ON REALIZING NAKAOKA'S COINCIDENCE POINT TRANSFER AS AN S-MAP

BY

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1. Introduction

In [6] the existence was asserted of a transfer for fibre bundles whose fibers were compact manifolds. This transfer is a homomorphism in singular homology or cohomology. In [1] this transfer was shown to be induced by an S-map, at least for fibre bundles with compact Lie groups as structure groups and finite complexes as base spaces. When a transfer is induced by an S-map then it exists in every cohomology and homology theory. In the case of the transfer in [1] the S-map led immediately to a simple topological proof of the Adams Conjecture.

The transfer map for finite covering spaces was known to be induced by an S-map. This fact was crucial in the proof of the celebrated Kahn-Priddy theorem [8].

At present there are several different transfers defined in various circumstances. Many are defined only for singular homology groups. Sometimes these are not realized by S-maps. For example the transfer for ramified coverings [10] does not commute with the Steenrod algebra [7] (also attributed to Dold). Hence it cannot be realized by an S-map. Recently, Ralph Cohen [4] has shown that the ramified covering transfer can be realized if the ramified covering is localized at certain primes. This was also known to Larry Smith.

In [9], Nakaoka obtained a transfer in the following situation: *Let*

$$E_1 \overset{p_1}{\to} B \quad and \quad E_2 \overset{p_2}{\to} B$$

be fibre bundles with closed oriented m-dimensional manifolds M_1 and M_2 as fibres. Suppose that $\pi_1(B)$ acts trivially on both $H^m(M_1; \mathbf{Z})$ and $H^m(M_2; \mathbf{Z})$. Let f and g: $E_1 \to E_2$ be fibre preserving maps covering the identity 1: $B \to B$. Then there is a homomorphism

$$\tau_{f,g}: H^*(E_1; \mathbf{Z}) \to H^*(B; \mathbf{Z})$$

such that $\tau_{f,g} \circ p_1^*$ is multiplication by $\lambda(\bar{f}, \bar{g})$ where $\bar{f}$ and $\bar{g}$ are the restrictions

of f and g to a fibre and $\lambda(\bar{f}, \bar{g})$ is the Lefschetz trace of $\bar{f}^*\bar{g}^!$, that is $\lambda(\bar{f}, \bar{g}) = \Sigma_i(-1)^i \mathrm{tr}(\bar{f}^*\bar{g}^!)_i$ where $g^!$ is the Umkehr map.

We shall show the following theorem.

THEOREM. $\tau_{f,g}$ is induced by an S-map $\tau: B \to E_1$ if B (and hence E_1 and E_2) is a closed oriented manifold, both fibre bundles p_1 and p_2 are smooth, and g is a normal map.

We require manifolds because we want to use Umkehr maps to define the transfer. We use smoothness to obtain bundles of tangents along the fibre easily.

Nakaoka's construction was a variation of the constructions in [6] and [2]. Our method here uses the techniques in [3]. The strategy is to define the transfer for singular cohomology as a composition of induced homomorphisms and Umkehr maps, in fact $p_1^! F^* G^!$ where F and G will be defined later. These maps are induced by maps of Thom spaces and then the condition that g is a normal map will yield a Thom space of a trivial bundle, which is a suspension. In order to express the transfer as $p_1^! F^* G^!$ we must give a somewhat different definition of the coincidence number transfer than Nakaoka's.

2. An S-map

Consider the following diagram of spaces and maps:

(1)
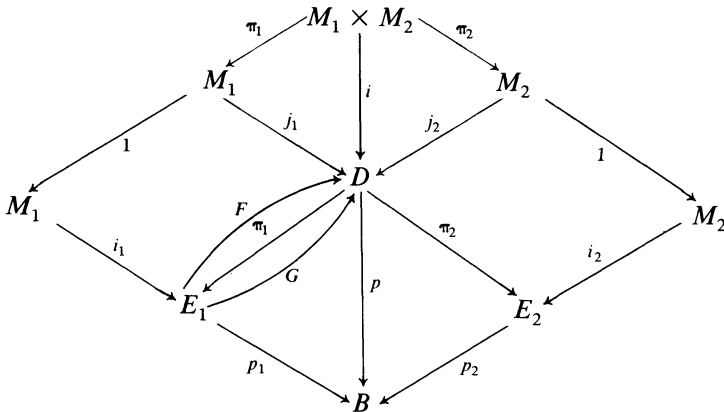
Here
$$E_1 \overset{p_1}{\to} B$$
is a fibre bundle with M_1 the fibre over a fixed b and
$$M_1 \overset{i_1}{\to} E_1.$$
Similarly
$$M_2 \overset{i_2}{\to} E_2 \overset{p_2}{\to} B$$

is a fibre bundle. Let D be the subspace of $E_1 \times E_2$ given by

$$D = \{(e_1, e_2) \in E_1 \times E_2 | p_1(e_1) = p_2(e_2)\}.$$

Let $\pi_1 \colon D \to E_1$ be projection on the first factor, i.e., $\pi_1(e_1, e_2) = e_1$, and let $\pi_2 \colon D \to E_2$ be projection on the second factor, $\pi_2(e_1, e_2) = e_2$. Then π_1 and π_2 are fibre bundles with fibres M_2 and M_1 respectively and the fibre inclusions are j_2 and j_1 respectively. The composition $p_1 \circ \pi_1 = p_2 \circ \pi_2 = p$ is itself a fibre bundle with fibre inclusion

$$M_1 \times M_2 \overset{i}{\to} D.$$

Now if $f \colon E_1 \to E_2$ is a fibre preserving map over the identity define the "graph" $F \colon E_1 \to D$ by

$$F(e_1) = (e_1, f(e_1)).$$

Similarly G is the graph of $g \colon E_1 \to E_2$. Note that

(2) $$\pi_1 \circ F = 1_{E_1}, \ \pi_2 \circ F = f \quad \text{and} \quad \pi_1 \circ G = 1_{E_1}, \ \pi_2 \circ G = g.$$

We now assume that E_1 and E_2 are smooth manifolds. Then D is a smooth manifold and all the bundles in (1) are smooth. Now let α_1 and α_2 be the bundle of tangents along the fibre

$$E_1 \overset{p_1}{\to} B \quad \text{and} \quad E_2 \overset{p_2}{\to} B$$

respectively. Also let β_1 and β_2 be tangents along the fibre for

$$D \overset{\pi_2}{\to} E_2 \quad \text{and} \quad D \overset{\pi_1}{\to} E_1$$

respectively. Then six pullback relations are easily seen to hold:

(3) $$\beta_1 = \pi_1^*\alpha_1, \quad \beta_2 = \pi_2^*\alpha_2,$$
$$F^*\beta_1 = \alpha_1, \quad F^*\beta_2 = f^*\alpha_2, \quad G^*\beta_1 = \alpha_1, \quad G^*\beta_2 = g^*\alpha_2.$$

For example, $F^*\beta_2 = F^*(\pi_2^*\alpha_2) = (\pi_2 \circ F)^*\alpha_2 = f^*\alpha_2$ by the equations in (2).

Now we imitate the approach of §5 of [3]. We embed E_1 in $\mathbf{R}^N$ by a smooth embedding k. Then we embed E_1 in $B \times \mathbf{R}^N$ by sending $e \mapsto (p_1(e), k(e))$. The normal bundle of E_1 in $B \times \mathbf{R}^N$ is $-\alpha_1$. Let $\hat{p}_1 \colon B^N = \Sigma^N B^+ \to E_1^{-\alpha_1}$ be the collapsing map from the Thom space B^N of the trivial N dimensional bundle N to the Thom space of the normal bundle of E_1 in $B \times \mathbf{R}^N$.

The embedding $F \colon E_1 \to D$ extends to a bundle map $\alpha_1 \to \beta_1$. So F induces a map $\bar{F} \colon E_1^{-\alpha_1} \to D^{-\beta_1}$.

Finally G: $E_1 \to D$ is an embedding. Note that $G^*\beta_1 = \alpha_1$. Also note that the normal bundle of $G(E_1)$ in D is $G^*\beta_2$. Thus the normal bundle of $G(E_1)$ in $D^{-\beta_1}$ is $G^*(-\beta_1) \oplus G^*(\beta_2) = -\alpha_1 + g^*(\alpha_2)$. Thus if $\hat{G}$ denotes the collapsing map of $D^{-\beta_1}$ onto the Thom space of the normal bundle of $G(E_1)$ in $D^{-\beta_1}$ then $\hat{G}$: $D^{-\beta_1} \to E_1^{-\alpha_1 + g^*(\alpha_2)}$.

The composition

$$(4) \qquad \hat{G} \circ \bar{F}\hat{p}_1 \colon \Sigma^N B^+ \to E_1^{-\alpha_1} \to D^{-\beta_1} \to E_1^{-\alpha_1 + g^*(\alpha)}$$

gives rise to the required S-map

$$\tau \colon \Sigma^N B^+ \to \Sigma^N E_1^+$$

if $-\alpha_1 + g^*(\alpha_2)$ is stably trivial, i.e., $-\alpha_1 + g^*(\alpha_2) = N$.

Now $g^*(\alpha_2)$ is stably equivalent to α_1 if and only if g: $E_1 \to E_2$ is a normal map. This follows since

$$g^*(-\tau_{E_2}) = g^*(-\alpha_2 - p_2^*(\tau_B))$$
$$= -g^*(\alpha_2) - g^* p_2^*(\tau_B) = -g^*(\alpha_2) - p_1^*(\tau_B)$$

and

$$-\tau_{E_1} = -\alpha_1 - p_1^*(\tau_B)$$

and since $g^*(-\tau_{E_2}) = -\tau_{E_1}$ we see that $g^*(\alpha_2) = \alpha_1$. Thus we have shown:

LEMMA 1.   *If g: $E_1 \to E_2$ is a normal map then*

$$\tau = \hat{G} \circ \bar{F} \circ \hat{p}_1 \colon \Sigma^N B^+ \to \Sigma^N E_1^+.$$

3. The induced homomorphism

First we shall show that τ: $\Sigma^N B^+ \to \Sigma^N E_1^+$ induces $p_1^! F^* G^!$ on homology. More precisely we have the following lemma where

$$S \colon H^*(X) \to H^*(\Sigma^N X)$$

denotes the suspension isomorphism.

LEMMA 2. $S^{-1}\tau^* S = \pm p_1^! F^* G^!$

Proof.   Since p_1 and p_2 are orientable fibrations so are π_1: $D \to E_1$ and π_2: $D \to E_2$. Hence α_1, α_2, β_1 and β_2 are orientable vector bundle. Hence their associated Thom spaces admit Thom isomorphisms, which we shall

denote by S. Then

$$S^{-1}\tau*S = S^{-1}\hat{p}_1^*\bar{F}*\hat{G}*S \quad \text{(by Lemma 1)}$$
$$= (S^{-1}\hat{p}_1^*S)(S^{-1}\bar{F}*S)(S^{-1}\hat{G}*S)$$
$$= (\pm p_1^!)(\pm F^*)(\pm G^!) \quad \text{(by (8), (9), (10) of §3 in [3])}$$
$$= \pm p_1^! F^*G^!$$

Now set $\tau_{f,g} = p_1^! F^*G^!$. We must show that $\tau_{f,g} \circ p_1^*$ is a multiplication by $\lambda(\bar{f}, \bar{g})$ where $\lambda(\bar{f}, \bar{g}) = \Sigma_i(-1)^i \text{tr}(\bar{f}^*\bar{g}^!)_i$, the Lefschetz trace of $\bar{f}^*\bar{g}^!$.

LEMMA 3. $\tau_{f,g} \circ p_1^*$ is multiplication by k where $k = \langle i_1^* F^*G^!(1), [M_1] \rangle$

Proof.

$$\tau_{f,g} p_1^*(x) = p_1^! F^*G^! p_1^*(x)$$
$$= p_1^! F^*G^!(G^*\pi_1^*) p_1^*(x)$$
$$= p_1^! F^*(\pi_1^* p_1^*(x) \cup G^!(1))$$
$$= p_1^! (F^*\pi_1^* p_1^*(x) \cup F^*G^!(1))$$
$$= p_1^! (p_1^*(x) \cup F^*G^!(1))$$
$$= x \cup p_1^! (F^*G^!(1)).$$

Now

$$p_1^! (F^*G^!(1)) \in H^0(B, \mathbf{Z}) \cong \mathbf{Z}$$

and so $\tau_{f,g} p_1^*$ is multiplication by $p_1^!(F^*G^!(1))$. But

$$p_1^! (F^*G^!(1)) = \langle i_1^*(F^*G^!(1)), [M_1] \rangle.$$

Finally we must show that $\langle i_1^* F^*G^!(1), [M_1] \rangle = \lambda(\bar{f}, \bar{g})$, the Lefschetz trace of $\bar{f}^*\bar{g}^!$ where $\bar{f}$ and $\bar{g}$ are the restrictions of f and g to the fibre $M_1 \to M_2$. Let $\bar{F}, \bar{G}: M_1 \to M_1 \times M_2$ be the graphs of $\bar{f}$ and $\bar{g}$ respectively, so that $\bar{F}$ and $\bar{G}$ are restrictions of F and G to the fibre M_1. Then $i \circ \bar{F} = F \circ i_1$ and $i \circ \bar{G} = G \circ i_1$ as can be seen by referring to diagram (1). By an argument similar to Lemma 1 of [3] we have that $\bar{G}^! i_1^* = \pm i^* G^!$. Thus

$$i_1^* F^*G^!(1) = \bar{F}^* i^* G^!(1) = \pm \bar{F}^*\bar{G}^!(i_1^*(1)) = \pm \bar{F}^*\bar{G}^!(1).$$

Thus we have shown:

LEMMA 4. $i_1^* F^*G^!(1) = \pm \bar{F}^*\bar{G}^!(1).$

Then the following lemma concludes the proof.

LEMMA 5. $\bar{F}^*\bar{G}^!(1) = \lambda(\bar{f}, \bar{g})[\overline{M_1}] \in H^m(M_1; \mathbf{Z}).$

Proof.   We will show this for rational coefficients $\mathbf{Q}$, and that will imply the lemma. Let B be a homogeneous basis for $H^*(M_1, \mathbf{Q})$ and let $\hat{B}$ be a dual basis defined by $\hat{b} \cup a = \delta_{ba}[\overline{M_1}]$ where $a \in B$. Let $\Delta: M_1 \to M_1 \times M_1$ be the diagonal map. Let μ be the unique element in $H^m(M_1 \times M_1; \mathbf{Q})$ defined by

$$\mu \cap [M_1 \times M_1] = \Delta_*[M].$$

Then $\mu = \Sigma_{b \in B}(-1)^{|b|}\hat{b} \times b$, (see p. 302, exercise 2 of [5]). Using the formula

$$(x \times y) \cap (\xi \times \eta) = (-1)^{|y||\xi|}(x \cap \xi) \times (y \cap \eta)$$

we obtain

$$
\begin{aligned}
\Delta_*[M_1] &= \mu \cap ([M_1] \times [M_1]) \\
&= \sum_{b \in B} (-1)^{|b|(1+m)}(\hat{b} \cap [M_1]) \times (b \cap [M_1])
\end{aligned}
$$

Note that $\overline{G} = (1 \times \bar{g}) \circ \Delta$. Hence

$$
\begin{aligned}
\overline{G}_*[M_1] &= (1 \times \bar{g})_* \Delta_*[M] \\
&= \sum_{b \in B} (-1)^{|b|(1+m)}((\hat{b} \cap [M_1]) \times (\bar{g}_*(b \cap [M_1])))
\end{aligned}
$$

Hence

$$
\begin{aligned}
\overline{G}^!(1) &= D^{-1}(\overline{G}_*[M_1]) \\
&= \sum_{b \in B} (-1)^{|b|(1+m)} D^{-1}((\hat{b} \cap [M_1] \times \bar{g}_*(b \cap [M_1]))
\end{aligned}
$$

where D is the Poincaré Duality isomorphism. We have

$$\overline{G}^!(1) = \sum_{b \in B} (-1)^{|b|(m+1)} \cdot (-1)^{m|b|}\hat{b} \times D^{-1}\bar{g}_*(b \cap [M_1]).$$

Hence $\overline{G}^!(1) = \Sigma_{b \in B}(-1)^{|b|}\hat{b} \times \bar{g}^!(b)$. Then

$$
\begin{aligned}
\overline{F}^*\overline{G}^!(1) &= \Delta^*(1 \times \bar{f})^*\overline{G}^!(1) \\
&= \Delta^*\left(\sum_{b \in B} (-1)^{|b|}\hat{b} \times \bar{f}^*\bar{g}^!(b) \right)
\end{aligned}
$$

So $\overline{F}^*\overline{G}^!(1) = \Sigma_{b \in B}(-1)^{|b|}\hat{b} \cup \bar{f}^*\bar{g}^!(b)$. Now $\bar{f}^*\bar{g}^!(b) = \Sigma_{c \in B}\Gamma_b^c c$ where $\Gamma_b^c \in \mathbf{Q}$. Hence

$$
\begin{aligned}
\overline{F}^*\overline{G}^!(1) &= \sum_{b \in B} (-1)^{|b|}\hat{b} \cup \left(\sum_c \Gamma_b^c c \right) \\
&= \left(\sum_{b \in B} (-1)^{|b|}\Gamma_b^b \right)[M_1] \\
&= \lambda(\bar{f}, \bar{g})[M_1].
\end{aligned}
$$

Remark. The usual definition of the coincidences index which Nakaoka used is based on comparing two maps,

$$M_1 \xrightarrow{\Delta} M_1 \times M_1 \xrightarrow{f \times g} M_2 \times M_2$$

and

$$M_2 \xrightarrow{\Delta} M_2 \times M_2.$$

In our proof we compare

$$M_1 \xrightarrow{\Delta} M_1 \times M_1 \xrightarrow{1 \times f} M_1 \times M_2$$

with

$$M_1 \xrightarrow{\Delta} M_1 \times M_1 \xrightarrow{1 \times g} M_1 \times M_2.$$

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