0 \\ 0 & 0 & a_4 \end{bmatrix}.$$

Then

$$\psi : M_2 \oplus M_1 \to M_2 \oplus M_1$$

is the restriction of the Schwarz map $\phi^{(3)} : M_3 \to M_3$ of Example 2.5 to the C^*-algebra formed of all elements of the type $T_2 \oplus T_1$. Define $\psi : A \to A$ by

$$\psi(T_2 \oplus T_1 \oplus T_{n_3} \oplus \ldots \oplus T_{n_m}) = \psi(T_2 \oplus T_1) \oplus T_{n_3} \oplus \ldots \oplus T_{n_m}.$$

Then ψ is a Schwarz map and

$$T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \oplus [0] \oplus 0 \oplus \ldots \oplus 0$$

belongs to D_ϕ, but T^* does not, Q.E.D.

Remark 2.7. In Example 2.1, we have considered the infinite dimensional noncommutative C^*-algebra $\beta(H)$, where H is a separable Hilbert space, and found a Schwarz map ϕ on $\beta(H)$ for which D_ϕ is not $*$-closed.

We can use Theorem 2.6 to show that if H is a not necessarily separable Hilbert space, then there is a Schwarz map ϕ on $\beta(H)$ for which D_ϕ is not $*$-closed. For this purpose, let G be a subspace of H of dimension 3, and let P denote the orthogonal projection of H onto G. Let

$$A = \{SP : S \in \beta(G)\}.$$

Then it is easy to see that A is a finite dimensional noncommutative C^*-subalgebra of $\beta(H)$ and that A is not isometrically $*$-isomorphic to M_2. Hence by Theorem 2.6, there is a Schwarz map $\psi : A \to A$ and some $S_0 \in \beta(G)$ such that $S_0 P \in D_\psi$ but $(S_0 P)^* \notin D_\psi$.

Define $\phi : \beta(H) \to \beta(H)$ by

$$\phi(T) = \psi((PT)_{|G}P), \quad T \in \beta(H).$$

Then ϕ is *linear. In fact, ϕ is a Schwarz map: Let $T \in \beta(H)$. Then since ψ is a Schwarz map,

$$\begin{aligned}\phi(T)^*\phi(T) &= \psi((PT)_{|G}P)^* \, \psi((PT)_{|G}P) \\ &\leq \psi(((PT)_{|G}P)^* \, (PT)_{|G}P).\end{aligned}$$

Now, it can be easily seen that

$$((PT)_{|G}P)^*(PT)_{|G}P \leq (PT^*T)_{|G}P).$$

Since ψ is positive, we see that

$$\phi(T)^*\phi(T) \leq \psi((PT^*T)_{|G}P) = \phi(T^*T).$$

Also, $\phi_{|A} = \psi$, since for $S \in \beta(G)$, we have

$$\phi(SP) = \psi((PSP)_{|G}P) = \psi(SP).$$

Hence $S_0 P \in D_\phi$, but $(S_0 P)^* \notin D_\phi$.

Again, since the range of ϕ is contained in the C^*-algebra $\varkappa(H)$ of all compact operators on H, we can consider the restriction of ϕ to $\varkappa(H)$ and obtain a Schwarz map on $\varkappa(H)$ for which the set D_ϕ is not $*$-closed.

Finally, if A is any infinite dimensional noncommutative C^*-subalgebra of $\varkappa(H)$, then by Theorem 1.4.5 of [1],

$$A = \bigoplus_\alpha \varkappa(H_\alpha),$$

where each H_α is a Hilbert space. Hence we can find a Schwarz map $\phi : A \to A$ for which D_ϕ is not $*$-closed. We have not been able to answer the question whether this can be done for any infinite dimensional noncommutative C^*-algebra A.

References

1. W. Arvenson, *An Invitation to C*-algebras,* Springer-Verlag, New York, 1976.
2. R.V. Kadison, *A generalized Schwarz inequality and algebraic invariants for operator algebras,* Ann. of Math., vol. 56(1952), pp. 494–503.
3. T.W. Palmer, *Characterizations of *homomorphisms and expectations,* Proc. Amer. Math. Soc., vol. 46(1974), pp. 265–272.
4. A.G. Robertson, *A Korovkin theorem for Schwarz maps on C*-algebras,* Math. Zeitschr., vol. 56(1977), pp. 205–207.
5. A.L.Shields and L.J. Wallen, *The commutants of certain Hilbert space operators,* Indiana Univ. Math. J., vol. 20(1971), pp. 777–788.
6. M. Takesaki, *Theory of operator algebras I,* Springer-Verlag, New York, 1979.

Indian Institute of Technology
Bombay, India.