ON THE MEASURABILITY OF STOCHASTIC PROCESSES

BY

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Let $x(t, \gamma)$ be a stochastic process over a probability space, Γ, which takes its values in I, the unit interval, and has a parameter set T, the unit interval provided with a regular Borel measure ν. Let Ω be the space of all I-valued functions over T. With the product topology, Ω is a compact Hausdorff space. The Kolmogoroff representation theorem gives a unique Borel measure on Ω having the property that, considering finitely many t's, the distributions generated by $x(t, \gamma)$ and $\xi(t, \omega) = \omega(t)$ are the same. By Kakutani's version of a stochastic process we mean that version $\xi(t, \omega)$ in which the probability space Ω is taken to be all functions from T to I provided with the product topology; the Borel field is taken to be all Borel sets with the extended Kolmogoroff measure, Pr; and $\xi(t, \omega) = \omega(t)$. For more details of this construction and some of its properties see [4].

DEFINITION. By a *measurable modification* of the stochastic process, $x(t, \gamma)$, we mean a measurable stochastic process, $y(t, \gamma)$ such that $x(t, \cdot) = y(t, \cdot)$ almost everywhere with respect to the measure of the probability space for each t.

Thus a measurable modification of $\xi(t, \omega) = \omega(t)$ is a function $\tilde{f}(t, \omega)$ such that $\tilde{f}(t, \cdot) = \xi(t, \cdot)$ a.e. [Pr] for every t. As a stochastic process, $\tilde{f}$ and f give the same finite distribution. The existence of a measurable modification for one version of a stochastic process implies the existence of a measurable modification for every version.

The problem, first raised by Doob in [2], as to whether Kakutani's version of a process $x(t, \omega)$ is jointly measurable when, say, $x(s, \cdot)$ converges to $x(t, \cdot)$ in measure as s goes to t, or, more generally, whether the existence of a measurable modification implies the measurability of Kakutani's version, was raised again in [4]. Conditions under which a process has a measurable modification are known (see [1]). The purpose of this paper is to prove the following

THEOREM. *If $x(t, \gamma)$ has a measurable modification, then $\xi(t, \omega)$, the Kakutani version, is measurable in the completed product measure.*

In the proof of the theorem we will use the following lemmas.

LEMMA 1. *If T is an uncountable parameter set, $\bar{\Omega} = \prod_{t \in T} I$, λ is a probability measure on the Borel sets of $\bar{\Omega}$, and $\tilde{f}$ is a measurable function on $\bar{\Omega}$, then*

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$\tilde{f}$ is equal a.e. [λ] to a function f which depends on only countably many coordinates.

Proof. By Lusin's theorem (see [3]), for every n there exists a compact set $C_n \subset \bar{\Omega}$ such that $\lambda(C_n) > 1 - 1/n$ and $\tilde{f}$, restricted to C_n, is continuous. We may assume $C_{n+1} \supset C_n$. By the Tietze extension theorem $\tilde{f}$ restricted to C_n may be extended to a continuous function g_n on $\bar{\Omega}$. Clearly $\{g_n\}$ converges a.e. [λ] to $\tilde{f}$. Each g_n depends on only countably many coordinates, and hence the sequence of functions $\{g_n\}$ depends on a countable set $S \subset T$ of coordinates. The function f defined by $f(\omega) = \lim g_n(\omega)$ if $\{g_n(\omega)\}$ converges and $f(\omega) = 0$ otherwise is a function depending on only countably many coordinates which is equal a.e. [λ] to $\tilde{f}(\omega)$.

Returning to a stochastic process which takes its values in the unit interval, we express $T \times \Omega$ as $\prod_{t \in T \cup \{a\}} I$ since T is the unit interval and let $\lambda = \nu \times \mathrm{Pr}$. Lemma 1 implies that the measurable modification, $\tilde{f}(t, \omega)$, of $\xi(t, \omega)$ is equal a.e. [λ] to a function $f(t, \omega)$ which depends on only countably many coordinates, $\{a\} \cup S$. Thus, except in a set T_0 of ν-measure 0, we have $f(t, \cdot) = \xi(t, \cdot)$ a.e. [Pr]. We may assume that the countable parameter set $S \subset T_1 = T - T_0$. For if $s \in S$, then we can set $f(s, \omega) = \xi(s, \omega) = \omega(s)$, and the resulting function still depends on only countably many coordinates. For the rest of this paper $f(t, \omega)$ refers to this function; let $\Omega' = \prod_{t \in T_1} I$ have the quotient measure Pr' coming from Pr.

LEMMA 2. *For every Borel measurable function $\tilde{f}$ on $T_1 \times \Omega'$ and every n there exists a measurable function f_n on $T_1 \times \Omega'$ which depends on only countably many coordinates such that on a subset F_n of $T_1 \times \Omega'$ having measure greater than $1 - 1/n$, $f_n(t, \omega') = f(t, \omega')$ a.e. $[\nu \times \mathrm{Pr}']$. Further if*

$$C_t^n = \{\omega' : f_n(t, \omega') = \omega'(t)\},$$

then $\mu_n(C_t^n) = 1$ where μ_n is the Kolmogoroff measure on Ω' determined by $f_n(t, \omega')$.

Proof. The function f depends on only countably many coordinates $\{a\} \cup S$ and hence may be considered as a measurable function on $T \times \Omega_S$. By Lusin's theorem there exists a compact subset $\bar{F}_n$ of $T \times \Omega_S$ such that the restriction of f to $\bar{F}_n$ is continuous in both variables together and for which $\nu \times \mathrm{Pr}_S(\bar{F}_n) \geqq 1 - 1/n$. By the Tietze extension theorem f restricted to $\bar{F}_n$ may be extended to a function $\tilde{f}_n$ continuous on all of $T \times \Omega_S$. Extend $\tilde{f}_n$ to a function $\bar{f}_n$ on $T \times \Omega'$ by $\bar{f}_n(t, \omega') = \tilde{f}(t, \pi\omega')$ where π is the projection of $\Omega' \to \Omega_S$. Let $\tilde{F}_n = p^{-1}\bar{F}_n$ where $p : T \times \Omega' \to T \times \Omega_S$ is the natural map. Then $F_n = \tilde{F}_n \cap (T_1 \times \Omega')$. Define f_n on $T_1 \times \Omega'$ by

$$f_n(t, \omega') = \bar{f}_n(t, \omega') \quad \text{if} \quad t \in T_1 - S,$$

$$f_n(t, \omega') = \omega'(t) \quad \text{if} \quad t \in S.$$

This is the required function.

Considering $f_n(t, \omega_S)$ as a stochastic process over Ω_S with Pr_S as measure, we can apply Theorem 1 of [4] to give a measure on Ω' which we call μ_n. We will now show that $\mu_n(C_t^n) = 1$ for each $t \in T_1$. This is clear for $t \in S$ since then $C_t^n = \Omega'$.

Suppose $t \in T_1 - S$. Note that since $f_n(t, \cdot)$ is continuous for each $t \in T_1 - S$ and depends only on S coordinates, C_t^n is a compact $(S \cup \{t\})$-cylinder. We will look at the finite joint distributions of the stochastic process $f_n(t, \omega_S)$ from a different point of view in order to see that $\mu_n(C_t^n) = 1$. For each finite set $K \subset T_1$, define $C_K^n = \bigcap_{t \in S} C_t^n$. For any K-cylinder B, let $\mu_K^n(B) = \Pr_S(\pi(B \cap C_K))$. The μ_K^n are consistent, that is, if $K_1 \subset K_2$ and B is a K_1-cylinder, then $\mu_{K_1}^n(B) = \mu_{K_2}^n(B)$. Further, the μ_K^n are merely the finite joint distributions of $f_n(t, \omega_S)$. Indeed if $B = \{\omega' : \omega'(t) < \lambda_t, t \in K\}$, then

$$\begin{aligned}\mu_K^n(B) &= \Pr_S(\pi(B \cap C_K^n)\\ &= \Pr_S(\pi(\{\omega' : \omega'(t) < \lambda_t, t \in K\} \cap \{\omega' : \omega'(t) = f_n(t, \omega'), t \in K\}))\\ &= \Pr_S(\pi\{\omega' : \omega'(t) = f_n(t, \omega') < \lambda_t, t \in K\})\\ &= \Pr_S(\{\omega_S : f_n(t, \omega_S) < \lambda_t, t \in K\}),\end{aligned}$$

which is the value given by the joint distribution of the random variables indexed by K. To get the last equality, note that if $\pi\omega'$ is in the set on the left, it is clearly in the set on the right, while if ω_S is in the set on the right, one can construct an ω' such that $\pi\omega' = \omega_S$ and $\pi\omega'$ is in the set on the left. Applying Theorem 1 of [4] to the $\{\mu_K^n\}$ we get the measure μ_n on Ω'. From this it is clear that $\mu_n(C_t^n) = 1$ for if B is any K-cylinder containing C_t^n, then

$$\begin{aligned}\mu_K^n(B) = \Pr_S(\pi(B \cap C_K^n) &\geqq \Pr_S(\pi(C_K^n \cap C_t^n))\\ &= \Pr_S(\pi(\textstyle\bigcap_{t \in K \cup \{t\}} C_t^n)) = \Pr_S(\Omega_S) = 1.\end{aligned}$$

(Given an arbitrary $\omega_S \in \Omega_S$ it is simple to construct an $\omega' \in \bigcap_{t \in K \cup \{t\}} C_t^n$ such that $\pi\omega' = \omega_S$.) This completes the proof of Lemma 2.

Proof of the theorem. Let $\tilde{f}(t, \omega)$ be the measurable modification of the Kakutani canonical version, and $f(t, \omega)$ the related function which depends on only countably many coordinates. It is sufficient to show that $\xi(t, \omega') = \omega'(t)$, defined on $T_1 \times \Omega'$, is measurable where the notation is that of Lemma 2.

We can assume that the sets F_n of Lemma 2 are increasing. Also, $D_n = \bigcap_{t \in T_1} C_t^n$ is a compact set having μ_n-measure one. Indeed, if $\mu_n(D_n) < 1$, then there is an open set $U \supset D_n$ such that $\mu_n(U) < 1$. Then $C_t^n - U$ is compact, and the collection has the finite intersection property. Hence the complete intersection is not empty. This is a contradiction.

Now on $T_1 \times D_n$, $f_n(t, \omega') = \omega'(t)$ everywhere. Hence on $F_i \cap (T_1 \times D_n)$ this equation holds. Since $\mu_n(D_n) = 1$,

$$\nu \times \mu_n(F_i \cap (T_1 \times D_n)) = \nu \times \mu_n(F_i).$$

But F_i is an $(\{a\} \cup S)$-cylinder; therefore $(\nu \times \mu_n)(F_i) = (\nu \times \mathrm{Pr}')(F_i)$ for all $n \geqq i$. Let $A_i = \{(t, \omega'); (t, \omega') \in F_i \cap (T_1 \times \Omega') \text{ and } f(t, \omega') \neq \omega'(t)\}$. Since $f(t, \omega')$ is equal to $f_n(t, \omega')$ if $(t, \omega') \in F_i$ and $n \geqq i$, $A_i \subset F_i - (T_1 \times D_n)$ for all $n \geqq i$. Hence $(\nu \times \mu_n)(A_i) = 0$ for all $n \geqq i$. If this would imply $(\nu \times \mathrm{Pr}')(A_i) = 0$, then, on F_i,

$$f(t, \omega') = f_n(t, \omega') = \omega'(t) \quad \text{a.e. } [\nu \times \mathrm{Pr}'],$$

and this implies the measurability of $\xi(t, \omega)$.

We will use a Fubini theorem to prove $(\nu \times \mathrm{Pr}')(A_i) = 0$. Let

$$T_2 = \{t \in T_1 ; \mu_n((A_i)_t) = 0; n \geqq i; \quad \text{and} \quad \chi(F_n)_t \to 1 \ [\mathrm{Pr}]\}.$$

Clearly $\nu(T_2) = 1$. The following lemma and Fubini's theorem complete the proof. We let $F'_n = F_n \cup \bigcup_{t \in S} \{t\} \times \Omega'$.

Lemma 3. *Let F'_n be a monotonically increasing family of measurable subsets of $T_2 \times \Omega'$, and let f_n be a sequence of functions such that*

(1) $\chi(F'_n)_{t_0} \to 1$ *a.e.* $[\mathrm{Pr}']$;

(2) $f_n/F'_n = f/F'_n$ *everywhere*;

(3) $(F'_n)_t = \Omega'$ *for all* $t \in S$.

Then if μ_n is the measure on Ω' induced by Pr' *with $f_n(t, \omega')$ and A is a* $(\{t_0\} \cup S)$-*cylinder in Ω' such that $\mu_n(A) \to 0$, then* $\mathrm{Pr}'(A) = 0$.

Proof. Let $\varepsilon > 0$ be given. Choose n large enough so that $\mathrm{Pr}'((F'_n)_{t_0}) > 1 - \varepsilon/2$ and $\mu_n(A) < \varepsilon/4$. Let $B \supset A$ be a finite or countable K-cylinder such that $\mu_n(B) < \varepsilon/2$. We can suppose that $t_0 \in K$ and $K \subset \{t_0\} \cup S$. Let $E = \bigcap_{t \in K} (F'_n)_t = (F'_n)_{t_0}$. Let

$$C = \{\omega; f(t, \omega) \in \pi_t B\} \quad \text{and} \quad D = \{\omega; f_n(t, \omega) \in \pi_t B\},$$

where π_t is the projection into the t^{th} coordinate. Then $\mu_n(B) = \mathrm{Pr}'(D)$, and $\mathrm{Pr}'(B) = \mathrm{Pr}'(C)$. Also note that $C \cap E = D \cap E$. Hence

$$\begin{aligned}\mathrm{Pr}'(C) &= \mathrm{Pr}'(C \cap E) + \mathrm{Pr}'(C \cap E') \\ &= \mathrm{Pr}'(D \cap E) + \mathrm{Pr}'(C \cap E') \leqq \varepsilon/2 + \varepsilon/2.\end{aligned}$$

Hence $\mathrm{Pr}'(A) \leqq \varepsilon$, but ε is arbitrary.

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