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ON THE EQUILIBRIUM MEASURE AND THE CAPACITY OF CERTAIN CONDENSERS

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ABSTRACT. We prove some geometric estimates for the equilibrium measure and the capacity of certain condensers. The proofs are based on the interpretation of the equilibrium measure as the distributional Laplacian of the corresponding potential, on a formula of T. Bagby, and on a method of Beurling and Nevanlinna that involves the transport of the Riesz mass of a superharmonic function.

1. Introduction

A *condenser* is a triple (R, A, B), where R is a domain in the extended complex plane $\mathbf{C}_\infty$, whose complement $\mathbf{C}_\infty \backslash R$ is the union of the nonempty, disjoint compact sets A and B. If $R \subset \mathbf{C}$ the *capacity* of (R, A, B) is defined by

$$\operatorname{cap}(R, A, B) = \operatorname{cap} R = \inf_u \int_R |\nabla u|^2 \, dm, \tag{1.1}$$

where dm denotes the Lebesgue area measure and the infimum is taken over all continuously differentiable functions u on R with boundary values 0 on A and 1 on B.

It follows from classical results of Ahlfors and Beurling [Ahl, p. 65] that the capacity of the condenser (R, A, B) is equal to the module of the family of curves that lie in R and join A and B. In particular, $\operatorname{cap} R$ is conformally invariant. The capacity of an arbitrary condenser ($R \subset \mathbf{C}_\infty$) may be defined by means of an auxiliary Möbius transformation.

Let $\mathcal{S}(R)$ denote the family of signed Borel measures of the form $\sigma = \sigma_A - \sigma_B$, where σ_A is a unit measure on A and σ_B is a unit measure on B. The *energy* (or *transfinite modulus*) of the condenser (R, A, B) with $\infty \notin \partial R$ (cf. [Bag, p. 318]) is defined by

$$\mathcal{E}(R) = \inf_{\sigma \in \mathcal{S}(R)} I(\sigma), \tag{1.2}$$

where

$$I(\sigma) = \int_{A \cup B} \int_{A \cup B} \log \frac{1}{|z - \zeta|} \, d\sigma(z) \, d\sigma(\zeta). \tag{1.3}$$

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If $\mathcal{E}(R) < \infty$, then the infimum in (1.2) is attained uniquely for a signed measure $\tau \in \mathcal{S}(R)$ which is called the *equilibrium measure* of the condenser [Bag, Lemma 4]. The logarithmic potential u of τ, defined by

$$u(z) = \int_{A \cup B} \log \frac{1}{|z-\zeta|} \, d\tau(\zeta), \tag{1.4}$$

is called the *equilibrium potential* of the condenser (R, A, B).

By Theorems 1 and 2 of [Bag], if $\mathcal{E}(R) < \infty$ and R is regular for the Dirichlet problem, then u is continuous on $\mathbf{C}_\infty$, harmonic in R, and there exist finite constants $V_B \le 0 \le V_A$ such that

$$V_B \le u(z) \le V_A, \quad z \in \mathbf{C}_\infty, \tag{1.5}$$

$$u(z) = V_A, \quad z \in A, \tag{1.6}$$

$$u(z) = V_B, \quad z \in B, \tag{1.7}$$

$$\mathcal{E}(R) = V_A - V_B. \tag{1.8}$$

The fundamental theorem of T. Bagby [Bag, Theorem 3] asserts that

$$\mathcal{E}(R) = \frac{2\pi}{\operatorname{cap} R}. \tag{1.9}$$

Geometric estimates for the capacity of condensers have been proved by various authors including H. Grötzsch, O. Teichmüler, G. Pólya, G. Szegö, V. Wolontis, and V. N. Dubinin. Our purpose is to use Bagby's identity (1.9) to prove inequalities which do not follow from the polarization and symmetrization results presented in [PóSz, Note A], [Dub].

Our first result concerns the equilibrium measure of certain condensers. Before stating it, we need to introduce some notation. If $F \subset \mathbf{C}_\infty$ we denote by $\widehat{F}$ the set symmetric to F with respect to the real axis $\mathbf{R}$, i.e., $\widehat{F} = \{\bar{z} \colon z \in F\}$; if $\infty \in F$, it is understood that $\infty \in \widehat{F}$. For $\zeta \in \mathbf{C}$ we write $\zeta F = \{\zeta z \colon z \in F\}$. Also $F_+ = \{z \in F \colon \Im z \ge 0\}$, $F_- = \{z \in F \colon \Im z \le 0\}$ and $-F = \{-z \colon z \in F\}$.

THEOREM 1. *Let A, B be two disjoint, compact sets in $\mathbf{C}_\infty$ such that*

(a) $\widehat{A_+} \subset A_-$,
(b) $\widehat{B_-} \subset B_+$,
(c) $R := \mathbf{C}_\infty \backslash (A \cup B)$ *is a domain regular for the Dirichlet problem with* $\infty \notin \partial R$.

Let $\tau = \tau_A - \tau_B$ be the equilibrium measure of the condenser (R, A, B). Then

$$\tau_A(E) \ge \tau_A(\widehat{E}), \quad \textit{for} \quad E \subset A_+ \tag{1.10}$$

and

$$\tau_B(F) \ge \tau_B(\widehat{F}), \quad \textit{for} \quad F \subset B_-. \tag{1.11}$$

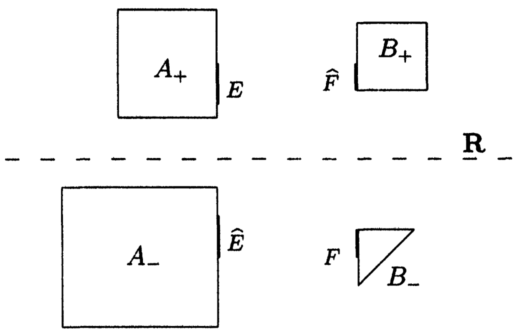

Figure 1. An illustration for Theorem 1.

Theorem 1 has an electrostatic interpretation. Assume for simplicity that A is symmetric, i.e., $\widehat{A_+} = A_-$, and that B lies in the upper half-plane, i.e., $B_- = \emptyset$. Then, since the negative charge $-\tau_B$ attracts the positive charge τ_A, the set $E \subset A_+$ has more charge than $\widehat{E}$ has. The components of A and B are assumed to be connected by very thin wires so that the charge can move from one component to the other towards equilibrium.

THEOREM 2. *Let* $0 \le \rho \le b_1 \le b_1' < b_2' \le b_2 < a_1 < a_2 < \infty$ *and consider the sets* $A = [-a_2, -a_1] \cup [a_1, a_2]$ *and* $S = \{z : |z| \le \rho\}$. *For* $\theta \in [0, \pi/2]$, *let* $I_\theta = e^{i\theta}[b_1, b_2] \cup e^{i(\theta+\pi)}[b_1', b_2']$ *and* $R_\theta = \mathbf{C}_\infty \setminus (A \cup I_\theta \cup S)$. *The function* $f(\theta) := \operatorname{cap}(R_\theta, A, S \cup I_\theta)$ *is strictly decreasing on* $[0, \pi/2]$.

For the proof of Theorem 2 we will need an inequality for harmonic measure. Let $\omega(z, K, D)$ denote the harmonic measure of a Borel set $K \subset \mathbf{C}$ with respect to a domain $D \subset \mathbf{C}_\infty$ at the point $z \in D\setminus K$; i.e., $\omega(z, K, D)$ is the Perron solution in $D\setminus K$ of the Dirichlet problem for the Laplacian with boundary values 1 on K and 0 on $\partial D \setminus K$.

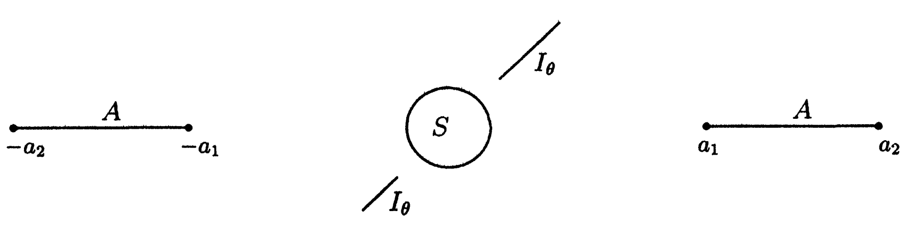

Figure 2. The condenser $(R_\theta, A, S \cup I_\theta)$ of Theorem 2.

THEOREM 3. *Let A, S, I_θ, R_θ be as in Theorem 2. For $z \in R_\theta$, define $w(z) = \omega(z, A, R_\theta)$. Then*

$$w(iy) \le w(-iy), \quad y > 0, \tag{1.12}$$

$$w(x) \le w(-x), \quad x > 0,\ x \in R_\theta, \tag{1.13}$$

$$w(\zeta) + w(\bar\zeta) \le w(-\zeta) + w(-\bar\zeta), \quad \zeta \in R_\theta,\ \Re\zeta \ge 0. \tag{1.14}$$

For the proof of Theorem 1 in Section 2 we will use the fact that the equilibrium measure of a condenser is given by the distributional Laplacian of the equilibrium potential. Theorem 3 will be proved in Section 3. It is a simple consequence of the Markov property of harmonic measure. The main tool in the proof of Theorem 2 (in Section 4) is Bagby's identity (1.9). We will also use Theorem 3 and a method of Beurling and Nevanlinna that involves the transport of the equilibrium measure.

We conclude this section with some comments on higher dimensional extensions of our results. The necessary theory for the equilibrium potential of space condensers has been developed in [AV1] and [PoSt, pp.194-195]. Theorem 1 holds in $\mathbf{R}^n$, $n \ge 3$, without any essential change: one only needs to replace the assumptions (a) and (b) with assumptions involving reflection with respect to an $(n-1)$-dimensional plane. With some obvious modifications Theorem 3 also holds in higher dimensions. Apropos of Theorem 2, we note that there are (at least) two different notions of capacity in $\mathbf{R}^n$, $n \ge 3$, the Newtonian capacity and the conformal capacity. A weaker version of Theorem 2 for conformal capacity has been proved in [Be2]. The proof of Theorem 2 can be modified to give monotonicity results for the Newtonian capacity or the α-moduli [AV2] of certain space condensers.

2. Proof of Theorem 1

We will prove (1.10). The proof of (1.11) is similar. Let u be the equilibrium potential of the condenser (R, A, B). Since u is harmonic in R and satisfies (1.5), (1.6), it is superharmonic in $D := R \cup A$. Let μ be the Riesz mass of u (that is, the measure appearing in the Riesz decomposition theorem for superharmonic functions; see [Lan, ch. 1, §5]).

LEMMA 1. $\mu = \tau_A$.

For now we assume that Lemma 1 is true and we defer its proof to the end of this section. From the proof of the Riesz theorem in [Lan] it follows that $\mu = -\Delta u/(2\pi)$, where Δu is the distributional Laplacian of u. For $r > 0$, we consider the functions $\Delta_r u$ defined in a neighborhood V of A by

$$\Delta_r u(z) = \frac{4}{r^2}\left[u(z) - \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{it})\,dt\right]. \tag{2.1}$$

Then [Lan, p. 102]

$$\Delta_r u \to 2\pi\mu, \tag{2.2}$$

in the sense [Lan, ch. 1, §1] that

$$\lim_{r\to 0}\int_V f\,\Delta_r u\,dm = 2\pi\int_V f\,d\mu$$

for all real, continuous functions f with compact support in V.

By Lemma 1 and (2.2), it suffices to prove that

$$\int_E \Delta_r u\,dm \geq \int_{\widehat{E}} \Delta_r u\,dm \tag{2.3}$$

for all sufficiently small r and all Borel sets $E \subset A_+$.

Taking into account (1.5), (1.6), and (2.1), we see that it suffices to show that

$$u(\zeta) \leq u(\bar{\zeta}) \quad \text{for} \quad \zeta \in R_+. \tag{2.4}$$

This inequality follows at once from the maximum principle applied to the function $u(\zeta) - u(\bar{\zeta})$ on the domain $R_+\setminus\widehat{A_-}$.

Therefore it remains only to prove Lemma 1.

Proof of Lemma 1. By the Riesz decomposition theorem [Lan, Theorem 1.22′],

$$u(z) = \int_A \log\frac{1}{|z-\zeta|}\,d\mu(\zeta) + h_1(z), \quad z \in \Omega, \tag{2.5}$$

where Ω is a neighborhood of A in D and h_1 is a function harmonic in Ω.

On the other hand, by definition,

$$u(z) = \int_A \log\frac{1}{|z-\zeta|}\,d\tau_A(\zeta) - \int_B \log\frac{1}{|z-\zeta|}\,d\tau_B(\zeta), \quad z \in D. \tag{2.6}$$

Since $B \cap \Omega = \emptyset$, the function $h_2(z) = \int_B \log|z-\zeta|\,d\tau_B(\zeta)$ is harmonic in Ω. Setting $\nu = \mu - \tau_A$, we infer from (2.5) and (2.6) that the potential

$$v(z) = \int_A \log\frac{1}{|z-\zeta|}\,d\nu(\zeta)$$

is equal to the harmonic function $h_2 - h_1$ in Ω, and hence [Lan, Th. 1.13] $\nu = 0$. Thus the lemma is proved.

3. Proof of Theorem 3

First we prove (1.12). Let $I_1 = e^{i\theta}[b_1, b_2]$, $I_2 = e^{i(\theta+\pi)}[b_1', b_2']$, $I_3 = (-I_1)\setminus I_2$, and $G = R_\theta\setminus I_3$. By the strong Markov property of harmonic measure (e.g., see

[Be1]) we have

$$w(iy) = \omega(iy, A, G) + \int_{I_3} \omega(iy, d\xi, G)\, \omega(\xi, A, R_\theta) \tag{3.1}$$

and similarly

$$w(-iy) = \omega(-iy, A, G) + \int_{I_3} \omega(-iy, d\xi, G)\, \omega(\xi, A, R_\theta). \tag{3.2}$$

Because of the symmetries $G = -G$ and $A = -A$,

$$\omega(iy, A, G) = \omega(-iy, A, G). \tag{3.3}$$

Also, by the Markov property again, for every interval $\Xi \subset I_3$,

$$\omega(iy, \Xi, G) = \int_{\mathbf{R}\setminus A} \omega(iy, dt, G_+)\, \omega(t, \Xi, G) \tag{3.4}$$

and

$$\omega(-iy, \Xi, G) = \omega(-iy, \Xi, G_-) + \int_{\mathbf{R}\setminus A} \omega(-iy, dt, G_-)\, \omega(t, \Xi, G). \tag{3.5}$$

By symmetry, for every interval $T \subset \mathbf{R}\setminus A$,

$$\omega(iy, T, G_+) + \omega(iy, -T, G_+) = \omega(-iy, T, G_-) + \omega(-iy, -T, G_-). \tag{3.6}$$

From (3.4), (3.5), and (3.6) we obtain

$$\omega(iy, \Xi, G) \le \omega(-iy, \Xi, G). \tag{3.7}$$

Finally, (1.12) follows from (3.1), (3.2), (3.3), and (3.7).

The proof of (1.13) is similar to the proof of (1.12). To prove (1.14) we apply the maximum principle to the function $w(\zeta) + w(\bar{\zeta}) - w(-\zeta) - w(-\bar{\zeta})$. Then (1.14) follows from (1.13).

4. Proof of Theorem 2

Let $0 \le \phi < \psi \le \pi/2$. By (1.4), it suffices to show that

$$\mathcal{E}(R_\phi) > \mathcal{E}(R_\psi). \tag{4.1}$$

Let $\tau^\psi = \tau_1^\psi - \tau_2^\psi$ be the equilibrium measure of $(R_\psi, A, S \cup I_\psi)$, and let u be the corresponding equilibrium potential. As in the proof of Theorem 1, we have $\tau^\psi = -\Delta u/(2\pi)$ in the sense of distributions. Because of properties (1.5)–(1.7) of u, we have

$$\omega(z, A, R_\theta) = \frac{u(z) - V_2}{V_1 - V_2} \tag{4.2}$$

for all $z \in R_\theta$ and some constants $V_1 \leq 0 \leq V_2$.

Theorem 3 and (4.2) imply

$$u(\zeta) + u(\bar{\zeta}) \leq u(-\zeta) + u(-\bar{\zeta}), \quad \zeta \in R_\theta,\ \Re\zeta \geq 0. \tag{4.3}$$

Using (4.3) and the method of proof of Theorem 1 we obtain

$$\tau_1^\psi(-E) \leq \tau_1^\psi(E), \quad E \subset A \cap \{z\colon \Re z \geq 0\}. \tag{4.4}$$

We define the signed measure $\tilde{\tau} = \tilde{\tau}_1 - \tilde{\tau}_2$ on $A \cup S \cup I_\phi$ as follows:

$$\tilde{\tau}_1(K) = \tau_1^\psi(K) \quad \text{for} \quad K \subset A, \tag{4.5}$$

$$\tilde{\tau}_2(L) = \tau_2^\psi(L) \quad \text{for} \quad L \subset S, \tag{4.6}$$

$$\tilde{\tau}_2(L) = \tau_2^\psi(e^{i(\psi-\phi)}L) \quad \text{for} \quad L \subset I_\phi. \tag{4.7}$$

A similar transport of the Riesz mass occurs in the proof of the Beurling-Nevanlinna projection theorem for harmonic measure; see [Nev, ch. IV, §5].

We will prove that

$$I(\tau^\psi) \geq I(\tilde{\tau}). \tag{4.8}$$

Note that $I(\tilde{\tau}) \geq \mathcal{E}(R_\phi)$ by the definition of $\mathcal{E}(R_\phi)$. Hence (4.8) implies (4.1), and therefore it remains only to prove (4.8).

By the definition of $I(\tau^\psi)$ we have

$$\begin{aligned} I(\tau^\psi) &= \int_{A \cup I_\psi \cup S} \int_{A \cup I_\psi \cup S} \log \frac{1}{|z-\zeta|}\, d\tau^\psi(z)\, d\tau^\psi(\zeta) \\ &= \int_{A \cup S} \int_{A \cup S} \log \frac{1}{|z-\zeta|}\, d\tau^\psi(z)\, d\tau^\psi(\zeta) \\ &\quad + \int_{I_\psi \cup S} \int_{I_\psi \cup S} \log \frac{1}{|z-\zeta|}\, d\tau_2^\psi(z)\, d\tau_2^\psi(\zeta) \\ &\quad - 2\int_A \int_{I_\psi} \log \frac{1}{|z-\zeta|}\, d\tau_1^\psi(z)\, d\tau_2^\psi(\zeta) \\ &=: J_1^\psi + J_2^\psi - 2J_3^\psi. \end{aligned}$$

Similarly, for $I(\tilde{\tau})$ we have

$$
\begin{aligned}
I(\tilde{\tau}) &= \int_{A\cup S}\int_{A\cup S} \log\frac{1}{|z-\zeta|}\,d\tilde{\tau}(z)\,d\tilde{\tau}(\zeta) \\
&\quad + \int_{I_\phi\cup S}\int_{I_\phi\cup S} \log\frac{1}{|z-\zeta|}\,d\tilde{\tau}_2(z)\,d\tilde{\tau}_2(\zeta) \\
&\quad - 2\int_A\int_{I_\phi} \log\frac{1}{|z-\zeta|}\,d\tilde{\tau}_1(z)\,d\tilde{\tau}_2(\zeta) \\
&=: \tilde{J}_1 + \tilde{J}_2 - 2\tilde{J}_3.
\end{aligned}
$$

By (4.5) and (4.6) we have $J_1^\psi = \tilde{J}_1$ and $J_2^\psi = \tilde{J}_2$. Therefore, to prove (4.8), it suffices to show that

$$
J_3^\psi \le \tilde{J}_3. \tag{4.9}
$$

We use (4.4) to write

$$
\begin{aligned}
J_3^\psi &= \int_{[a_1,a_2]}\int_{I_\psi}\left[\log\frac{1}{|-x-\zeta|} + \log\frac{1}{|x-\zeta|}\right] d\tau_1^\psi(x)\,d\tau_2^\psi(\zeta) \\
&\quad + \int_{[a_1,a_2]}\int_{I_\psi} \log\frac{1}{|x-\zeta|}\,d\sigma(x)\,d\tau_2^\psi(\zeta),
\end{aligned} \tag{4.10}
$$

where σ is a *positive* measure on $[a_1, a_2]$.

Also, by (4.7) and a change of variables,

$$
\begin{aligned}
\tilde{J}_3 &= \int_{[a_1,a_2]}\int_{I_\psi}\left[\log\frac{1}{|-x-\zeta e^{i(\phi-\psi)}|} + \log\frac{1}{|x-\zeta e^{i(\phi-\psi)}|}\right] d\tau_1^\psi(x)\,d\tau_2^\psi(\zeta) \\
&\quad + \int_{[a_1,a_2]}\int_{I_\psi} \log\frac{1}{|x-\zeta e^{i(\phi-\psi)}|}\,d\sigma(x)\,d\tau_2^\psi(\zeta),
\end{aligned} \tag{4.11}
$$

Now (4.9) follows from (4.10), (4.11), and the following lemma whose elementary proof is omitted.

LEMMA 2. *Let $0 < t < x$. For $\theta \in [0, \pi/2]$. Define*

$$
\begin{aligned}
g_1(\theta) &= \log\frac{1}{|-x-te^{i\theta}|} + \log\frac{1}{|x-te^{i\theta}|}, \\
g_2(\theta) &= \log\frac{1}{|x-te^{i\theta}|}.
\end{aligned}
$$

The functions g_1, g_2 are both strictly decreasing.

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