APPLICATION OF THE DOMAIN OF ACTION METHOD TO $|xy| \leqq 1$

BY

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1. Introduction

In his doctoral thesis, M. Rahman developed what he called the domain of action method in order to answer the question about the closest packing of certain star domains. Before we discuss this method, we need the following definitions:

Let $\mathcal{S}$ be a star domain in the ordinary affine plane, symmetric about O. A set of points $\mathcal{P}$ is said to provide a *packing for* $\mathcal{S}$ if the domains $\{\mathcal{S} + P\}$, where $P \epsilon \mathcal{P}$, have the property that no domain $(\mathcal{S} + P_0)$ contains the center of another in its interior. We shall also say that $\mathcal{P}$ is an *admissible* point set for $\mathcal{S}$.

As definition of the *density* of a point set $\mathcal{P}$ we accept the definition given in [3, p. 5] which is as follows:

Consider the square $|x| < t$, $|y| < t$. Let $A(t)$ denote the number of points of a set $\mathcal{P}$ in the square; then the density of $\mathcal{P}$ (denoted $\mathfrak{D}(\mathcal{P})$) is defined as $\limsup_{t\to\infty} A(t)/4t^2$.

From the definition it follows that for any 2-dimensional lattice $\mathcal{L}$ the density $\mathfrak{D}(\mathcal{L})$ is just the reciprocal of its mesh.

A *norm-distance* is a real-valued function, [1, p. 103], $N(X) = N(OX)$, defined on the plane, such that $N(X)$ is

(1) nonnegative; i.e., $N(X) \geqq 0$;

(2) continuous;

(3) homogeneous; i.e., $N(tX) = |t| N(X)$, where t is any real number.

A convex distance function or *Minkowski distance*, M, is a norm-distance with the additional properties:

(1) $M(PQ) = 0$ implies $P = Q$.

(2) $M(PQ) \leqq M(PR) + M(RQ)$.

Let $\mathcal{P}$ be a point set in the plane and M be a Minkowski distance. The *domain of action* [2, p. 16], $D(P) = D(P, M, \mathcal{P})$ of a point P, relative to M and $\mathcal{P}$, is the set of all points X in the plane for which

$$M(PX) \leqq M(QX) \qquad \text{where } Q \epsilon \mathcal{P}, \quad Q \neq P,$$

when this set is the closure of the set of all points in the plane which are closer to P than any other point of $\mathcal{P}$.

We must note here, however, that the closure of the set of points X such that $M(OX) < M(PX)$ may not always be the same as the set of points X such that $M(OX) \leqq M(PX)$. For there may be a point Y with $M(OY) = M(PY)$ such that for all X in a whole neighborhood of Y, $M(OX) = M(PX)$.

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This will occur, for example, when $M(X) = M(OX)$ is a Minkowski distance defined by a centrally symmetric convex body having a straight line segment contained in its boundary. For, let the ends of the segment be A_1 and A_2. We can assume that $A_1 A_2$ is parallel to the x-axis for we can bring this about by a rotation if necessary. Using the center as origin, let $A_1 = (a_1, b)$; $A_2 = (a_2, b)$ where $b > 0$. (See Figure 1.) For $X = (x, y)$ in the region $G_1 \cup (-G_1)$ where G_1 is defined by the two inequalities

$$bx - a_1 y < 0, \qquad bx - a_2 y > 0$$

and $(-G_1)$ is the reflection of G_1 in O, we have $M(OX) = (1/b)|y|$.

Let $P = (p, 0)$ be a point on the x-axis; that is, OP is parallel to $A_1 A_2$. Then for $X = (x, y), M(PX) = (1/b)|y|$ when X is in the region $G_2 \cup (-G_2)$ where G_2 is defined by

$$bx - a_1 y - bp < 0,$$

$$bx - a_2 y - bp > 0,$$

and $(-G_2)$ is the reflection of G_2 in P.

$G_1 \cap G_2$ is defined by

$$bx - a_2 y - bp < 0, \qquad bx - a_1 y > 0.$$

Therefore, for any point X in $[G_1 \cap G_2] \cup [(-G_1) \cap (-G_2)]$ we have $M(OX) = M(PX)$.

We see then that when the line through two points P' and Q' is parallel to a straight segment of the boundary of the convex body which determines the distance M, the union of the sets

$$(1) \qquad \{X \mid M(P'X) \leqq M(Q'X)\} \quad \text{and} \quad \{X \mid M(Q'X) \leqq M(P'X)\}$$

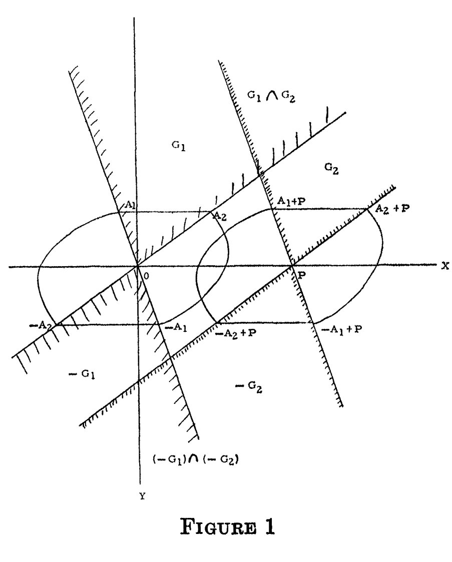

FIGURE 1

covers the plane but the sets have interior points in common. Also, the closures of the sets

$$(2) \qquad \{X \mid M(P'X) < M(Q'X)\} \quad \text{and} \quad \{X \mid M(Q'X) < M(P'X)\}$$

have no interior points in common but the union of the closures is not the whole plane.

Since neither of these situations is desirable, when such cases arise an adjustment will be needed in the definition. In all other cases, with either of the definitions above, the domain of action of a point R with respect to a single point S and the domain of action of S with respect to R together cover the plane, while $D(R) \cap D(S)$ contains no interior points. The adjustment we make should preserve these properties in the exceptional case discussed above; for example, in (1) where $D(R) \cap D(S)$ contains interior points, by apportioning the common region equally in some consistent manner.

Let $\mid D(P) \mid$ denote the area of $D(P)$. If m is the greatest lower bound of $\mid D(P) \mid$ for $P \in \mathcal{P}$, then it follows that the density $\mathfrak{D}(\mathcal{P})$ of the point set $\mathcal{P}$ is less than or equal to $1/m$.

2. Domain of action for $\mid xy \mid \leqq 1$

Consider the domain $\mathcal{S} : \mid xy \mid \leqq 1$. The norm-distance N determined by $\mathcal{S}$ is $N(OP)$ or

$$N(P) = \mid \sqrt{\mid xy \mid} \mid, \qquad \text{where } P = (x, y).$$

In general, for $P_1 = (x_1, y_1)$ and $P = (x_2, y_2)$,

$$N(P_1 P_2) = \mid \sqrt{\mid (x_2 - x_1)(y_2 - y_1) \mid} \mid.$$

Let the Minkowski distance M be defined by a maximal convex polygon[1] inscribed in $\mathcal{S}$ say, $\mid x \mid + \mid y \mid = 2$. Then

$$M(P) = \tfrac{1}{2}(\mid x \mid + \mid y \mid).$$

In general $M(P_1 P_2) = \frac{1}{2}(\mid x_2 - x_1 \mid + \mid y_2 - y_1 \mid)$. Note that $M(P) \geqq N(P)$ and that this will always be the case if N and M are defined in such a way. Therefore, if $\mathcal{P}$ is an $\mathcal{S}$-admissible point set, then

$$M(PQ) \geqq N(PQ) \geqq 1$$

for any two distinct points P and Q in $\mathcal{P}$.

Let O be an arbitrary point of $\mathcal{P}$ and be taken as origin. Then $D(O) = \bigcap_P D(O, M, P)$ for $P \in \mathcal{P}$, $P \neq O$, where $D(O, M, P)$ is determined as follows:

Let $P = (x_1, y_1)$.

[1] It is possible that in this case other inscribed convex bodies would give the desired results since $\mathcal{S}$ has a high degree of symmetry. However, in general, it seems that some care should be exercised in selecting the inscribed convex body which will be used to define the M-distance if a sharp estimate is to be obtained. It is easy to construct examples of point sets for which two distinct M-distances give different minimum values of $D(O)$.

I. If $x_1 \geqq y_1 > 0$, $D(O, M, P)$ is defined by the following inequalities:

$$y \leqq 0, \qquad x \leqq (x_1 + y_1)/2;$$
$$0 \leqq y \leqq y_1, \qquad x + y \leqq (x_1 + y_1)/2;$$
$$y_1 \leqq y, \qquad x \leqq (x_1 - y_1)/2. \qquad \text{(See Figure 2a.)}$$

In the notation of the discussion above for $x_1 = y_1$, this is equivalent to assigning the region $G_1 \cap G_2$ to O and $(-G_1) \cap (-G_2)$ to P.

II. If $y_1 > x_1 > 0$, $D(O, M, P)$ is determined by the following inequalities:

$$x \leqq 0, \qquad x \leqq (x_1 + y_1)/2;$$
$$0 \leqq x \leqq x_1, \qquad x + y \leqq (x_1 + y_1)/2;$$
$$x_1 \leqq x, \qquad y \leqq (y_1 - x_1)/2. \qquad \text{(See Figure 2b.)}$$

These definitions are for a point in the first quadrant. For P in any other quadrant, the definitions are analogous. Figure 3 is a typical domain of action. Note that $D(O, M, P)$ so defined is a continuous function of $P = (x, y)$ when (i) $|x| \geqq |y|$ and also when (ii) $|y| > |x|$. But $D(O, M, P)$ is not continuous if we allow equality in (ii), for the domain of action undergoes a sudden change in shape as P moves onto the bisector from a region $|y| > |x|$ or off the bisector into such a region. (See Figure 4.) This discontinuity will not present any problem, however, since if a point is on a quadrant bisector we can always approach it from the region where $|x| \geqq |y|$.

Let the four quadrants of the plane be denoted by Q_1, Q_2, Q_3, Q_4. As above $|D(O)|$ will be used to denote the area of the domain of action $D(O)$. Since for all points $P \in \mathcal{P}$, $P \neq O$, $M(OP) \geqq 1$, the minimum of $|D(O) \cap Q_i|$ is $\frac{1}{2}$.

Draw lines $y = x$ and $y = -x$ bisecting Q_1 and Q_2. Each half-quadrant

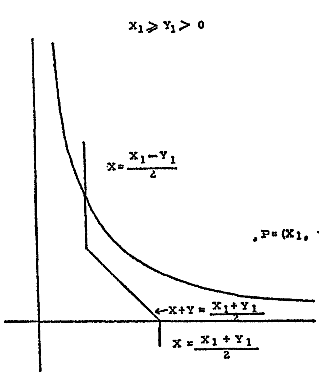

FIGURE 2a

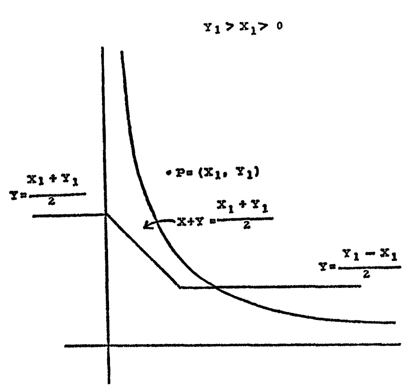

FIGURE 2b

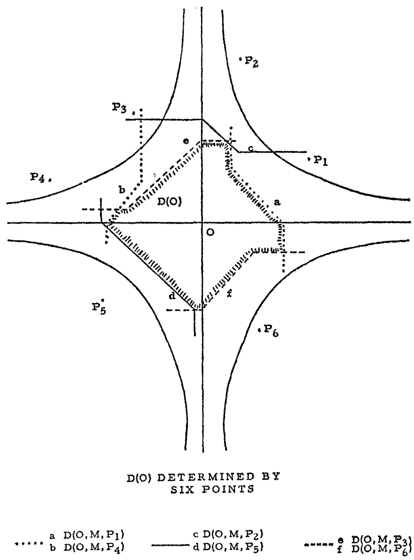

FIGURE 3

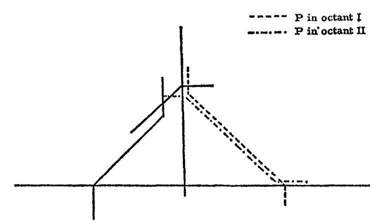

FIGURE 4

will be called an *octant* and will be denoted by I, III, $\cdots$, VIII in counterclockwise order from the positive x-axis.

3. Statement of the problem

Norman Smith [3] proved that a critical lattice gives the closest packing of $\mathcal{S} : |xy| \leqq 1$, or to put it another way, if $\mathcal{P}$ is any $\mathcal{S}$-admissible point set, $\mathfrak{D}(\mathcal{P}) \leqq 1/\sqrt{5}$. M. Rahman indicates that one might be able to get as sharp a result by using the domain of action method. The question is: if $\mathcal{P}$ is an $\mathcal{S}$-admissible point set, is $|D(P)| \geqq \sqrt{5}$ for any $P \in \mathcal{P}$? The question is answered in the affirmative and in the process methods are employed which we hope will be useful in applying the domain of action method to other stars.

We prove then the following

THEOREM. *If O is any point of an admissible point set for the region $\mathcal{S} : |xy| \leqq 1$, then $|D(O)| \geqq \sqrt{5}$.*

We will show that for any arrangement of points P_i of $\mathcal{P}$ in say, Q_1 and Q_2 such that the P_i influence the domain of action of O, the part of $D(O)$ in $Q_1 \cup Q_2$ has area greater than or equal to $\frac{1}{2}\sqrt{5}$. (Note that we do not assume that the part of $D(O)$ in $Q_1 \cup Q_2$ is completely determined by the points P_i but only that they influence the domain of action.) By the symmetry of $\mathcal{S}$, the same will hold for the contribution in $Q_3 \cup Q_4$ and hence the domain of action of O will always be greater than or equal to $\sqrt{5}$.

If there is to be an exception, i.e., if we can find a point O such that

$|D(O)| < \sqrt{5}$, then certainly the part of $D(O)$ in either $Q_1 \cup Q_2$ or $Q_3 \cup Q_4$ is less than $\frac{1}{2}\sqrt{5}$. We can call this pair of quadrants Q_1 and Q_2, for we could put the points determining the part of $D(O)$ in question in these quadrants by reflection in the x-axis if necessary. Observe that it follows from the definition that if the set of points $\{P_i\}$, $i = 1, 2, \cdots, n$ which determines $D(O) = \bigcap_i D(O, M, P_i)$ is reflected in either coordinate axis into points $\{P_i'\}$, then $\overline{D(O)} = \bigcap_i D(O, M, P_i')$ is the reflection of $D(O)$ in the axis of reflection.

We will prove the theorem by means of the following lemmas.

LEMMA 1. *If there are two points in a given octant*[2] *of* Q_1 *influencing* $D(O)$, *the area of* $D(O) \cap (Q_1 \cup Q_2)$ *is greater than or equal to* $\frac{1}{2}\sqrt{5}$.

LEMMA 2. *If there is one point in each octant of* Q_1 *and* Q_2, *then*

$$|D(O) \cap (Q_1 \cup Q_2)| \geqq \tfrac{1}{2}\sqrt{5}.$$

LEMMA 3. *If there are points in any three octants of* $Q_1 \cup Q_2$, *then*

$$|D(O) \cap (Q_1 \cup Q_2)| \geqq \tfrac{1}{2}\sqrt{5}.$$

LEMMA 4. *If there is only one point in each of* Q_1 *and* Q_1, *then*

$$|D(O) \cap (Q_1 \cup Q_2)| \geqq \tfrac{1}{2}\sqrt{5}.$$

A few remarks will show that the arrangements of points in these four lemmas are the only ones which must be considered. If there were more than two points in an octant, it will be clear from the proof in Lemma 1 that additional points influencing $D(O)$ would make this area still larger. Also, there must be at least one point of $\mathcal{P}$ in each quadrant, for if some quadrant contained no point of $\mathcal{P}$, the part of $D(O)$ in it would contain at least a square of area 1. Allowing only the minimum area of $\frac{1}{2}$ for either adjacent quadrant gives $|D(O) \cap (Q_1 \cup Q_2)| > \frac{1}{2}\sqrt{5}$.

The theorem then follows easily from these four lemmas, since the points influencing $D(O)$ in each half-plane will fall in one of the above categories; hence, the whole domain of action will be greater than or equal to $\sqrt{5}$.

4. Proof of Lemma 1

Suppose that there are two points of $\mathcal{P}$ in a given octant which influence $D(O)$. We may assume that either

(a) two points (P' and P_1) are in I,

or

(b) two points (P'' and P_2) are in II since the points could be placed in these octants by a reflection, if necessary.

We will assume that there is at most one point of $\mathcal{P}$ in each of the remaining

[2] Where a specific quadrant is mentioned, it is understood that the points could be put there by a reflection if necessary; hence, there is no loss of generality.

octants. It will be clear from the following discussion that if there were more, $|D(O) \cap (Q_1 \cup Q_2)|$ would be made larger.

Note that if two octants are reflections in one of the coordinate axes (e.g., II and III), at least one of them must contain a point of $\mathcal{P}$ for otherwise $D(O)$ would contain an infinite strip.

(a) Let $P' = (x', y')$, $P_i = (x_1, y_1)$ be points of I which contribute to $D(O)$ such that $M(OP') \leqq M(OP_1)$. Then $x' - y' > x_1 - y_1$, for if not, the point P_1 does not influence $D(O)$ since $D(O, M, P')$ would be strictly contained in $D(O, M, P_1)$.

Let K be the intersection of $x + y = (x_1 + y_1)/2$ and $x = (x' - y')/2$. Then the ordinate of K, $y_k = ((x_1 + y_1) - (x' - y'))/2$.

$P_2 \in \mathrm{II}$ or $P_3 \in \mathrm{III}$ can influence $D(O)$ only if the polygonal line which bounds $D(O, M, P_j)$, $j = 2$ or 3, intersects the line $x + y = (x_1 + y_1)/2$ above K. (See Figure 5.) In the case of P_2, this means that

$$y = (y_2 - x_2)/2 > y_k$$

which also implies that

$$x + y = (y_2 + x_2)/2 > y_k .$$

For P_3 we must have

$$y = (|x_3| + |y_3|)/2 > y_k .$$

Hence, P_2 or P_3 can influence $D(O)$ only if

$$M(OP_j) = (|x_j| + |y_j|)/2 > y_k \qquad (j = 2, 3).$$

Further note that we can assume that if there are three points in Q_1 influencing $D(O)$, the octant which contains two of them must also contain the point in Q_1 closest to O. This being the case, we may assume

$$(*) \qquad 1 \leqq M(OP') < \tfrac{1}{2}\sqrt{5},$$

for if $M(OP') \geqq \frac{1}{2}\sqrt{5}$, then

$$|D(O) \cap Q_1| \geqq (-1 + \sqrt{5})/2, \qquad |D(O) \cap (Q_2)| \geqq \tfrac{1}{2},$$

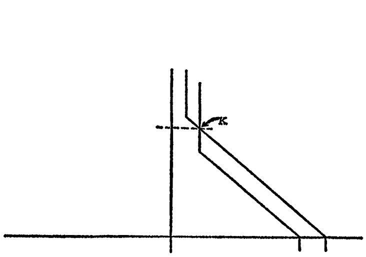

FIGURE 5

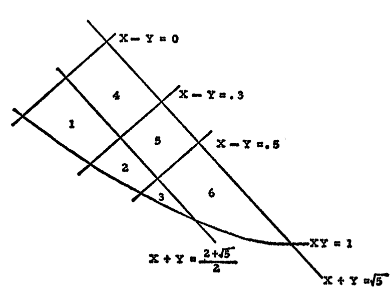

FIGURE 6

and

$$|D(O) \cap (Q_1 \cup Q_2)| \geqq \tfrac{1}{2}\sqrt{5},$$

and Lemma 1 would be proved.

Hence, P' is restricted to the closed region in Q_1 (actually in I) bounded by the lines $xy = 1$, $x - y = 0$, and $x + y = \sqrt{5}$. In the proof, P' is further restricted to the sectors indicated in Figure 6 which shows the region enlarged.

If there are points in the respective octants, we use the following notation: P' denotes the point in I with smallest M-distance from O, restricted as in $(*)$. P_1 denotes the second point in I; hence $M(OP') < M(OP_1)$.

P_2 denotes a point in II; P_3 a point in III; P_4 or $P_{4'}$ a point in IV.

LEMMA 1.1. *If P' and P_1 are two points of $\mathcal{P}$ in* I, *using the above notation, and if both influence $D(O)$, then $M(OP_1) \geqq 2$ or $x_1 + y_1 \geqq 4$.*

As noted above, the hypothesis implies that

$$x' - y' > x_1 - y_1,$$

and also

$$y_1 - y' > x_1 - x' > 0. \tag{1}$$

Lemma 1.1 is proved by indirect argument. Suppose $M(OP_1) < 2$ or

$$x_1 + y_1 < 4. \tag{2}$$

But $x' + y' \geqq 2$ which implies

$$-x' - y' \leqq -2. \tag{3}$$

Combining (2) and (3) we obtain

$$(x_1 - x') + (y_1 - y') < 2. \tag{4}$$

$N(P', P_1) \geqq 1$ and (1) imply

$$(x_1 - x')(y_1 - y') \geqq 1$$

or

$$y_1 - y' \geqq 1/(x_1 - x') > 0. \tag{5}$$

Then (4) and (5) imply

$$2 > (x_1 - x') + (y_1 - y') \geqq (x_1 - x') + 1/(x_1 - x') \geqq 2$$

which is a contradiction.

Hence, $M(OP_1) \geqq 2$.

Further, the lemma implies that if $x' + y' = m$ and P' is restricted as in $(*)$ then $x_1 + y_1 \geqq m + 2$ or $M(OP_1) \geqq M(OP') + 1$.

Let P' vary successively in the sectors of the region in Figure 6. The above considerations enable us to determine lower bounds on the M-distances of P_1, P_2, and P_3 from P' when P' is restricted to a given sector. We determine $M(OP_4)$ and $M(OP_4')$ as follows:

Let P' be in the i^{th} sector. Determine a point P_{4i} in the region labeled R_2 (see Figure 7) such that

$$N(P_{4i}Q_4) = 1 = N(OP_{4i}).$$

Then every point P in the i^{th} sector satisfies the condition $N(PP_{4i}) \leqq 1$. Therefore, as P' varies in the i^{th} sector, P_4 in R_2 satisfies

$$M(OP_4) \geqq M(OP_{4i}).$$

Further, the absolute value of the difference of the coordinates of a point in R_2 is smallest at P_{4i}.

Determine a point P_{4i}' in the region labeled R_1 such that

$$N(P_{4i}'Q_2) = 1$$

and P_{4i}' is on the line $x + y = 0$. Then, as P' varies in the i^{th} sector, P_4' in R_1 satisfies

$$M(OP_4') \geqq M(OP_{4i}').$$

These statements can be verified by determining the position of the i^{th} sector with respect to the region

$$H_i : |(x - x_{ji})(y - y_{ji})| \leqq 1.$$

For each j, $j = 2, 3, 4$, the i^{th} sector is in H_j as can be proved by considering the tangent to $|(x - x_{ji})(y - y_{ji})| = 1$ at the point Q_j. (See Figure 8.)

Using these facts, we obtained lower estimates of $|D(O) \cap (Q_1 \cup Q_2)|$ as follows.

The general shape of the domain of action remains unchanged throughout this discussion. (See Figure 9a.) For P' in the i^{th} sector ($i = 1, \cdots, 6$) the areas of the shaded parts were estimated using appropriate lower estimates

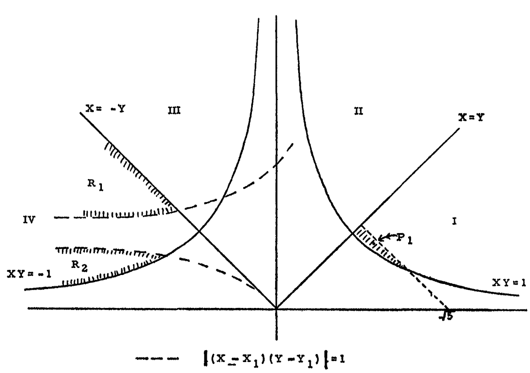

FIGURE 7

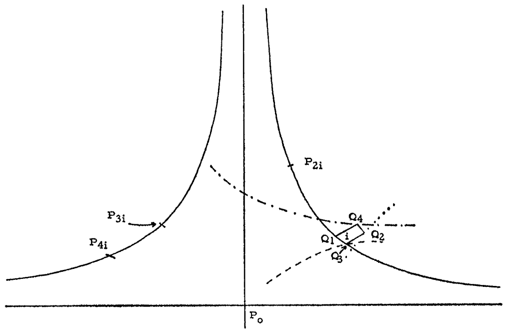

FIGURE 8

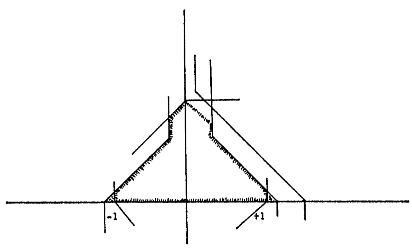

FIGURE 9a

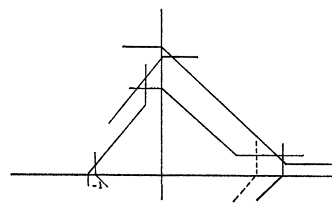

FIGURE 9b

of $M(OP_j)$, $j = 1, 2, 3, 4$, and $|D(O) \cap (Q_1 \cup Q_2)|$ was found to be greater than $\frac{1}{2}\sqrt{5}$. (See Tables 1 and 2.)

(b) Let P'' and P_2 be points in II which influence $D(O)$. Observe that $D(O) \cap (Q_1 \cup Q_2)$ is in general not completely determined by the points of $\mathcal{P}$ in $(Q_1 \cup Q_2)$. In the preceding argument we estimated $|D(O) \cap (Q_1 \cup Q_2)|$ by means of the shaded polygon in Figure 9a. We could explain the lines $x = \pm 1$ by saying that we allowed for the effect of points in Q_3 and Q_4 on $D(O) \cap (Q_1 \cup Q_2)$ or that we allowed for the *cut-off* from Q_3 and Q_4.

If two points, P'' and P_2, in II influence $D(O)$, then, by the argument used above in part (a), a point in Q_4 which influences $D(O)$ must have M-distance from O greater than or equal to the value of $M(OP_3)$ in part (a).

In the present case it is clear that the cut-off from Q_4 could be closer to O if P_2 did not influence $D(O)$. (See Figure 9b.) This means that $|D(O) \cap (Q_1 \cup Q_2)|$ with two points in II is greater than $|D(O) \cap (Q_1 \cup Q_2)|$ under the conditions of Lemma 2 or Lemma 3. Therefore, the proof of this case will follow from these lemmas.

5. Proof of Lemma 2

Assume that $P_1 \in \mathrm{I}$, $P_2 \in \mathrm{II}$, $P_3 \in \mathrm{III}$, $P_4 \in \mathrm{IV}$ are the only points of P in $Q_1 \cup Q_2$ that influence $D(O)$. Then $|D(O) \cap (Q_1 \cup Q_2)| \geqq \frac{1}{2}\sqrt{5}$.

We need the following results.

TABLE 1

P' in	$M(OP')$	$\frac{1}{2}(x' - y')$	$M(OP_1)$	$M(OP_2)$, $M(OP_3)$	$M(OP_4')$	$M(OP_4)$	$\frac{1}{2}(\lvert x_4 \rvert - y_4)$
1	1.0	.0	2.0	1.85	1.271	1.082	.427
2	1.011	.15	2.011	1.75	1.185	1.161	.605
3	1.030	.25	2.030	1.68	1.105	1.242	.755
4	1.059	.0	2.059	1.9	1.305	1.024	.335
5	1.059	.15	2.059	1.8	1.0*	1.0*	**
6	1.059	.25	2.059	1.5	1.0*	1.0*	**

* Only minimum values needed. ** Estimate not needed.

TABLE 2

P' in	Case	$\lvert D(O) \cap Q_1 \rvert$	$\lvert D(O) \cap Q_2 \rvert$	$\lvert D(O) \cap (Q_1 \cup Q_2) \rvert$
1	i	.5	.771	1.2
	ii	.5	.86	1.3
2	i	.511	.685	1.18
	ii	.511	.66	1.17
3	i	.530	.605	1.13
	ii	.530	.74	1.2
4	i	.559	.805	1.3
	ii	.559	.764	1.3
5	i	.664	.5	1.15
	ii	.664	.5	1.15
6	i	.659	.5	1.15
	ii	.659	.5	1.15

Case i lists the estimates when the point in IV is in R_1, Case ii when it is in R_2. Note that the estimates in the first line of this table imply that we need not consider the possibility of points on the bisectors of Q_1 and Q_2 simultaneously.

LEMMA 2.1. *Let P_1 and P_2 be two points on $\lvert xy \rvert = 1$, such that $N(P_1 P_2) = 1$; then P_1 and P_2 generate a critical lattice of the star $\lvert xy \rvert \leqq 1$.*

Proof. There is no loss of generality in assuming that P_1 is in Q_1 and P_2 is in either Q_1 and Q_2 for the points could be placed in these positions by reflections if necessary.

Let σ be an affine transformation defined by

$$(x, y) \rightarrow (x/x_1, y/y_1).$$

This transformation preserves norm-distance since

$$xy = xy/x_1 y_1 .$$

It also preserves area since it is unimodular. Therefore, it suffices to prove the lemma for P_1' and P_2', the images of P_1 and P_2 respectively under σ.

$$P_1\sigma = P_1' = (x_1/x_1, y_1/y_1) = (1, 1),$$
$$P_2\sigma = P_2' = (x_2/x_1, y_2/y_1),$$

where P_2' has norm-distance 1 from P_1' and from the origin.

Note that P_1' is in Q_1 and P_2' is in the same quadrant as P_2.

If P_2' is in Q_1, then x_2', y_2' satisfy either

$$xy = 1 \quad \text{and} \quad (x-1)(y-1) = 1$$

or

$$xy = 1 \quad \text{and} \quad (x-1)(y-1) = -1.$$

Therefore,

$$P_2' = ((3-\sqrt{5})/2, (3+\sqrt{5})/2) \quad \text{or} \quad ((3+\sqrt{5})/2, (3-\sqrt{5})/2).$$

If P_2' is in Q_2, then x_2', y_2' satisfy either

$$xy = -1 \quad \text{and} \quad (x-1)(y-1) = 1$$

or

$$xy = -1 \quad \text{and} \quad (x-1)(y-1) = -1.$$

Then

$$P_2' = ((1-\sqrt{5})/2, (1+\sqrt{5})/2) \quad \text{or} \quad ((-1+\sqrt{5})/2, (-1-\sqrt{5})/2).$$

In any case, $P_1' = (1, 1)$ and P_2', for any of the above possibilities, do generate a critical lattice as is well known.

It is easily verified that for any lattice $\mathfrak{L}$ of mesh $d(\mathfrak{L})$, $|D(O)| = d(\mathfrak{L})$, for any point $O \in \mathfrak{L}$. Further, $D(O)$ is symmetric in O and therefore

$$|D(O) \cap (Q_1 \cup Q_2)| = \tfrac{1}{2}|D(O)| = \tfrac{1}{2}d(\mathfrak{L}).$$

Further, let $\mathcal{P}$ be an admissible point set. If the points of $\mathcal{P}$ in $Q_1 \cup Q_2$ which influence $D(O)$ are part of a critical lattice, then the cut-off from Q_3 can be no closer to O than when it is determined by a point of the same critical lattice. As noted above, the cut-off depends on P_3. Then the point P which causes the cut-off nearest to O satisfies

$$(\#) \qquad N(PP_3) = N(OP) = 1.$$

For P_3 near to O there is a unique point P in octant V satisfying condition (#). But if P_3 is a point of a critical lattice, the lattice point $-P_1$ satisfies $N(-P_1) = N(-P_1, P_3) = 1$. Hence $P = -P_1$.

We further note that any additional points of the critical lattice in $Q_1 \cup Q_2$ could only decrease the domain of action. Hence in this case we have that $|D(O) \cap (Q_1 \cup Q_2)|$ is at least $\tfrac{1}{2}\sqrt{5}$.

This remark is used repeatedly in the following proofs.

LEMMA 2.2. *Let there be a point of an admissible point set in each of the octants,* I, II, III, *and* IV. *If the points of* $\mathcal{P}$ *in* $Q_1 \cup Q_2$ *which influence* $D(O)$

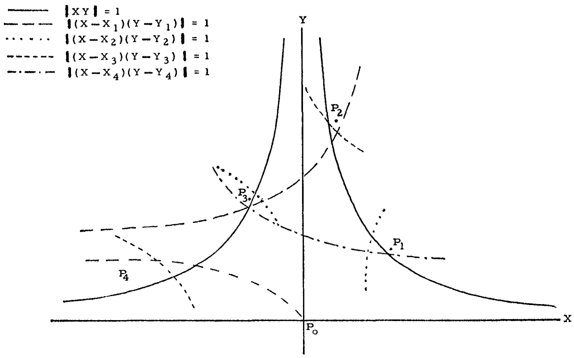

FIGURE 10a

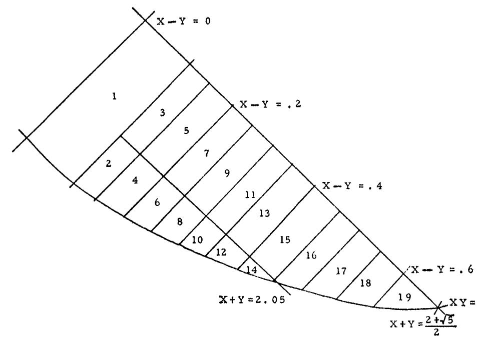

FIGURE 10b

are "close" to O and if P_1 has minimum M-distance from O among them, then the following statements always hold:

(a) *P_1 can be moved so as to have norm-distance 1 from at least two other points (allowing $P_0 = O$ to be one of them) while all other points are kept fixed and $|D(O)|$ is decreased.*[3]

(b) *The same holds for P_4.*

(c) *If P_2 can be moved so as to decrease $|D(O)|$ until it has norm-distance 1 from P_0 or P_1, or P_4, then it can be moved so that it has norm-distance 1 from another of the points also.*

(d) *If P_3 can be moved so as to decrease $|D(O)|$ until it has norm-distance 1 from P_0, P_2, or P_4, then it can be moved so that it has norm-distance 1 from another of the points also.*

Before proceeding with the proof of Lemma 2.2 we must explain what is meant by "close". As in Lemma 1 we divide the region in I bounded by $xy = 1$, $x = y$, and $x + y = (2 + \sqrt{5})/2$ into nineteen sectors and allow P_1 to vary successively in these sectors. (See Figure 10b.) We obtain, as before, estimates of $M(OP_i)$, $i = 2, 3, 4$, and $|D(O) \cap (Q_1 \cup Q_2)|$. Note, however, that while this method is not good enough to yield Lemma 2, since for P_1 in any of the sectors 14–18 the estimated $|D(O) \cap (Q_1 \cup Q_2)|$ is less than $\frac{1}{2}\sqrt{5}$, it does enable us to place sufficiently good bounds on the M-distances of the point P_i, $i = 1, 2, 3, 4$. (See Tables 3 and 4. Table 4 does not show the estimated areas for sectors 14–18.) Using this information we say that the points P_i are "close" to P_0 if the following inequalities hold:

$$M(OP_1) \leqq (2 + \sqrt{5})/4, \qquad M(OP_2) \leqq 1.5,$$

$$\min\,(M(OP_3), M(OP_4)) \leqq \tfrac{1}{2}\sqrt{5}, \qquad \max\,(M(OP_3), M(OP_4)) \leqq 1.5.$$

[3] For simplicity we continue to call the domain of action being considered $D(O)$ instead of $D(P_0)$.

TABLE 3

P_1 in	$M(OP_1)$	$\frac{1}{2}(x_1 - y_1)$	$M(OP_2)$	$M(OP_3)$*	$M(OP_3)$**	$M(OP_4)$
1	1.0	.0	1.424	1.374	1.094	1.082
2	1.001	.05	1.412	1.337	1.083	1.126
3	1.025	.05	1.400	1.325	1.065	1.104
4	1.002	.075	1.389	1.287	1.073	1.140
5	1.025	.075	1.379	1.279	1.077	1.116
6	1.004	.1	1.366	1.241	1.063	1.156
7	1.025	.1	1.358	1.233	1.067	1.130
8	1.007	.125	1.345	1.195	1.050	1.173
9	1.025	.125	1.337	1.187	1.058	1.144
10	1.011	.15	1.324	1.149	1.047	1.191
11	1.025	.15	1.317	1.142	1.049	1.161
12	1.015	.175	1.305	1.105	1.040	1.212
13	1.025	.175	1.298	1.098	1.041	1.178
14	1.019	.2	1.285	1.060	1.033 (54)	1.234
15	1.025	.2	1.280	1.055	1.033 (48)	1.198
16	1.025	.225	1.262	1.012	1.027	1.219
17	1.030	.25	1.245	1.0	1.022	1.242
18	1.037	.275	1.229	1.0	1.017	1.268
19	1.044	.3	1.198	1.0	1.009	1.296

* Estimate when P_2 influences $D(O)$.
** Estimate when there is no point in II which influences $D(O)$.

TABLE 4
Lower Estimates of $|D(O) \cap (Q_1 \cup Q_2)|$ for Lemma 2

P_1 in	$\lvert D(O) \cap Q_1 \rvert$	$\lvert D(O) \cap Q_2 \rvert$	$\lvert D(O) \cap (Q_1 \cup Q_2) \rvert$
1	.5	.699*	1.19
2	.501	.626	1.12
3	.525	.604	1.12
4	.502	.640	1.14
5	.525	.611	1.13
6	.504	.656	1.15
7	.525	.630	1.15
8	.507	.673	1.17
9	.525	.644	1.16
10	.511	.649	1.15
11	.525	.642	1.16
12	.515	.605	1.12
13	.525	.598	1.12
19	.577	.544**	1.12

* Estimate of $\frac{1}{2}(y_4 - x_4)$ needed here. For some arrangements of points in Lemmas 3 and 4 estimates of $\frac{1}{2}(y_3 - x_3)$ are also needed for P_1 in 1.
** $M(OP_1)$ is assumed to be less than or equal to $M(OP_i)$, $i = 2, 3, 4$.

Other estimates used in the proof may be obtained as a result of the foregoing inequalities.

The conclusions of Lemmas 2.2 and 2.3 (below) hold also in the cases where only two or three points of $Q_1 \cup Q_2$ influence the domain of action with slight adjustments in parts of the proofs.

We assume for the present that as P_i is moved it does not meet the quadrant bisector before it reaches the second hyperbola. We treat the case in which this occurs after Lemma 2.3.

Denote by $H_j : |(x - x_i)(y - y_j)| = 1$ where x_j and y_j are the coordinates of P_j.

$$|x_i| + |y_i| = s(x_i), \qquad ||x_i| - |y_i|| = d(x_i)$$
$$s_i = s(x_i)/2, \qquad d_i = d(x_i)/2.$$

Proof of Lemma 2.2. Either P_i, $i = 1, 2, 3, 4$, is on one of the hyperbolas H_j; $j \neq i, j = 0, 1, 2, 3, 4$, or it can be moved parallel to the quadrant bisector and toward P_0 until it does lie on one. This displacement decreases s_i while leaving d_i unchanged. Hence, the domain of action is decreased (see Figure 11).

(a) Given P_1 on only one of the hyperbolas, we show in the following in which direction it should be moved to decrease the domain of action. In all cases, it can be moved until it lies on the intersection of at least two hyperbolas.

(1) Let P_1 be on H_0, i.e., $P_0 P_1 = 1$ or $x_1 y_1 = 1$. Set

$$x + y = x_1 + 1/x_1 = s(x_1)$$

and

$$x - y = x_1 - 1/x_1 = d(x_1).$$

Differentiating s and d with respect to x_1, we obtain

$$s'(x_1) = 1 - 1/x_1^2 > 0 \qquad \text{for } x_1 > 1$$

which is always true for $P_1 \in \mathrm{I}$, and

$$d'(x_1) = 1 + 1/x_1^2 > 0.$$

Since both the sum and the differences of coordinates are increasing with

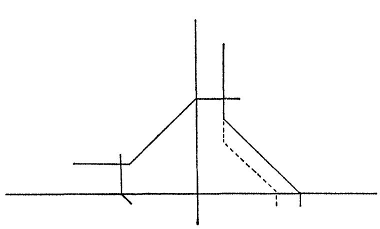

FIGURE 11

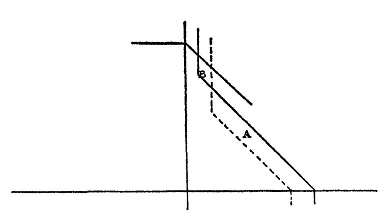

FIGURE 12

x_1, we could move P_1 along H_0 in the direction of decreasing x until it lies on a second hyperbola.

(2) Let P_1 be on $H_2 : |(x_1 - x_2)(y_1 - y_2)| = 1$.

$$s(x_1) = x_1 + y_2 - 1/(x_1 - x_2) = x + y.$$

$$s'(x_1) = 1 + 1/(x_1 - x_2)^2 > 0,$$

i.e., the sum of the coordinates decreases in the direction of decreasing x.

$$d(x_1) = x_1 - y_2 + 1/(x_1 - x_2) = x - y.$$

$$d'(x_1) = 1 - 1/(x_1 - x_2)^2 < 0 \qquad \text{if } 0 < x_1 - x_2 < 1.$$

This will be the case if $(y_1 - y_2)/(x_1 - x_2) < -1$, i.e., if

$$y_1 - y_2 < -x_1 + x_2 \quad \text{or} \quad y_1 + x_1 < y_2 + x_2$$

which is so since $M(OP_1) < M(OP_2)$.

Since the argument used above fails here, we show that the decrease in the domain of action because of the decrease in the sum of the coordinates overbalances the increase due to the increase in the difference of the coordinates. We use the following notation:

As x_i moves along H_j in a given direction

$$x_i \to x_i \pm \delta = x_i^* \qquad (\delta > 0).$$

Let $1/(x_i - x_j) = a$ and $1/(x_i \pm \delta - x_j) = b$ for $i, j = 1, 2, 3, 4; i \neq j$.

In the present case the original domain of action is increased by polygon B and decreased by polygon A (see Figure 12).

The area of B is

$$|B| = (d_1^* - d_1)(s_2 - s_1)$$

while the area of A is

$$|A| = (s_1 - s_1^*)(s_1^* - d_1^*) + \tfrac{1}{2}(s_1^* - s_1)^2.$$

Note that

$$s_1 - s_1^* = \tfrac{1}{2}(\delta - a + b),$$

$$d_1^* - d_1 = \tfrac{1}{2}(-\delta - a + b).$$

Therefore

$$s_1 - s_1^* > d_1^* - d_1 .$$

Also

$$s_1^* - d_1^* \geqq 1 - .35 = .65,$$

$$s_2 - s_1 \leqq 1.5 - 1 = .5$$

and

$$s_1^* - d_1^* > s_2 - s_1 .$$

Therefore, the area of A is greater than the area of B and the overall effect is a decrease in the domain of action.

(3) For P_1 on H_3 or H_4, the method of (a)(1) shows that both sum and

difference of the coordinates of P_1 decrease as P_1 is moved in the direction of decreasing x, hence $|D(O)|$ decreases also.

This completes the proof of Lemma 2.2(a).

(b) Let P_4 be on one of the hyperbolas H_0, H_1, H_3. In each case it can be moved until it lies on the intersection of at least two hyperbolas. We use the method of (a)(1) for P_4 on H_0 or H_1 and obtain the following:

(1) As P_4 moves along H_0 in the direction of increasing x, $|D(O)|$ decreases.

(2) As P_4 is moved along H_1 in the direction of increasing x, $|D(O)|$ decreases.

For P_4 on H_3 we use the methods of (a)(1) if $M(OP_3) \leqq M(OP_4)$ and obtain that $|D(O)|$ decreases as P_4 is moved along H_3 in the direction of increasing x.

If $M(OP_3) > M(OP_4)$, we use the method of (a)(2) and obtain that $|D(O)|$ decreases as P_4 is moved along H_3 in the direction of increasing x.

(c) Let P_2 be on only one of the hyperbolas H_0, H_1. In each case, it can be moved until it lies on the intersection of at least two hyperbolas.

Since $M(OP_1) < M(OP_2)$, only the sum of the coordinates of P_2 affects $D(O)$. We use the method in (a)(1) and obtain the following:

(1) If P_2 is moved downward along H_0 in the direction of increasing x, $|D(O)|$ decreases.

(2) If P_2 is moved along H_1 in the direction of decreasing x, $|D(O)|$ decreases.

(d) Let P_3 be on only one of the hyperbolas H_0, H_2, H_4.

(1) For P_3 on H_0 or H_4 using the method of (a)(1) we obtain that
   - (i) $|D(O)|$ decreases as P_3 is moved downward along H_0 in the direction of decreasing x;
   - (ii) $|D(O)|$ decreases as P_3 is moved along H_4 in the direction of increasing x.

(2) For P_3 on H_2 we use the method of (a)(2) and obtain that $|D(O)|$ decreases as P_3 is moved along H_2 in the direction of increasing x.

LEMMA 2.3. *Let there be a point of an admissible point set in each of the octants* I, II, III, *and* IV. *If* $D(O) \cap (Q_1 \cup Q_2)$ *is determined by points "close" to* P_0, *and if* P_2 *has minimum* M*-distance from* P_0, *then the following statements hold*:

(a) P_1 *can be moved so as to have norm-distance* 1 *from at least two other points (allowing* P_0 *to be one of them) while all other points are kept fixed and* $|D(O)|$ *is decreased.*

(b) *The same holds for* P_4.

(c) *The same holds for* P_3.

(d) *If* P_2 *can be moved so as to decrease* $|D(O)|$ *until it has norm-distance*

1 *from* P_0, P_1, *or* P_3, *then it can be moved so that it has norm-distance* 1 *from another point as well.*

The proof is similar to that of Lemma 2.2.

It remains for us to consider the case in which P_i moves along some H_j but meets the quadrant bisector before reaching the intersection with another H_k, i.e., before P_i has norm-distance 1 from two points under consideration.

P_2 and P_3 are never considered to be located on a quadrant bisector. Thus it is impossible for either to come to a quadrant bisector before reaching a second hyperbola in moving along some H_j. This can be verified by considering the intersection of $y = x$ or $y = -x$ with the branch of H_j in question.

P_1 or P_4 on the quadrant bisector constitute symmetric cases so without loss we can consider P_1 on the line $y = x$. As observed in Lemma 1, this prevents P_4 from being on $y = -x$ or rather, makes $| D(O) \cap (Q_1 \cup Q_2) | > \frac{1}{2}\sqrt{5}$, if it is on $y = -x$. By the calculations used in proving Lemma 2 (when P_1 is in the first sector—Table 3), we can conclude that whenever P_1 meets $y = x$ before it meets a second hyperbola, $| D(O) \cap (Q_1 \cup Q_2) |$ is already greater than or equal to $\frac{1}{2}\sqrt{5}$.

In the proof of Lemma 2 we will use the following notation:

$H_i : | (x - x_i)(y - y_i) | = 1$ where x_i and y_i are the coordinates of the point P_i.

$P_i P_j$ denotes that the norm-distance between P_i and P_j is 1.

$P_i P_j P_k$ denotes a normed triangle; all three sides have norm-length 1.

$P_i = H_j H_k$. P_i is *the* point of intersection of H_j and H_k. We say *the* for we use it as referring to the intersection of a specific branch of H_j with a specific branch of H_k, this being clear from the context. We observe that $P_i = H_j H_k$ implies that $N(P_i P_j) = N(P_i P_k) = 1$; similarly for $P_i = H_j H_k H_m$.

Proof of Lemma 2. As above, P_1 denotes the point in I, P_2 in II, P_3 in III, P_4 in IV influencing $D(O)$.

(a) Let $M(OP_1) \leqq M(OP_i)$ $(i = 2, 3, 4)$.

The points P_1, P_2, P_3, P_4, and $P_0 = O$ determine ten segments of norm-length greater than or equal to 1. $N(P_2 P_4)$, however, can never equal 1 so we have nine possible norm-lengths 1. We will show first that if there are at least seven norm-distances equal to 1, then the five points are part of a critical lattice for $\mathcal{S}$ and therefore, $| D(O) \cap (Q_1 \cup Q_2) | \geqq \frac{1}{2}\sqrt{5}$. The number of ways in which we can have seven norm-distances equal to 1 is the same as the number of ways we can have two norm-distances greater than or equal to 1. If we number the nine possible joins as in Figure 13 and use the symbol ij to mean that the joins marked i and j have norm-lengths greater than or equal to 1 while the others are equal to 1, then there are the following pos-

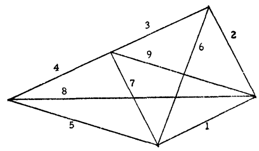

FIGURE 13

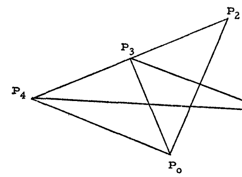

FIGURE 14

sibilities:

12	16	23	27	35	39	48	58	69
13	17	24	28	36	45	49	59	78
14	18	25	29	37	46	56	67	79
15	19	26	34	38	47	57	68	89

Each of these determines a certain seven-lined configuration, in which the lines represent norm-distances equal to 1. We will discuss a typical case, 12, in detail. The others are treated similarly and yield the same results.

Let $\mathfrak{L}$ be the critical lattice generated by the normed triangle $P_0 P_3 P_4$. (See Figure 14.) We will show that $P_1 \in \mathfrak{L}$, $P_2 \in \mathfrak{L}$ by showing that $P_1 = P_3 - P_4$ and $P_2 = P_1 + P_3$. By simple computation we have that

(i) the lattice point $P_3 - P_4 = H_0 H_3 H_4$ since it has norm-distance 1 from P_3, P_4 and also from P_0 as can be verified by computing the product of the coordinates of the point $(P_3 - P_4)$.

We are given that

(ii) $P_1 = H_3 H_4$.

(i) and (ii) imply that $P_1 = P_3 - P_4$. Similarly $P_1 + P_3 = H_0 H_1 H_3$ and $P_2 = H_0 H_3$ imply that $P_2 = P_1 + P_3$.

Hence the five points P_0, P_1, P_2, P_3, P_4, all belong to the same critical lattice.

Suppose now that there are six norm-distances equal to 1. In the above notation we have the following list of ways to have three norm-distances greater than or equal to 1.

12—(3 to 9)	23—(4 to 9)	35—(6 to 9)	48—9
13—(4 to 9)	24—(5 to 9)	36—(7 to 9)	56—(7 to 9)
14—(5 to 9)	25—(6 to 9)	37—(8 and 9)	57—(8 and 9)
15—(6 to 9)	26—(7 to 9)	38—9	58—9
16—(7 to 9)	27—(8 and 9)	45—(6 to 9)	67—(8 and 9)
17—(8 and 9)	28—9	46—(7 to 9)	68—9
18—9	34—(5 to 9)	47—(8 and 9)	78—9

Using the results of Lemma 2.2 we need not consider the possibilities containing the following combinations:

128	189	23	36	48	349	479
129	289	45	58	379		

where 128, for example, implies that P_1 is on H_3 only. But by Lemma 2.2(a) we know that $|D(O)|$ can be decreased until P_1 lies on at least one other hyperbola.

In the remaining cases we show that $|D(O) \cap (Q_1 \cup Q_2)|$ determined by the given configuration is at least as great as that determined by the points of a critical lattice and hence is greater than or equal to $\frac{1}{2}\sqrt{5}$.

When comparing two domains of action we always assume that

(1) the two sets of points which determine them differ in only one point, e.g., $(RP_i P_j P_k)$ and $(QP_i P_j P_k)$,
(2) R and Q both are on the same branch of some hyperbola.

When these conditions are satisfied, Lemma 2.2 and the admissibility of the point set enable us to determine which domain of action is smaller. For brevity we speak of the domains as corresponding to R and Q since the effect of P_i, P_j, P_k is the same in both instances.

This *comparison method* is used repeatedly in the following proofs. Consider the case 124. (See Figure 15.) $P_0 P_2 P_3$ generates a critical lattice $\mathcal{L}$. Let $P_1' \in \mathcal{L}$ be such that $P_1' = P_2 - P_3$; then $P_1' = H_0 H_2 H_3$. But $P_1 = H_3 H_4$. Then P_1 and P_1' are on H_3. By Lemma 2.2(a), $|D(O)|$ decreases as a point on H_3 is moved toward P_0 (in the direction of decreasing x). Since P_1' is on H_0, if P_1 is to determine a smaller domain of action for O than P_1' then it is necessary that $N(OP)$ be less than 1. But P_1 is a point of an $\mathcal{S}$-admissible set. Therefore, P_1 must determine a domain of action at least as great as that determined by P_1'. (We denote this by $P_1' \leqq P_1$.)

Let $P_3 - P_1' = P_4' \in \mathcal{L}$.

Then $P_4' = H_0 H_3$ while $P_4 = H_0 H_1$. Thus both P_4 and P_4' are on H_0. By Lemma 2.2(c), $|D(O)|$ decreases as a point is moved along H_0 toward P_0 (i.e., in the direction of increasing x). If P_4 is to determine a smaller domain of action for O than P_4' does then $N(P_4 P_3) < 1$ since P_4' is on H_3. Hence by the admissibility of the point set $\mathcal{P}$, we have $P_4' \leqq P_4$.

Therefore, the configuration in question determines a domain of action in $Q_1 \cup Q_2$ which is at least as great as that determined by points of the critical lattice $\mathcal{L}$ and therefore is greater than or equal to $\frac{1}{2}\sqrt{5}$.

Suppose now that there are only five norm-distances equal to 1. Consider all possible ways in which we can have four norm-distances greater than or equal to 1. By using the results of Lemma 2.2 to reduce the number of cases

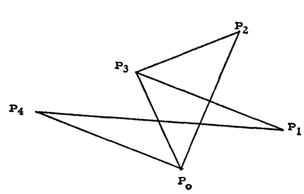

FIGURE 15

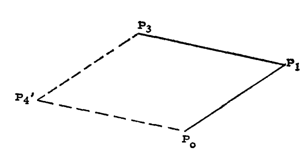

FIGURE 16

to be considered, the remaining cases can be handled as in the preceding discussion. In each case we obtain that the given configuration determines

$$| D(O) \cap (Q_1 \cup Q_2) | \geqq \tfrac{1}{2}\sqrt{5}.$$

Lemmas 2.2 and 2.3 imply that if there are fewer than five norm-distances equal to 1, then $| D(O) \cap (Q_1 \cup Q_2) |$ can be decreased by moving certain of the points until at least five norm-distances are 1.

This completes part (a) of the present case.

Part (b), with P_2 the point of $\mathcal{P}$ in $Q_1 \cup Q_2$ having smallest M-distance from O, follows by similar arguments.

6. Proof of Lemma 3

Now consider the case of three points in the upper half-plane influencing the domain of action of O.

These three points and $O = P_0$ determine six segments each of norm-length greater than or equal to 1. If all six of these norm-distances are 1, then the four points belong to one and the same critical lattice. The same conclusion can be reached if we have five norm-distances equal to 1. Here, however, there are six ways in which this can occur for each choice of the three points. In the proof the methods of the preceding section are used.

Lemmas 2.2 and 2.3 imply that if there are fewer than four norm-distances equal to 1 (among the points $O = P_0$ and the three points of $Q_1 \cup Q_2$ influencing $D(O)$), then $| D(O) \cap (Q_1 \cup Q_2) |$ can be decreased by moving certain of the points until at least four norm-distances are 1.

If there are only four norm-distances equal to 1, we use the comparison method to prove that in each instance $| D(O) \cap (Q_1 \cup Q_2) | \geqq \frac{1}{2}\sqrt{5}$.

7. Proof of Lemma 4

Suppose now that there is only one point of $\mathcal{P}$ in Q_1 and one point in Q_2 influencing $D(O)$. We have the following possibilities:

A. The closest point is in I and the second point is (1) in III; (2) in IV.

B. The closest point is in II and the second point is (1) in III; (2) in IV.

Case A. (1) The two points are P_1 and P_3. By the arguments of Lemma 2.2(a), we can decrease $D(O)$ by moving P_1 until it has norm-distance 1 from P_0 and P_3. (See Figure 16.)

N. Smith, [3, p. 7] proved that a triangle with vertices belonging to an $\mathcal{S}$-admissible point set and satisfying the condition that the slopes of the three sides are not all positive has area greater than or equal to $\frac{1}{2}\sqrt{5}$. (Smith called such a triangle a type (a) triangle.)

$P_0 P_1 P_3$ is a type (a) triangle and hence its area is greater than or equal to $\frac{1}{2}\sqrt{5}$. Hence, the mesh of the lattice $\mathcal{L}$ generated by P_1 and P_3 is greater than or equal to $\sqrt{5}$.

By hypothesis, $N(P_0 P_3) \geqq 1$. If $N(P_0 P_3) > 1$ simple computational arguments show that $P_2 = P_1 + P_3$ is inside $\mathcal{S}$, i.e., $x_2 y_2 < 1$. Hence $\mathfrak{L}$ is admissible if and only if $N(P_0 P_3) = 1$. But in any case the cut-off from Q_3 and Q_4 could be no closer to O than when determined by a lattice point; hence, $| D(O) \cap (Q_1 \cup Q_2) | \geqq \frac{1}{2}\sqrt{5}$.

(2) The two points are P_1 and P_4. In this case there would be an infinite strip about the y-axis contained in $D(O)$; hence, we need not consider it further.

Case B. (1) The two points are P_2 and P_3. By the arguments of Lemma 2.3(c), we can move P_2 to decrease $D(O)$ until $N(P_2 P_3) = N(P_0 P_2) = 1$. Let P_4' be a point of the lattice generated by $P_0 P_2 P_3$. It is clear that the domain of action determined by P_0, P_2, P_3, P_4, is less than or equal to that determined by P_0, P_2, P_3 only. However, in Lemma 3 we proved that P_0, P_2, P_3, P_4' determine

$$| D(O) \cap (Q_1 \cup Q_2) | \geqq \tfrac{1}{2}\sqrt{5}.$$

(2) Again by arguments of Lemma 2.3, we can move P_4 to decrease $D(O)$ until $N(P_0 P_4) = N(P_2 P_4) = 1$ and the argument of A(1) holds.

This completes the proof of Lemma 4.

8. Conclusion

With the completion of Lemma 4 the proof of the theorem stated in Section 3 is also complete, namely, if O is any point of an admissible point set $\mathcal{P}$ for $\mathcal{S} : | xy | \leqq 1$, then $| D(O) | \geqq \sqrt{5}$. Thus, using the domain of action method on $\mathcal{S}$, we find that $\mathfrak{D}(\mathcal{P}) \leqq 1/\sqrt{5}$ which is best possible.

Note also that $| D(O) | = \sqrt{5}$ only if the points P_i which determine $D(O)$ are part of a critical lattice. This is seen as follows.

If the points P_i are moved as explained in Lemmas 2.2 and 2.3, $| D(O) \cap (Q_1 \cup Q_2) |$ decreases strictly. In the proof of Lemma 2, all cases involving only five norm-distances equal to 1 can be shown to determine $| D(O) \cap (Q_1 \cup Q_2) |$ greater than or equal to that determined by a configuration with six norm-distances equal to 1.

Therefore, it suffices to prove that any configuration with exactly six norm-distances equal to 1 determines

$$| D(O) \cap (Q_1 \cup Q_2) | > \tfrac{1}{2}\sqrt{5}.$$

Consider the configuration denoted by the symbol 124 (above). If $P_1' < P_1$ then $| D(O) \cap (Q_1 \cup Q_2) |$ determined by the critical lattice is strictly less than that determined by the configuration 124, and hence the latter domain is strictly greater than $\frac{1}{2}\sqrt{5}$.

If $P_1' = P_1$ then P_1' and P_1 coincide and $N(P_0 P_1) = 1$. Then there are seven norm-distances equal to 1 and, by a previous argument, the points P_i, $i = 0, 1, 2, 3, 4$, all belong to the same critical lattice.

A similar argument will give the same result for the remaining configurations involving five points and for the configurations for three and four points.

If the points in $Q_1 \cup Q_2$ and the points in $Q_3 \cup Q_4$ which determine $D(O)$ were part of two different critical lattices, or if a point of the corresponding critical lattice which influences $D(O)$ has no counterpart in the given set of points, then $|D(O)| > 5$ as can be verified by simple computation.

Therefore, if $|D(O)| = \sqrt{5}$, $D(O)$ must be determined by points which all belong to the same critical lattice and further must be identical with a typical domain of action of a point of such a critical lattice.

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