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ON MEROMORPHIC SOLUTIONS OF A LINEAR DIFFERENTIAL EQUATION WITH DOUBLY PERIODIC COEFFICIENTS

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ABSTRACT. In this paper we treat a linear differential equation with doubly periodic coefficients. We examine value distribution properties of meromorphic solutions. Some examples are presented to illustrate our results.

1. Introduction

Consider an equation of the form

$$(\mathrm{E}_n) \qquad w^{(n)}+p_{n-1}(z)w^{(n-1)}+\cdots+p_1(z)w'+p_0(z)w=0 \qquad ('=d/dz,\ n\in\mathbf{N}),$$

where the coefficients $p_0(z)$ ($\not\equiv 0$), $p_1(z),\ldots,p_{n-1}(z)$ are doubly periodic meromorphic functions with the common periods ω, ω' ($\mathrm{Im}(\omega'/\omega)\neq 0$). Denote by $\mathcal{P}_k$ ($0\le k\le n-1$) the set of all the poles of $p_k(z)$, and put

$$\mathcal{P}=\bigcup_{k=0}^{n-1}\mathcal{P}_k\subset\mathbf{C}.$$

Throughout this paper we suppose that every point $a\in\mathcal{P}$ is a regular singular point of (E_n) with the properties:

(P1) *All the characteristic exponents $q(a,j)$ ($j=1,\ldots,n$) are integers.*

(P2) *There exist linearly independent solutions expressible in the form*

$$\varphi_{a,j}(z)=(z-a)^{q(a,j)}h_{a,j}(z), \qquad j=1,\ldots,n,$$

where $h_{a,j}(z)$ is analytic around $z=a$ and satisfies $h_{a,j}(a)=1$.

Let $w=\psi(z)$ be an arbitrary solution of (E_n) analytic around the point $z=z_0\in\mathbf{C}-\mathcal{P}$. For every curve $C(z_0,z_1)\subset\mathbf{C}-\mathcal{P}$ starting from z_0 and ending at z_1, the solution $\psi(z)$ is continued analytically along $C(z_0,z_1)$. If the endpoint $z=z_1$ is near a point $a\in\mathcal{P}$, then, in the disk $|z-z_1|<|a-z_1|$, the analytic continuation of $\psi(z)$ is expressible in the form $\sum_{j=1}^{n}c_j\varphi_{a,j}(z)$ for some $c_j\in\mathbf{C}$, which implies that $\psi(z)$

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is meromorphic at $z = a$. Therefore, *all the solutions of* (E_n) *are meromorphic in the whole complex plane.* (For basic facts concerning linear differential equations and singular points of them; see [4].) In particular, every $a \in \mathcal{P}$ of (E_2) with $p_1(z) \equiv 0$ possesses the properties (P1) and (P2) if the coefficient $p_0(z)$ is a doubly periodic meromorphic function such that, around every pole $z = a \in \mathcal{P}_0 = \mathcal{P}$,

$$p_0(z) = (z-a)^{-2} \sum_{l=0}^{\infty} b_l (z-a)^l, \tag{1.1}$$

where the coefficients b_l ($l \geq 0$) have the following properties:

(a) $b_0 = -q(a)(q(a)+1)$, *where* $q(a)$ *is a positive integer.*

(b) The set of b_l ($l = 1, \ldots, 2q(a)+1$) *satisfies*

$$D(a) = \begin{vmatrix} \mu_1 & 0 & \cdots & & 0 & b_1 \\ b_1 & \mu_2 & \ddots & (0) & \vdots & b_2 \\ b_2 & b_1 & \ddots & \ddots & \vdots & b_3 \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ b_{2q(a)-1} & b_{2q(a)-2} & \cdots & b_1 & \mu_{2q(a)} & b_{2q(a)} \\ b_{2q(a)} & b_{2q(a)-1} & \cdots & b_2 & b_1 & b_{2q(a)+1} \end{vmatrix} = 0, \tag{1.2}$$

$$\mu_l = l^2 - (2q(a)+1)l \quad (1 \leq l \leq 2q(a))$$

(see [6], [7]). This is regarded as a generalization of Lamé's equation

$$w'' - (q(q+1)\wp(z) + B)w = 0, \quad q \in \mathbf{N}, \quad B \in \mathbf{C}, \tag{1.3}$$

where $\wp(z)$ is Weierstrass' $\wp$-function (see [8]). (Examples of equation (E_n) other than (1.3) are given in Section 4. All the solutions of them are meromorphic in $\mathbf{C}$.) In general, for linear differential equations with meromorphic coefficients, meromorphic solutions are not studied so much (see [2], [7]).

The purpose of this paper is to clarify value distribution properties of meromorphic solutions of equation (E_n). Throughout this paper, we use basic facts in the value distribution theory and the standard notation such as $m(r, f)$, $N(r, f)$, $T(r, f)$, $N_1(r, f) = N(r, f) - \overline{N}(r, f)$ (see [5], [6]); in addition, for functions $g(r)$ and $h(r)$ ($r \geq r_0$), we write $g(r) \asymp h(r)$, if $g(r) = O(h(r))$ and $h(r) = O(g(r))$ simultaneously hold as $r \to \infty$.

Let $\phi(z)$ be an arbitrary meromorphic solution of (E_n). Our results are stated as follows.

THEOREM 1.1. $m(r, \phi) = O(r)$, $T(r, \phi) = O(r^2)$.

THEOREM 1.2. *For every* $\alpha \in \mathbf{C} - \{0\}$, $m(r, 1/(\phi - \alpha)) = O(\log r)$, *and* $m(r, 1/\phi) = O(r)$.

THEOREM 1.3. *We have*

$$m(r,\phi)+m(r,1/\phi)+N(r,1/\phi')+N_1(r,\phi)=2T(r,\phi)+O(\log r). \tag{1.4}$$

THEOREM 1.4. *If there exists a point* $a_0 \in \mathcal{P}$ *such that*

$$P_0=\lim_{z\to a_0}(z-a_0)^n p_0(z)\neq 0, \tag{1.5}$$

then

$$T(r,\phi)\asymp r^2, \tag{1.6}$$

$$N(r,\phi)\asymp r^2, \tag{1.7}$$

and for every $\alpha\in\mathbf{C}$,

$$N(r,1/(\phi-\alpha))\asymp r^2. \tag{1.8}$$

Remark 1. Estimates (1.6) and (1.8) imply that the growth order and the exponent of convergence of zeros are finite:

$$\sigma(\phi)=\limsup_{r\to\infty}\frac{\log T(r,\phi)}{\log r}=2,\qquad \lambda(\phi)=\limsup_{r\to\infty}\frac{\log N(r,1/\phi)}{\log r}=2.$$

These properties are quite different from those of equations with simply periodic entire coefficients (cf. [1], [3]).

Remark 2. Theorem 1.4 is applicable to (E_2) whose coefficients satisfy $p_1(z)\equiv 0$ and (1.1) with (a), (b), especially to the equations of Examples 4.1 through 4.3, and also to that of Example 4.4.

Remark 3. If every $a\in\mathcal{P}$ satisfies $\lim_{z\to a}(z-a)^n p_0(z)=0$, then in some cases there exists a solution ϕ_0 such that $T(r,\phi_0)\asymp r$, and in other cases every solution ϕ satisfies $T(r,\phi)\asymp r^2$ (cf. Examples 4.5 and 4.6).

2. Preliminaries

We use the following notation in this section.

(1) For a matrix $A=(a_{ij})\in M_n(\mathbf{C})$ $(1\le i\le n,\ 1\le j\le n)$, we write $\|A\|=\max_{1\le i\le n}(\sum_{j=1}^n|a_{ij}|)$. Then for $A, B\in M_n(\mathbf{C})$, $\|AB\|\le\|A\|\|B\|$.

(2) For a set S, $|S|$ denotes the cardinal number of it.

Let $\psi_j(z)$ $(j=1,\dots,n)$ be arbitrary linearly independent solutions of (E_n). Consider the row vector function $\Psi(z)=(\psi_1,\dots,\psi_n)$. By $M_1, M_2\in GL(n,\mathbf{C})$ we denote Floquet matrices given by

$$\Psi(z+\omega)=\Psi(z)M_1,\quad \Psi(z+\omega')=\Psi(z)M_2. \tag{2.1}$$

Since every entry of $\Psi(z)$ is meromorphic,

$$M_1 M_2 - M_2 M_1 = 0. \tag{2.2}$$

LEMMA 2.1. $m(r, \psi_j) = O(r)$ $(j = 1, \dots, n)$.

Proof. By (2.1) and (2.2), for every pair of integers $(\mu, \nu) \in \mathbf{Z}^2$,

$$\Psi(z + \mu\omega + \nu\omega') = \Psi(z) M_1^{\mu} M_2^{\nu}. \tag{2.3}$$

We put $\Delta = \{\sigma\omega + \tau\omega' \mid 0 \le \sigma < 1, 0 \le \tau < 1\}$, $\Delta(\mu, \nu) = \{z + \mu\omega + \nu\omega' \mid z \in \Delta\}$. Cover the circle $\Gamma_r = \{z \mid |z| = r\}$ with the smallest number of these sets; $\Gamma_r \subset \bigcup_{(\mu,\nu)\in I(r)} \Delta(\mu, \nu)$ with $I(r) = \{(\mu, \nu) \in \mathbf{Z}^2 \mid \Delta(\mu, \nu) \cap \Gamma_r \ne \emptyset\}$. Then

(i) $(\mu, \nu) \in I(r)$ implies $|\mu| + |\nu| = O(r)$;

(ii) $|I(r)| = O(r)$.

Since each parallelogram $\Delta(\mu, \nu)$ is congruent with Δ,

(iii) for every $(\mu, \nu) \in I(r)$, [*the length of the arc* $\Gamma_r \cap \Delta(\mu, \nu)$] $\le |\omega| + |\omega'| + O(1/r) = O(1)$.

In Δ, the solutions $\psi_j(z)$ $(j = 1, \dots, n)$ are written in the form

$$\psi_j(z) = \eta_j(z) \prod_{\sigma=1}^{\kappa(j)} (z - a_{j,\sigma})^{-1},$$

where $a_{j,\sigma}$ $(1 \le \sigma \le \kappa(j))$ are the poles of ψ_j in Δ, each counted according to its multiplicity, and $\eta_j(z)$ $(j = 1, \dots, n)$ are functions analytic and bounded in Δ. Suppose that $\Gamma_r \cap \Delta(\mu, \nu) \ne \emptyset$. Every point on $\Gamma_r \cap \Delta(\mu, \nu)$ is written as $s = re^{i\theta} = z + \mu\omega + \nu\omega'$ $(z \in \Delta)$. Then, using (2.3), we have

$$\overset{+}{\log} |\psi_j(re^{i\theta})| \le \overset{+}{\log} \left(\sum_{j=1}^{n} |\psi_j(z)| \, \|M_1^{\mu} M_2^{\nu}\| \right) \le \rho(z) + \gamma_0(|\mu| + |\nu|), \tag{2.4}$$

$$\rho(z) = \sum_{j=1}^{n} \left(\overset{+}{\log} |\eta_j(z)| + \sum_{\sigma=1}^{\kappa(j)} \overset{+}{\log} \frac{1}{|z - a_{j,\sigma}|} \right) + \log n,$$

where $\overset{+}{\log} x = \max\{\log x, 0\}$, $\gamma_0 = \max\{\log(\|M_k\| + \|M_k^{-1}\|) \mid k = 1, 2\}$. Putting $\Theta(r, \mu, \nu) = \{\theta \mid re^{i\theta} \in \Gamma_r \cap \Delta(\mu, \nu),\ 0 \le \theta < 2\pi\}$, $\Gamma_r^0(\mu, \nu, \Delta) = \{z = s - \mu\omega - \nu\omega' \mid s \in \Gamma_r \cap \Delta(\mu, \nu)\} \subset \Delta$, and using (i), (iii), (2.4), we have

$$\begin{aligned} r\int_{\Theta(r,\mu,\nu)} \overset{+}{\log} |\psi_j(re^{i\theta})| \, d\theta &\le \int_{\Gamma_r^0(\mu,\nu,\Delta)} \rho(z)\, |dz| + \gamma_0(|\mu|+|\nu|) \int_{\Gamma_r \cap \Delta(\mu,\nu)} |ds| \\ &\le K \left(1 + r\int_{\Gamma_r \cap \Delta(\mu,\nu)} |ds| \right), \end{aligned}$$

where K is a positive constant independent of r and (μ, ν). This inequality and (ii) yield

$$m(r, \psi_j) = \frac{1}{2\pi} \sum_{(\mu,\nu)\in I(r)} \int_{\Theta(r,\mu,\nu)} \overset{+}{\log} |\psi_j(re^{i\theta})|\, d\theta = \frac{K}{2\pi} \left(\frac{|I(r)|}{r} + \int_{\Gamma_r} |ds| \right) = O(r).$$

Thus the lemma is verified. □

LEMMA 2.2. *Let $\varpi(z)$ be an arbitrary doubly periodic meromorphic function with periods ω, ω'. Then, $m(r, \varpi) = O(1)$, $N(r, \varpi) = C_\varpi r^2 + O(r)$, where C_ϖ is a positive constant.*

Proof. For every $(\mu, \nu) \in \mathbf{Z}^2$, $\varpi(z + \mu\omega + \nu\omega') = \varpi(z)$. From this relation instead of (2.3), we derive $m(r, \varpi) = O(1)$, by the same argument as in the proof of Lemma 2.1. Recall $\Delta(\mu, \nu)$ of the proof of Lemma 2.1, and write $D_r = \{z \mid |z| < r\}$. We have $D_{r_-} \subset \bigcup_{(\mu,\nu)\in K_-(r)} \Delta(\mu, \nu) \subset D_r \subset \bigcup_{(\mu,\nu)\in K_+(r)} \Delta(\mu, \nu) \subset D_{r_+}$ with $K_-(r) = \{(\mu, \nu) \mid \Delta(\mu, \nu) \subset D_r\}$, $K_+(r) = \{(\mu, \nu) \mid \Delta(\mu, \nu) \cap D_r \neq \emptyset\}$, $r_\pm = r \pm (|\omega| + |\omega'|)$. Hence $|K_\pm(r)| = (\pi/s_0)r^2 + O(r)$, where s_0 denotes the area of Δ. This implies $N(r, \varpi) = C_\varpi r^2 + O(r)$, which completes the proof. □

LEMMA 2.3 [6, Corollary 2.3.4]. *Let f be an arbitrary meromorphic function satisfying $\sigma(f) < \infty$. Then, for each positive integer j, we have $m(r, f^{(j)}/f) = O(\log r)$.*

3. Proofs of theorems

3.1. *Proof of Theorem* 1.1. By Lemma 2.1, for an arbitrary solution $\phi(z)$ of (E_n),

$$m(r, \phi) = O(r). \tag{3.1}$$

Each pole of $\phi(z)$ is a pole of some coefficient $p_k(z)$ $(0 \le k \le n-1)$. By the double periodicity of $p_k(z)$, $Q_0 = \max\{|q(a, j)| \mid a\text{: } \textit{regular singular point}, j = 1, \dots, n\}$ (cf. (P2)) is bounded. By Lemma 2.2, we have

$$N(r, \phi) \le Q_0 \sum_{k=0}^{n-1} N(r, p_k) = O(r^2). \tag{3.2}$$

Thus Theorem 1.1 is verified.

3.2. *Proof of Theorem* 1.2. For every $\alpha \in \mathbf{C} - \{0\}$, the function $\chi(z) = \phi(z) - \alpha$ satisfies $-\alpha/\chi = 1 + (1/p_0)(p_1\chi'/\chi + \cdots + p_{n-1}\chi^{(n-1)}/\chi + \chi^{(n)}/\chi)$. By Lemmas 2.2, 2.3 and Theorem 1.1, we have

$$m(r, 1/(\phi - \alpha)) = O\left(\log r + \sum_{k=1}^{n-1} m(r, p_k) + m(r, 1/p_0) \right) = O(\log r).$$

In addition to $\phi(z)$, take other solutions $\phi_2(z), \dots, \phi_n(z)$ of (E_n) in such a way that $\phi, \phi_2, \dots, \phi_n$ are linearly independent. Note that the Wronskian determinant $\Phi(z) = W(\phi, \phi_2, \dots, \phi_n)$ is a meromorphic function and that $v = 1/\Phi(z)$ satisfies $v' - p_{n-1}(z)v = 0$. By Lemma 2.1, $m(r, 1/\Phi) = O(r)$. From Theorem 1.1, Lemma 2.3 and the relation

$$\frac{1}{\phi} = \frac{1}{\Phi(z)} \begin{vmatrix} 1 & \phi_2 & \cdots & \phi_n \\ \phi'/\phi & \phi_2' & \cdots & \phi_n' \\ \vdots & \vdots & & \vdots \\ \phi^{(n-1)}/\phi & \phi_2^{(n-1)} & \cdots & \phi_n^{(n-1)} \end{vmatrix},$$

it follows that $m(r, 1/\phi) = O(r)$. Thus the proof is complete.

3.3. *Proof of Theorem* 1.3. Observe that $\phi/\phi' = -(1/p_0)(p_1 + p_2\phi''/\phi' + \cdots + p_{n-1}\phi^{(n-1)}/\phi' + \phi^{(n)}/\phi')$. By Lemmas 2.2, 2.3 and Theorem 1.1, we have

$$m(r, \phi/\phi') \le m(r, 1/p_0) + \sum_{k=2}^{n} \left(m(r, p_{k-1}) + m(r, \phi^{(k)}/\phi')\right) = O(\log r). \tag{3.3}$$

Since $N(r, 1/\phi') + N_1(r, \phi) = N(r, \phi') + N_1(r, \phi) + m(r, \phi') - m(r, 1/\phi') + O(1) = 2T(r, \phi) - 2m(r, \phi) + m(r, \phi') - m(r, 1/\phi') + O(1)$, the left-hand side of (1.4) is written in the form

$$2T(r, \phi) + \sigma(r) + O(1) \tag{3.4}$$

with $\sigma(r) = -m(r, \phi) + m(r, 1/\phi) + m(r, \phi') - m(r, 1/\phi')$. Then, $\sigma(r) \le 2m(r, \phi'/\phi) = O(\log r)$, and by (3.3), $-\sigma(r) \le 2m(r, \phi/\phi') = O(\log r)$. Hence $\sigma(r) = O(\log r)$. Substitution of this estimate into (3.4) yields (1.4).

3.4. *Proof of Theorem* 1.4. By (P2), around $z = a_0$, the solution $\phi(z)$ is written in the form

$$\phi(z) = \sum_{j=1}^{n} c_j^0 \varphi_{a_0, j}(z) = (z - a_0)^{q_*} h_0(z), \qquad q_* \in \mathbf{Z}, \qquad c_j^0 \in \mathbf{C},$$

where $h_0(z)$ is analytic at $z = a_0$ and satisfies $h_0(a_0) \ne 0$. Note that the exponent q_* is a root of the equation

$$\sum\nolimits_{k=1}^{n} P_k \lambda(\lambda - 1) \cdots (\lambda - k + 1) + P_0 = 0,$$

where

$$P_n = 1, \qquad P_k = \lim_{z \to a_0} (z - a_0)^{n-k} p_k(z) \quad (0 \le k \le n - 1).$$

By (1.5), we have $q_* \in \mathbf{Z} - \{0\}$. This implies that, at $z = a_0$, the solution $\phi(z)$ has either a zero or a pole. By this fact and the double periodicity of $p_0(z)$, we derive

$$2T(r, \phi) \ge N(r, \phi) + N(r, 1/\phi) + O(1) \ge (1/i_0)N(r, p_0) + O(r), \tag{3.5}$$

where i_0 is the sum of the multiplicities of all the poles of $p_0(z)$ in a period parallelogram. From (3.5), Lemma 2.2 and Theorem 1.1, it follows that $T(r, \phi) \asymp r^2$. Combining this estimate with Theorems 1.1, 1.2, we immediately obtain (1.7) and (1.8). Thus the proof is complete.

4. Examples

Let $\wp(z)$ be Weierstrass' $\wp$-function with periods ω, ω', $\mathrm{Im}(\omega'/\omega) \neq 0$, and let $\zeta(z)$ be Weierstrass' ζ-function such that $-\zeta'(z) = \wp(z)$ (see [8]). We write $\omega_1 = \omega/2$, $\omega_3 = \omega'/2$, $\omega_2 = \omega_1 + \omega_3$, $e_\nu = \wp(\omega_\nu)$, $\eta_\nu = \zeta(\omega_\nu)$ $(\nu = 1, 2, 3)$. Then, around $z = 0$,

$$\wp(z) = z^{-2} + \sum_{l=1}^{\infty} a_l z^{2l}, \tag{4.1}$$

$$\wp(z + \omega_\nu) = \sum_{l=0}^{\infty} \alpha_l^{(\nu)} z^{2l}, \quad \alpha_0^{(\nu)} = e_\nu, \tag{4.2}$$

$$\zeta(z + \omega_\nu) - \zeta(z) = -z^{-1} + \eta_\nu - \sum_{l=0}^{\infty} \beta_l^{(\nu)} z^{2l+1}, \quad \beta_0^{(\nu)} = e_\nu. \tag{4.3}$$

In what follows we call a regular singular point satisfying (P1), (P2) a *non-branching regular singularity*. In Examples 4.1, 4.2, 4.3 below, we consider equation (E_2) with $p_1(z) \equiv 0$, $p_0(z) = -p(z)$.

Example 4.1. For arbitrary $q_0, q_\nu \in \mathbf{N}$ $(\nu = 1, 2, 3)$, and for arbitrary $B \in \mathbf{C}$, put

$$p(z) = q_0(q_0 + 1)\wp(z) + \sum_{\nu=1}^{3} q_\nu(q_\nu + 1)\wp(z + \omega_\nu) + B.$$

By (4.1) and (4.2), the poles $z = 0$, $-\omega_\nu$ $(\nu = 1, 2, 3)$ are non-branching regular singularities with the characteristic exponents $\{-q_0, q_0 + 1\}$, $\{-q_\nu, q_\nu + 1\}$ respectively.

Example 4.2. For $q_0, \mu \in \mathbf{N}$ such that $q_0 < \mu$, and for arbitrary $B \in \mathbf{C}$, put

$$p(z) = q_0(q_0 + 1)\wp(z) + K\wp(z/2) + B, \quad K = \frac{1}{4}(\mu(\mu + 1) - q_0(q_0 + 1)),$$

which has the periods $2\omega = 4\omega_1$, $2\omega' = 4\omega_3$. By (4.1) and (4.2), the pole $z = 0$ is a non-branching regular singularity with the characteristic exponents $-\mu$, $\mu + 1$, and the poles $z = \omega, \omega', \omega + \omega'$ are ones with the characteristic exponents $-q_0, q_0 + 1$.

Example 4.3. For arbitrary $q_0 \in \mathbf{N}$, and for arbitrary $B \in \mathbf{C}$, put

$$p(z) = q_0(q_0+1)(\wp(z)+\wp(z+\omega_1)) + \gamma(\zeta(z+\omega_1)-\zeta(z)) + B.$$

If γ is an arbitrary root of a certain algebraic equation of degree $2q_0$ depending on B, a_l, $\alpha_l^{(1)}$, $\beta_l^{(1)}$ $(0 \le l \le q_0 - 1)$, then $z = 0$ and $z = -\omega_1$ are non-branching regular singularities. For instance, consider the case where $q_0 = 1$. It is easy to see that $p(z)$ has the periods ω, ω'. By (4.1), (4.2) and (4.3), near $z = 0$ we have

$$p(z) = 2z^{-2} - \gamma z^{-1} + (2e_1 + \gamma\eta_1 + B) - \gamma e_1 z + O(z^2),$$

and near $z = -\omega_1$,

$$p(z) = 2(z+\omega_1)^{-2} + \gamma(z+\omega_1)^{-1} + (2e_1+\gamma\eta_1+B) + \gamma e_1(z+\omega_1) + O((z+\omega_1)^2).$$

Then $D(0) = -D(-\omega_1) = \gamma(\gamma^2 - 4\eta_1\gamma - 4(e_1 + B))$ (cf. (1.2)). Hence, if γ satisfies $\gamma^2 - 4\eta_1\gamma - 4(e_1 + B) = 0$, then $z = 0$ and $z = -\omega_1$ are non-branching regular singularities.

Example 4.4. Let $p(z)$ be one of the doubly periodic functions given above, and let w_1, w_2 be linearly independent solutions of (E_2) with $p_1(z) \equiv 0$, $p_0(z) = -p(z)$. Then every pole of $p(z)$ is a non-branching regular singularity of (E_3) with $p_2(z) \equiv 0$, $p_1(z) = -4p(z)$, $p_0(z) = -2p'(z)$, which has linearly independent meromorphic solutions w_1^2, w_1w_2, w_2^2.

It is quite easy to construct equation (E_2) such that there exists no point $a \in \mathcal{P}$ satisfying (1.5).

Example 4.5. For an arbitrary nontrivial doubly periodic function $\pi(z)$, equation (E_2) with

$$p_0(z) = (\pi(z) - \pi'(z))'/(\pi(z) - \pi'(z)), \quad p_1(z) = -1 - p_0(z)$$

has linearly independent solutions $\phi_0 = e^z$, $\phi_1 = \pi(z)$. Clearly every point $a \in \mathcal{P}$ is a non-branching regular singularity and is a simple pole of $p_0(z)$.

Example 4.6. The functions $\wp(z)$ and $\wp(z+\omega_1)$ are linearly independent solutions of equation (E_2) with

$$p_1(z) = -W(\wp(z), \wp(z+\omega_1))'/W(\wp(z), \wp(z+\omega_1)), \quad W(f,g) = fg' - f'g,$$

$$p_0(z) = -\frac{\wp''(z)}{\wp(z)} - p_1(z)\frac{\wp'(z)}{\wp(z)} = -\frac{\wp''(z+\omega_1)}{\wp(z+\omega_1)} - p_1(z)\frac{\wp'(z+\omega_1)}{\wp(z+\omega_1)}.$$

Now take the periods ω, ω' of $\wp(z)$ so that $\wp(\omega_1) \ne 0$, $\wp(\omega_1/2) \ne 0$, $\wp(\omega_1/2 + \omega_3) \ne 0$. Then $\wp(z)$ and $\wp(z+\omega_1)$ do not simultaneously vanish. Hence every pole or every zero of these solutions belongs to $\mathcal{P}$ and is at most a simple pole of $p_0(z)$. Suppose that there exists a point $a \in \mathcal{P}$ other than a pole or a zero of these solutions. Then, $W(\wp(a), \wp(a+\omega_1)) = 0$, so that there exists a solution of the form $\wp(z) - c\wp(z+\omega_1) = O((z-a)^2)$ $(c \neq 0)$ around $z = a$. Since $\wp(z)$ and $\wp(z+\omega_1)$ satisfy $w'' = 6w^2 - g_2/2$, we have $\wp(z) \equiv c\wp(z+\omega_1)$, which is a contradiction. Therefore every point $a \in \mathcal{P}$ is a non-branching regular singularity and is at most a simple pole of $p_0(z)$.

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