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ON SPECTRAL DECOMPOSITION OF CLOSED OPERATORS ON BANACH SPACES

BY

WANG SHENGWANG AND I. ERDELYI

This paper is concerned with presenting some necessary and sufficient conditions for a closed operator to have the spectral decomposition property. We refer to [2] for notations and terminology, but for convenience we repeat some definitions.

Throughout this paper, T is a closed operator with domain D_T and range in a Banach space over the complex field $\mathbf{C}$. Let $\mathbf{N}$ denote the set of natural numbers and let $Z^+ = \mathbf{N} \cup \{0\}$. If S is a set then $\bar{S}$ is the closure, S^c is the complement, Int S is the interior and we denote by $\operatorname{cov} S$ the collection of all finite open covers of S. Without loss of generality, we assume that for $S \subset \mathbf{C}$, every $\{G_i\}_{i=0}^n \in \operatorname{cov} S$ has, at most, one unbounded set G_0. A set $S \subset \mathbf{C}$ is said to be a neighborhood of ∞, in symbols $S \in V_\infty$, if $\overline{S^c}$ is compact in $\mathbf{C}$. Given T, $\sigma(T)$ is the spectrum, $\sigma_a(T)$ is the approximate point spectrum, $\rho(T)$ is the resolvent set and $R(\cdot\,; T)$ is the resolvent operator. If A is a bounded operator then $\rho_\infty(A)$ denotes the unbounded component of $\rho(A)$. If T has the single valued extension property (SVEP), then $\sigma_T(x)$, $\rho_T(x)$ and $x(\cdot)$ denote the local spectrum, the local resolvent set and the local resolvent function, respectively, at $x \in X$.

For $S \subset \mathbf{C}$, we shall make an extensive use of the spectral manifold

$$X(T, S) = \{x \in X \colon \sigma_T(x) \subset S\}. \tag{1}$$

We write $\operatorname{Inv} T$ for the lattice of the subspaces of X which are invariant under T. For $Y \in \operatorname{Inv} T$, $T|Y$ is the restriction of T to Y and $\hat{T} = T/Y$ denotes the coinduced operator by T on the quotient space X/Y. The coset $\hat{x} = x + Y$ is a vector in X/Y and $\hat{x} \in D_{\hat{T}}$ iff $\hat{x} \cap D_T \neq \varnothing$. If f is an X-valued function then the function $\hat{f}$ has the range in X/Y.

1. Introduction

In this section, certain basic notions pertaining to the spectral theory will be touched upon and some preliminary results will be established to be used in

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the subsequent theory. First, we re-examine the single valued extension property in the spirit of an earlier work by Finch [3].

1.1. THEOREM. *Given T, for every $x \in X$ and $\lambda_0 \in \mathbf{C}$, the following assertions are equivalent*:

(i) *There is a neighborhood δ of λ_0 and an analytic function $f\colon \delta \to D_T$ such that*

$$(\lambda - T)f(\lambda) = x \text{ on } \delta. \tag{1.1}$$

(ii) *There are numbers $M > 0$, $R > 0$ and a sequence $\{a_n\}_{n=0}^{\infty} \subset D_T$, with the following properties*:

$$\begin{gathered}\text{(a) } (\lambda_0 - T)a_0 = x; \text{ (b) } (\lambda_0 - T)a_{n+1} = a_n; \\ \text{(c) } \|a_n\| \le MR^n,\ n \in Z^+.\end{gathered} \tag{1.2}$$

Proof. (i) $\Rightarrow$ (ii). We may assume that

$$\delta = \{\lambda \in \mathbf{C}\colon |\lambda - \lambda_0| < r\}$$

for some $r > 0$. Let

$$f(\lambda) = \sum_{n=0}^{\infty} a_n(\lambda_0 - \lambda)^n, \quad \lambda \in \delta \tag{1.3}$$

be the power series expansion of f. By decreasing r, we may assume that (1.3) holds on $\bar{\delta}$ and $r < 1$. Then, for $\lambda \in \partial\delta$,

$$\|a_n\| r^n \to 0 \quad \text{as } n \to \infty.$$

Hence, there is $M > 0$ such that

$$\|a_n\| r^n \le M, \quad n \in Z^+. \tag{1.4}$$

For $R = r^{-1}$, (1.4) implies (1.2)(c). By taking $\lambda = \lambda_0$ in (1.1) and (1.3), one obtains (1.2)(a). Furthermore, it follows from (1.3) that

$$a_n = -\frac{1}{2\pi i}\int_{\partial\delta} \frac{f(\lambda)\, d\lambda}{(\lambda_0 - \lambda)^{n+1}}, \quad n \in Z^+.$$

In view of (1.1), one can write

$$(\lambda_0 - T)f(\lambda) = (\lambda_0 - \lambda)f(\lambda) + (\lambda - T)f(\lambda) = (\lambda_0 - \lambda)f(\lambda) + x.$$

T being closed, one obtains $a_n \in D_T$ $(n \in Z^+)$ and

$$\begin{aligned}(\lambda_0 - T)a_{n+1} &= -\frac{1}{2\pi i}\int_{\partial\delta}\frac{(\lambda_0 - T)f(\lambda)\,d\lambda}{(\lambda_0 - \lambda)^{n+2}} \\ &= -\frac{1}{2\pi i}\int_{\partial\delta}\frac{f(\lambda)\,d\lambda}{(\lambda_0 - \lambda)^{n+1}} - \frac{1}{2\pi i}\int_{\partial\delta}\frac{x d\lambda}{(\lambda_0 - \lambda)^{n+2}} = a_n.\end{aligned}$$

This proves (1.2)(b).

(ii) $\Rightarrow$ (i). In view of (1.2)(c), the power series (1.3) defines a function f, analytic on

$$\delta = \left\{\lambda \in \mathbf{C}: |\lambda - \lambda_0| < \frac{1}{R}\right\}.$$

Thus, for

$$f_k(\lambda) = \sum_{n=0}^{k} a_n(\lambda_0 - \lambda)^n, \quad \lambda \in \delta, k \in \mathbf{N},$$

one obtains

$$\begin{aligned}(\lambda - T)f_k(\lambda) &= \sum_{n=0}^{k}(\lambda - T)a_n(\lambda_0 - \lambda)^n \\ &= \sum_{n=0}^{k}(\lambda_0 - T)a_n(\lambda_0 - \lambda)^n - \sum_{n=0}^{k} a_n(\lambda_0 - \lambda)^{n+1} \\ &= x + \sum_{n=1}^{k} a_{n-1}(\lambda_0 - \lambda)^n - \sum_{n=0}^{k} a_n(\lambda_0 - \lambda)^{n+1} \\ &= x - a_k(\lambda_0 - \lambda)^{k+1}.\end{aligned}$$

Since T is closed and for all $\lambda \in \delta$,

$$f_k(\lambda) \to f(\lambda) \text{ and } a_k(\lambda_0 - \lambda)^{k+1} \to 0 \quad \text{as } k \to \infty,$$

we have

$$f(\lambda) \in D_T \text{ and } (\lambda - T)f(\lambda) = x \quad \text{for all } \lambda \in \delta. \qquad \square$$

1.2. COROLLARY. *T does not have the SVEP iff there exists $\lambda_0 \in \mathbf{C}$ and there are numbers $M > 0$, $R > 0$ and a sequence $\{a_n\}_{n=0}^{\infty} \subset D_T$ such that*

$$(\lambda_0 - T)a_0 = 0; \quad (\lambda_0 - T)a_{n+1} = a_n; \quad \|a_n\| \le MR^n, \quad n \in Z^+$$

and $a_n \neq 0$ for n sufficiently large.

Proof. T does not have the SVEP iff there is a neighborhood δ of some $\lambda_0 \in \mathbf{C}$ such that

$$(\lambda - T)f(\lambda) = 0 \quad \text{and} \quad f(\lambda) \not\equiv 0 \text{ on } \delta. \tag{1.5}$$

In view of Theorem 1.1, the situation described by (1.5) occurs iff the properties expressed by the corollary hold, for $x = 0$. □

Another consequence of Theorem 1.1 is [3, Theorem 2], asserting that T does not have the SVEP if there exists $\lambda_0 \in \mathbf{C}$ such that $\lambda_0 - T$ is surjective but not injective.

Next, we recall some definitions and related properties. The spectral maximal space concept [4] admits two distinct extensions to the case of unbounded operators.

1.3. Definition. Given T, $Y \in \operatorname{Inv} T$ is called a spectral maximal space of T if, for any $Z \in \operatorname{Inv} T$, the inclusion $\sigma(T|Z) \subset \sigma(T|Y)$ implies $Z \subset Y$.

$Y \in \operatorname{Inv} T$ with $Y \subset D_T$ is said to be a T-bounded spectral maximal space if, for any $Z \in \operatorname{Inv} T$, the inclusions $Z \subset D_T$, $\sigma(T|Z) \subset \sigma(T|Y)$ imply $Z \subset Y$.

If T has the SVEP and if, for compact $F \subset \mathbf{C}$, $X(T, F)$ is closed then

$$X(T, F) = \Xi(T, F) \oplus X(T, \varnothing). \tag{1.6}$$

Here, $\Xi(T, F)$ is a T-bounded spectral maximal space [2]. If T has the SVEP and if, for closed $F \subset \mathbf{C}$, $X(T, F)$ is closed then $X(T, F)$ is a spectral maximal space of T. In this case, for disjoint closed F_1 and compact F_2 we have [2]

$$X(T, F_1 \cup F_2) = X(T, F_1) \oplus \Xi(T, F_2) \tag{1.7}$$

and if both F_1 and F_2 are compact, then

$$\Xi(T, F_1 \cup F_2) = \Xi(T, F_1) \oplus \Xi(T, F_2). \tag{1.8}$$

Two additional types of invariant subspaces will be useful later on.

1.4 Definition [7], [5]. Given T, $Y \in \operatorname{Inv} T$ is said to be T-absorbent if, for any $y \in Y$ and any $\lambda \in \sigma(T|Y)$, the condition $(\lambda - T)x = y$ implies $x \in Y$.

$Y \in \operatorname{Inv} T$ is called analytically invariant under T if, for every analytic D_T-valued function f defined on an open $D \subset \mathbf{C}$, $(\lambda - T)f(\lambda) \in Y$ implies $f(\lambda) \in Y$ on D.

In particular, every spectral maximal space of T as well as each T-bounded spectral maximal space is T-absorbent [2]. If Y is any of the subspaces introduced by Definitions 1.3 and 1.4, then $\sigma(T|Y) \subset \sigma(T)$.

2. The spectral decomposition property

In this section we study some local properties of a general type of spectral decomposition.

2.1 DEFINITION. T is said to have the spectral decomposition property (SDP) if, for every $\{G_i\}_{i=0}^n \in \operatorname{cov}\sigma(T)$ with $G_0 \in V_\infty$, there exists $\{Y_i\}_{i=0}^n \subset \operatorname{Inv} T$ satisfying the following conditions:

(I) $Y_i \subset D_T$, if G_i is relatively compact ($1 \le i \le n$);

(II) $\sigma(T|Y_i) \subset G_i$, $0 \le i \le n$;

(III) $X = \sum_{i=0}^n Y_i$.

Remarks Without any deviation from the above defined notion, we may consider

$$\{G_i\}_{i=0}^n \in \operatorname{cov}\mathbf{C},$$

and replace (II) by

(II′) $\sigma(T|Y_i) \subset \overline{G}_i, 0 \le i \le n$.

In terms of spectral maximal and T-bounded spectral maximal spaces, the decomposition (III) can be expressed by

$$X = X(T, \overline{G}_0) + \sum_{i=1}^n \Xi(T, \overline{G}_i). \tag{2.1}$$

Note that (2.1) implies conditions (I), (II) and (III) above. Even a two-summand decomposition ($n = 1$) of T implies the SDP [6].

2.2 DEFINITION. An operator T is said to have property (κ) if

(a) T has the SVEP;

(b) $X(T, F)$ is closed for all closed $F \subset \mathbf{C}$.

A slightly strengthened version of Bishop's condition (β) [1, Definition 8], as expressed by the following definition, will greatly enhance the study of the spectral decomposition problem.

2.3 DEFINITION. T has property (β) if, for any sequence $\{f_n: G \to D_T\}$ of functions, analytic on an open $G \subset \mathbf{C}$, $(\lambda - T)f_n(\lambda) \to 0$ (as $n \to \infty$) in the strong topology of X and uniformly on every compact subset of G implies that $f_n(\lambda) \to 0$ in the strong topology of X and uniformly on every compact subset of G.

In contrast to [8, 4.16–4.18], the above definition does not require that Tf_n be analytic for each n. Property (β) implies property (κ). Clearly, it implies the SVEP. Also, if T has property (β) then, for closed $F \subset \mathbf{C}$, $X(T, F)$ is closed (e.g., [8, Corollary 4.18]).

2.4 LEMMA. *Given T, let $Y \in \operatorname{Inv} T$ be such that $Y \subset D_T$ and $\hat{T} = T/Y$ is closed. If T has the SVEP and $\sigma(T|Y) \cap \sigma(\hat{T})$ is nowhere dense in $\mathbf{C}$, then Y is analytically invariant under T.*

Proof. Let f: $D_f \to D_T$ be analytic on an open $D_f \subset \mathbf{C}$ and satisfy condition

$$(\lambda - T)f(\lambda) \in Y \quad \text{for all } \lambda \in D_f.$$

We may assume that D_f is connected. By the canonical map $X \to X/Y$, we have

$$(\lambda - \hat{T})\hat{f}(\lambda) = \hat{0} \text{ on } D_f.$$

By [6, Lemma 3.2], there is an analytic function h: $D_h(\subset D_f) \to D_T$ such that $\hat{h}(\lambda) = \hat{f}(\lambda)$ and $(\lambda - T)h(\lambda)$ is analytic on D_h. Likewise D_f, D_h can be assumed to be a connected open set.

First, suppose that $D_h \cap \rho(T|Y) \neq \varnothing$. The function g: $D_h \cap \rho(T|Y) \to X$, defined by $g(\lambda) = (\lambda - T)h(\lambda)$, is analytic and

$$\hat{g}(\lambda) = (\lambda - \hat{T})\hat{h}(\lambda) = (\lambda - \hat{T})\hat{f}(\lambda) = \hat{0}$$

implies that $g(\lambda) \in Y$ on $D \cap \rho(T|Y)$. Then

$$(\lambda - T)[h(\lambda) - R(\lambda; T|Y)g(\lambda)] = 0$$

and the SVEP of T implies

$$h(\lambda) = R(\lambda; T|Y)g(\lambda) \in Y \quad \text{on } D_h \cap \rho(T|Y).$$

Thus, $h(\lambda) \in Y$ on D_h, by analytic continuation. Since $\hat{f}(\lambda)$ and $\hat{h}(\lambda)$ agree on D_h, we have $f(\lambda) - h(\lambda) \in Y$ on D_h. Thus, $f(\lambda) \in Y$ on D_h and hence $f(\lambda) \in Y$ on D_f, by analytic continuation.

Next, assume that $D_h \subset \sigma(T|Y)$. Since, by hypothesis, $D_h \cap \rho(\hat{T}) \neq \varnothing$ it follows from $(\lambda - \hat{T})\hat{h}(\lambda) = \hat{0}$ that $\hat{h}(\lambda) = \hat{0}$ on $D_h \cap \rho(\hat{T})$. Therefore, $\hat{f}(\lambda) = \hat{0}$, i.e. $f(\lambda) \in Y$ on D_f, by analytic continuation. □

2.5. LEMMA. *Given a subspace Y of X, let H, K be open disks with $\overline{K} \subset H$. If $\hat{f}$: $V \to X/Y$ is an analytic function on a neighborhood V of $\overline{H}$, then there exists an analytic function h: $H \to X$ such that*

$$\max_{\lambda \in \overline{K}} \|h(\lambda)\| \leq A \max_{\lambda \in \overline{H}} \|\hat{f}(\lambda)\|,$$

where A is a constant.

Proof. Let

$$H = \{\lambda: |\lambda - \lambda_0| < R\}, \quad K = \{\lambda: |\lambda - \lambda_1| < r\}$$

for $\lambda_0 \in \mathbf{C}$, $\lambda_1 \in H$, $0 < r \leq r + |\lambda_0 - \lambda_1| < R$ and let

$$\hat{f}(\lambda) = \sum_{n=0}^{\infty} \hat{a}_n(\lambda - \lambda_0)^n \quad \text{with } \{\hat{a}_n\} \subset X/Y$$

be the power series expansion of $\hat{f}$. Choose ρ to satisfy $r + |\lambda_0 - \lambda_1| < \rho < R$. By the Cauchy inequality, we have $\|\hat{a}_n\| \leq MR^{-n}$, where $M = \max_{\lambda \in \bar{H}} \|\hat{f}(\lambda)\|$. For every n, choose $a_n \in \hat{a}_n$ such that $\|a_n\| \leq 2\|\hat{a}_n\|$ and define

$$h(\lambda) = \sum_{n=0}^{\infty} a_n(\lambda - \lambda_0)^n.$$

Then h is analytic on H and since $\bar{K} \subset \{\lambda: |\lambda - \lambda_0| < \rho\}$, we have

$$\max_{\lambda \in \bar{K}} \|h(\lambda)\| \leq \sum_{n=0}^{\infty} \|a_n\|\rho^n \leq 2M \sum_{n=0}^{\infty} \left(\frac{\rho}{R}\right)^n = A \max_{\lambda \in \bar{H}} \|\hat{f}(\lambda)\|,$$

where $A = 2R/(R - \rho)$. □

2.6. THEOREM. *Given T, suppose that for every pair of open disks G, H with $\bar{G} \subset H$, there exists $Z \in \operatorname{Inv} T$ such that*
(a) $\sigma(T|Z) \subset G^c$;
(b) $\hat{T} = T/Z$ *is bounded in X/Z and* $\sigma(\hat{T}) \subset H$.
Then T has property (β).

Proof. Let $\{f_n\}$ be a sequence of D_T-valued analytic functions on an open G_0 such that

$$(2.2) \qquad (\lambda - T)f_n(\lambda) \to 0 \quad (n \to \infty)$$

in the strong topology of X and uniformly on every compact subset of G_0. We may assume that $G_0 = \{\lambda: |\lambda| < R\}$ for some $R > 0$. Choose the numbers R_0, R_1, R_2 such that $0 < R_0 < R_1 < R_2 < R$ and let

$$K = \{\lambda: |\lambda| \leq R_0\}, G = \{\lambda: |\lambda| < R_1\}, \quad H = \{\lambda: |\lambda| < R_2\}.$$

By hypothesis, there exists $Z \in \operatorname{Inv} T$ satisfying conditions (a) and (b) for G and H. It follows from (2.2) that

$$(\lambda - \hat{T})\hat{f}_n(\lambda) \to \hat{0}$$

in the strong topology of X/Z and uniformly on $\overline{H}$. Since $\partial H \subset \rho(\hat{T})$, we have

$$\hat{f}_n(\lambda) \to \hat{0} \tag{2.3}$$

in the strong topology of X/Z and uniformly on ∂H. By the maximum principle, the convergence (2.3) is uniform on $\overline{H}$. Furthermore, since $\hat{T}$ is bounded,

$$\hat{T}\hat{f}_n(\lambda) \to \hat{0} \tag{2.4}$$

in the strong topology of X/Z and uniformly on $\overline{H}$.

The graph $G(T)$ of T is closed in $X \oplus X$ and $G(T|Z)$ is closed in $Z \oplus Z$. The mapping

$$\tau\colon [x \oplus Tx + G(T|Z)] \to (x + Z) \oplus (Tx + Z)$$

of $G(T)/G(T|Z)$ into $G(\hat{T}) \subset (X/Z) \oplus (X/Z)$ is both injective and surjective. Since $\hat{T}$ is bounded, $G(\hat{T})$ is closed and hence it follows from the inequalities

$$\begin{aligned}\|x \oplus Tx + G(T|Z)\| &= \inf\{\|x \oplus Tx + z \oplus Tz\| : z \in Z \cap D_T\}\\ &\geq \inf\{\|(x + z_1) \oplus (Tx + z_2)\| : z_1, z_2 \in Z\}\\ &= \|(x + Z) \oplus (Tx + Z)\|\end{aligned}$$

that τ is a topological isomorphism.

Since $\lambda \to \hat{f}_n(\lambda) \oplus \hat{T}\hat{f}_n(\lambda)$ is analytic on a neighborhood of $\overline{H}$, so is

$$\lambda \to \tau^{-1}[\hat{f}_n(\lambda) \oplus \hat{T}\hat{f}_n(\lambda)].$$

Evidently, $\tau^{-1}(\hat{f}_n \oplus \hat{T}\hat{f}_n)$ is a $G(T)/G(T|Z)$-valued function. Consequently, one can find a $G(T)$-valued analytic function $h_n \oplus Th_n$ defined on H such that

$$h_n(\lambda) \oplus Th_n(\lambda) \in \tau^{-1}[\hat{f}_n(\lambda) \oplus \hat{T}\hat{f}_n(\lambda)].$$

It follows from Lemma 2.5 that one can choose h_n such that

$$\max_{\lambda \in K} \|h_n(\lambda) \oplus Th_n(\lambda)\| \leq A \max_{\lambda \in \overline{H}} \|\tau^{-1}[\hat{f}_n(\lambda) \oplus \hat{T}\hat{f}_n(\lambda)]\| \tag{2.5}$$

for some $A > 0$. Clearly, both h_n and Th_n are analytic and $\hat{h}_n(\lambda) = \hat{f}_n(\lambda)$ on H. Thus,

$$h_n(\lambda) - f_n(\lambda) \in Z \text{ on } H.$$

In view of (2.3) and (2.4),

$$\hat{f}_n(\lambda) \oplus \hat{T}\hat{f}_n(\lambda) \to \hat{0} \quad (n \to \infty) \tag{2.6}$$

uniformly on $\overline{H}$. By (2.5) and (2.6),

$$h_n(\lambda) \to 0 \text{ and } Th_n(\lambda) \to 0 \quad (n \to \infty) \tag{2.7}$$

uniformly on K. It follows from (2.2) and (2.7) that

$$(\lambda - T)[h_n(\lambda) - f_n(\lambda)] \to 0 \quad (n \to \infty) \tag{2.8}$$

uniformly on K. Since $K \subset G$ and $\sigma(T|Z) \subset G^c$, we have $K \subset \rho(T|Z)$ and then (2.8) implies that

$$h_n(\lambda) - f_n(\lambda) \to 0 \quad (n \to \infty)$$

uniformly on K. Thus, by (2.7), $f_n(\lambda) \to 0$ uniformly on K. Since $R_0 < R$ is arbitrary, it follows that T has property (β). □

Remark. In Theorem 2.6, the inclusion in (b) can be replaced by $\sigma(\hat{T}) \subset \overline{G}$.

2.7. COROLLARY. *If T has the SDP then T has property (β).*

Proof. Let G, H be open disks with $\overline{G} \subset H$. Since $\{(\overline{G})^c, H\} \in \operatorname{cov} \sigma(T)$, there exist $Y, Z \in \operatorname{Inv} T$ such that

$$X = Y + Z, \quad \sigma(T|Y) \subset H, \quad Y \subset D_T, \quad \sigma(T|Z) \subset (\overline{G})^c \subset G^c.$$

Since $\rho(T|Y \cap Z) \supset \rho_\infty(T|Y) \supset H^c$, we have $\sigma(T|Y \cap Z) \subset H$. The coinduced operators $\hat{T} = T/Z$ and $\tilde{T} = (T|Y)/Y \cap Z$ being similar, $\hat{T}$ is bounded and

$$\sigma(\hat{T}) = \sigma(\tilde{T}) \subset \sigma(T|Y) \cup \sigma(T|Y \cap Z) \subset H.$$

Thus, T satisfies the hypotheses of Theorem 2.6 and hence T has property (β). □

Next, we quote a property which will be used in characterizations of operators with the SDP (Theorems 2.9, 2.10).

2.8. THEOREM [10]. *Let T have the SDP. Then, for every $Y \in \operatorname{Inv} T$ with $\sigma(T|Y) \neq \mathbf{C}$, the coinduced operator $\hat{T} = T/Y$ is closed. In particular, if Y is a spectral maximal space of T or a T-bounded spectral maximal space, then*

$$\sigma(\hat{T}) = \overline{\sigma(T) - \sigma(T|Y)}.$$

Moreover, if Y is a spectral maximal space and $\overline{\sigma(T) - \sigma(T|Y)}$ is compact, then $\hat{T}$ is bounded.

Some characterizations of closed operators with the SDP now follow.

2.9. THEOREM. *Given T, the following assertions are equivalent.*

(I) *T has the SDP.*

(II) (a) *T has property* (κ);

(b) *for every compact $F \in V_\infty$, $\hat{T} = T/X(T, F)$ is closed and*

$$\sigma(\hat{T}) \subset (\operatorname{Int} F)^c.$$

(III) *For every relatively compact open $G \subset \mathbf{C}$, there is $Y \in \operatorname{Inv} T$ such that*

(2.9) *$Y \subset D_T$, $\sigma(T|Y) \subset \bar{G}$, $\hat{T} = T/Y$ is closed and $\sigma(\hat{T}) \subset G^c$.*

Proof (I) $\Rightarrow$ (II). Corollary 2.7 implies (II)(a). Let $F \subset \mathbf{C}$ be compact. If $\operatorname{Int} F = \varnothing$, then $\sigma(\hat{T}) \subset \mathbf{C} = (\operatorname{Int} F)^c$. Suppose that $\operatorname{Int} F \neq \varnothing$. By [9, Theorem 1.6], we have

$$\operatorname{Int} F \cap \sigma(T) \subset \overline{\operatorname{Int} F \cap \sigma(T)} \subset \sigma[T|\Xi(T, F)].$$

Then, with the help of Theorem 2.8, we obtain

$$\sigma(\hat{T}) = \overline{\sigma(T) - \sigma[T|\Xi(T, F)]} \subset \overline{\sigma(T) - [\operatorname{Int} F \cap \sigma(T)]} \subset (\operatorname{Int} F)^c.$$

(II) $\Rightarrow$ (III). This follows for $Y = \Xi(T, \bar{G})$

(III) $\Rightarrow$ (I). First, we show that T has the SVEP. Let $f\colon D_f \to D_T$ be analytic and such that

$$(\lambda - T)f(\lambda) = 0 \quad \text{on an open } D_f \subset \mathbf{C}. \tag{2.10}$$

We may suppose that D_f is connected. Choose $G \subset \mathbf{C}$ open and relatively compact such that $\bar{G} \subset D_f$. By hypothesis, there exists $Y \in \operatorname{Inv} T$ satisfying (2.9). In view of (2.10), we have

$$(\lambda - \hat{T})\hat{f}(\lambda) = \hat{0}, \quad \lambda \in D_f. \tag{2.11}$$

Since $G \subset \rho(\hat{T})$, (2.11) implies that $\hat{f}(\lambda) = 0$ on G and hence $\hat{f}(\lambda) = \hat{0}$ on D_f, by analytic continuation. Thus, $f(\lambda) \in Y$ on D_f. It follows from (2.10) and the inclusion $\sigma(T|Y) \subset \bar{G}$ that $f(\lambda) = 0$ on $D_f - \bar{G}$ and hence $f(\lambda) = 0$ on D_f, by analytic continuation. Therefore, T has the SVEP.

Now, let $\{G_0, G_1\} \in \operatorname{cov} \sigma(T)$ with $G_0 \in V_\infty$. Put $G = G_0 \cap G_1$ and note that there is $Y \in \operatorname{Inv} T$ satisfying the conditions in (2.9). By Lemma 2.4, Y is analytically invariant under T and hence $\sigma(T|Y) \subset \sigma(T)$. Then,

$$\sigma(\hat{T}) \subset \sigma(T) \cup \sigma(T|Y) = \sigma(T)$$

and, by the last of (2.9),

$$\sigma(\hat{T}) \subset G^c \cap \sigma(T) = [G_0^c \cap \sigma(T)] \cup [G_1^c \cap \sigma(T)].$$

The spectral sets $G_0^c \cap \sigma(T)$, $G_1^c \cap \sigma(T)$ are disjoint and the former is compact. By the functional calculus, there are $\hat{Z}_0, \hat{Z}_1 \in \operatorname{Inv} \hat{T}$ satisfying conditions

$$X/Y = \hat{Z}_0 \oplus \hat{Z}_1, \hat{Z}_1 \subset D_{\hat{T}}, \sigma(\hat{T}|\hat{Z}_i) \subset G_j^c \cap \sigma(T), \quad j \neq i; i, j = 0, 1.$$

The subspaces $Z_i = \{x \in X, x \in \hat{x}, \hat{x} \in \hat{Z}_i\}$ $(i = 0, 1)$ are invariant under T and $X = Z_0 + Z_1$. Furthermore, we have

$$\sigma(T|Z_i) \subset \sigma(T|\dot{Y}) \cup \sigma(\hat{T}|\hat{Z}_i) \subset \bar{G} \cup [G_j^c \cap \sigma(T)] \subset \bar{G}_i, \quad j \neq i; i, j = 0, 1.$$

Since $Y \subset D_T$ and $\hat{Z}_1 \subset D_{\hat{T}}$, it follows from the definition of Z_1 that $Z_1 \subset D_T$. Thus, T has the SDP. □

2.10. THEOREM. *Given T, the following assertions are equivalent*:
(i) *T has the SDP.*
(ii) (a) *T has property (κ)*;
(b) *for every closed $F \in V_\infty$, $\hat{T} = T/X(T, F)$ is bounded and*

$$\sigma(\hat{T}) \subset (\operatorname{Int} F)^c;$$

(iii) *For every relatively compact open $G \subset \mathbf{C}$, there is $Y \in \operatorname{Inv} T$ such that*
(2.12) *$\hat{T} = T/Y$ is bounded, $\sigma(T|Y) \subset G^c$ and $\sigma(\hat{T}) \subset \bar{G}$.*

Proof. (i) ⇒ (ii). T has property (κ), by Corollary 2.7. We quote [9, Theorem 1.6] to write

$$\operatorname{Int} F \cap \sigma(T) \subset \overline{\operatorname{Int} F \cap \sigma(T)} \subset \sigma[T|X(T, F)]. \tag{2.13}$$

It follows from (2.13) that $\overline{\sigma(T) - \sigma[T|X(T, F)]}$ is compact and hence $\hat{T}$ is bounded by Theorem 2.8. Furthermore, with the help of (2.13) we obtain

$$\sigma(\hat{T}) = \overline{\sigma(T) - \sigma[T|X(T, F)]} \subset \overline{\sigma(T) - [\operatorname{Int} F \cap \sigma(T)]} \subset (\operatorname{Int} F)^c.$$

(ii) ⇒ (iii). This follows directly for $G = F^c$ and $Y = X(T, F)$.

(iii) ⇒ (i). T has property (κ), by Theorem 2.6. Let $\{G_0, G_1\} \in \operatorname{cov} \mathbf{C}$ with $G_0 \in V_\infty$. Select an open $H \in V_\infty$ such that $\bar{H} \subset G_0$ and $\{H, G_1\} \in$

cov **C**. The open set

$$G = (\overline{G}_0)^c \cup (H \cap G_1)$$

is relatively compact and, by hypothesis, there exists $Y \in \operatorname{Inv} T$ satisfying the conditions in (2.12). Since

$$\overline{\mathbf{C} - \overline{G}_0} \cap \overline{H \cap G_1} = \varnothing,$$

there are $\hat{Z}_0, \hat{Z}_1 \in \operatorname{Inv} \hat{T}$ producing the decomposition

$$X/Y = \hat{Z}_0 + \hat{Z}_1, \quad \sigma(\hat{T}|\hat{Z}_0) \subset \overline{H \cap G_1}, \quad \sigma(\hat{T}|\hat{Z}_1) \subset \overline{\mathbf{C} - \overline{G}_0} \subset G_0^c.$$

Define the subspaces $Z_i = \{x \in X, x \in \hat{x}, \hat{x} \in \hat{Z}_i\} \in \operatorname{Inv} T \ (i = 0, 1)$ and obtain

$$X = Z_0 + Z_1; \tag{2.14}$$

$$\sigma(T|Z_0) \subset \sigma(\hat{T}|\hat{Z}_0) \cup \sigma(T|Y) \subset \overline{G}_0; \tag{2.15}$$

$$\sigma(T|Z_1) \subset \sigma(\hat{T}|\hat{Z}_1) \cup \sigma(T|Y) \subset H^c \cup G_1^c. \tag{2.16}$$

Hence $Z_0 \subset X(T, \overline{G}_0)$, by (2.15) and it follows from (2.16) and (1.7) that

$$Z_1 \subset X(T, H^c \cup G_1^c) = \Xi(T, H^c) \oplus X(T, G_1^c) \subset \Xi(T, \overline{G}_1) + X(T, \overline{G}_0).$$

Thus, we infer from (2.14) that

$$X = X(T, \overline{G}_0) + \Xi(T, \overline{G}_1)$$

and hence T has the SDP. □

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NANJING UNIVERSITY
NANJING, THE PEOPLE'S REPUBLIC OF CHINA

TEMPLE UNIVERSITY
PHILADELPHIA, PENNSYLVANIA