ON KLOOSTERMAN SUMS CONNECTED WITH MODULAR FORMS OF HALF-INTEGRAL DIMENSION

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1. Introduction

In this paper we consider certain exponential sums $W(c, n, \mu, v)$ which are intimately related to the well-known Kloosterman sums and which arise naturally in the theory of modular forms. It is our purpose to show that when the dimension of the modular form is half-integral we can obtain for these sums the asymptotic estimate

$$W(c, n, \mu, v) = O(c^{1/2+\varepsilon}), \qquad \varepsilon > 0, \text{ as } c \to +\infty, \tag{1}$$

where the constant involved depends upon μ and v, but is independent of n (see §2 for definitions and an explanation of the notation).

We use a method of Petersson [4, pp. 16–19] to reduce $W(c, n, \mu, v)$ to a finite sum of sums K_c, for which the estimate (1) has recently been obtained by Malishev [3]. In this way we obtain (1) for all multiplier systems v connected with the modular group and any half-integral dimension. For *integral* dimension the Petersson method alone suffices to derive (1), no use being made in this case of Malishev's result.

The estimate (1) was obtained by Lehmer [1] in the particular case when $W(c, n, \mu, v)$ is the sum connected with $\eta^{-1}(\tau)$, the well-known modular form of dimension $\frac{1}{2}$. It is conceivable that his method could be extended to give the estimate in all the cases for which we obtain it here.

In §5 we remark on the impossibility of obtaining (1) for certain dimensions and choices of the parameters n, μ, v, and conclude with an application of (1) to the estimation of the Fourier coefficients of cusp forms.

2. Preliminaries

Let $\Gamma(1)$ denote the modular group, that is, the set of all 2×2 matrices with rational integral entries and determinant one. Let $\Gamma(n)$ be the principal congruence subgroup of level n, the set of all elements of $\Gamma(1)$ which are congruent, elementwise, to the identity matrix modulo n. If r is a real number, we define a *modular form of dimension r* to be a function $F(\tau)$ meromorphic in the upper half-plane, Im $(\tau) > 0$, such that $\lim_{y\to+\infty} | F(iy) |$ exists (possibly $+\infty$), and satisfying

$$F(M\tau) = v(M)(c\tau + d)^{-r}F(\tau), \tag{2}$$

for each $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1)$. Here $v(M)$ is complex-valued, independent of

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τ, and satisfies $|v(M)| = 1$, for each $M \epsilon \Gamma(1)$. By $M\tau$ we mean

$$(a\tau + b)/(c\tau + d).$$

In order to fix the branch of $(c\tau + d)^{-r}$ when r is not an integer, for any complex number z and real s we define $z^s = |z|^s \exp(i \arg z)$, with $-\pi \leqq \arg z < \pi$. A modular form $F(\tau)$ is called a *cusp form* if $F(\tau)$ is regular in Im $(\tau) > 0$, and $\lim_{y\to+\infty} F(iy) = 0$.

When a function exists satisfying (2) it follows that if

$$M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \epsilon \Gamma(1), \qquad M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \epsilon \Gamma(1),$$

then

(3) $v(M_1 M_2)(c_3 \tau + d_3)^{-r} = v(M_1)v(M_2)(c_1 M_2 \tau + d_1)^{-r}(c_2 \tau + d_2)^{-r}$,

where $M_1 M_2 = \left(\begin{smallmatrix} * & * \\ c_3 & d_3 \end{smallmatrix}\right)$. Any complex-valued function $v(M)$ defined on $\Gamma(1)$ such that $|v(M)| = 1$, for all $M \epsilon \Gamma(1)$, and satisfying (3) is called a *multiplier system for* $\Gamma(1)$ *and the dimension* r. Let $U = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, and define κ by

(4) $$v(U) = e^{2\pi i\kappa}, \qquad 0 \leqq \kappa < 1.$$

We observe from (3) that when r is an integer v is a character on $\Gamma(1)$. Also it follows from (3) that

(5) $$v(MU) = v(UM) = v(M)v(U), \qquad M \epsilon \Gamma(1).$$

From now on we assume that v is connected with a half-integral dimension $r = s/2$. Let $u = 2(r - [r])$; $u = 0$ if s is even, and $u = 1$ if s is odd. Then it is immediate that

(6) $$v(M) = v_1(M)v_2^u(M), \qquad M \epsilon \Gamma(1),$$

where v_1 is a multiplier system for the dimension $[r]$ (and hence a character on $\Gamma(1)$), and v_2 is the multiplier system for $\eta^{-1}(\tau)$. Let κ, κ_1, κ_2 be associated with v, v_1, v_2, respectively, as in (4). By (6), $\kappa_1 + u\kappa_2 \equiv \kappa \pmod 1$. It follows from [6, p. 445] that $\kappa_1 = l/6$ $(l = 0, 1, 2, 3, 4, 5)$ if $[r]$ is even, and $\kappa_1 = l/12$ $(l = 1, 3, 5, 7, 9, 11)$ if $[r]$ is odd. In either case we can write

(7) $$\kappa_1 = l/12 \qquad (0 \leqq l \leqq 11).$$

Furthermore it is known that $\kappa_2 = 23/24$.

It is shown in [2] that any character on $\Gamma(1)$ is identically 1 on $\Gamma(12)$. Hence,

(8) $$v_1(M) = 1, \qquad \text{for } M \epsilon \Gamma(24).$$

This fact will be critical later.

We now come to the definition of the exponential sums $W(c, n, \mu, v)$. Let c be a positive integer and let n and μ be any integers. We put

(9) $$W = W(c, n, \mu, v) = \sum_{d=1}^{c} \bar{v}(M_{c,d}) \exp\left[\frac{2\pi i}{c}\{(n + \kappa)a + (\mu + \kappa)d\}\right]$$

where $M_{c,d} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is any element of $\Gamma(1)$ with lower row (c, d), v is any multiplier system connected with $\Gamma(1)$, and κ is defined by (4). (We are here restricting ourselves to v connected with half-integral dimension, but the definition makes sense for arbitrary dimension. In §5 we will have occasion to discuss sums W arising from arbitrary real dimension.)

Writing

$$g(a, b, c, d) = \bar{v}(M_{c,d}) \exp\left[\frac{2\pi i}{c}\{(n+\kappa)a + (\mu+\kappa)d\}\right],$$

we observe from (5) that

$$g(a + c, b + d, c, d) = g(a, b, c, d) \tag{10}$$

and

$$g(a, b + a, c, d + c) = g(a, b, c, d). \tag{11}$$

By (10) we see that $g(a, b, c, d)$ is a function of (c, d) only and does not depend upon the particular choice of a and b in $M_{c,d}$. Hence W is well defined.

Finally we state Malishev's result [3] of which we will make important use. Let

$$K_c(\mu, n; q) = \sum_{\substack{x \pmod q \\ (x,q)=1}} \left(\frac{x}{c}\right) \exp\left\{\frac{2\pi i}{q}(\mu x + n x')\right\},$$

where μ and n are integers, q is a positive integer and c is an odd positive integer all of whose prime factors divide q. Furthermore x' is any integral solution of the congruence $xx' \equiv 1 \pmod q$ and $\left(\frac{x}{c}\right)$ is the Jacobi symbol. Then

$$| K_c(\mu, n; q)| \leqq A(\varepsilon)\cdot q^{1/2+\varepsilon} \min\{ (\mu, q)^{1/2}, (n, q)^{1/2}\} \tag{12}$$

for each $\varepsilon > 0$, where $A(\varepsilon) > 0$ depends only on ε.

3. Reduction of the sum W

By (11), we have

$$24W = \sum_{d=1}^{24c} \bar{v}(M) \exp\left[\frac{2\pi i}{24c}(ma + \omega d)\right], \tag{13}$$

where $m = 24(n + \kappa)$ and $\omega = 24(\mu + \kappa)$ are integers. Notice that we have written M for $M_{c,d}$. We will follow this practice for the remainder of the paper.

We write

$$\Gamma(1) = \sum_{\substack{1 \leqq s \leqq \nu \\ 0 \leqq l < 24}} \Gamma(24) U^l K_s,$$

a coset decomposition of $\Gamma(1)$ modulo $\Gamma(24)$, where the K_s form a complete set of representatives modulo $\Gamma(24)$, subject to the added restriction that the second rows of the K_s be distinct modulo 24. If we put

$$K_s = \begin{pmatrix} \alpha_s & \beta_s \\ \gamma_s & \delta_s \end{pmatrix}, \qquad (s = 1, \cdots, \nu),$$

it follows from (10) that (13) can be rewritten as

$$24W = \sum_{1 \leqq s \leqq \nu}' \sum_{d=1}^{24c} {}^{K_s} \bar{v}(M) \exp\left[\frac{2\pi i}{24c}(ma + \omega d)\right],$$

where $\sum^{K_s}$ indicates that the inner sum is restricted by the condition $M \equiv K_s \pmod{24}$, and the prime on the outer sum indicates that we are restricted to those s such that (13) can be satisfied with our fixed c. Using (6), (8), and the fact that v_1 is a character on $\Gamma(1)$, we obtain

$$(14) \qquad \begin{aligned} 24W &= \sum_{1 \leqq s \leqq \nu}' \bar{v}_1(K_s) \sum_{d=1}^{24c} {}^{K_s} \bar{v}_2^u(M) \exp\left[\frac{2\pi i}{24c}(ma + \omega d)\right] \\ &= \sum_{1 \leqq s \leqq \nu} \bar{v}_1(K_s) \cdot W(K_s). \end{aligned}$$

Let

$$K_s' = U^l K_s U^k = \begin{pmatrix} \alpha_s' & \beta_s' \\ \gamma_s' & \delta_s' \end{pmatrix} = \begin{pmatrix} \alpha_s + l\gamma_s & \beta_s + l\delta_s + k(\alpha_s + l\gamma_s) \\ \gamma_s & k\gamma_s + \delta_s \end{pmatrix}.$$

Then

$$(15) \qquad \begin{aligned} W(K_s') &= \sum_{d=1}^{24c} {}^{K_s'} \bar{v}_2^u(M) \exp\left[\frac{2\pi i}{24c}(ma + \omega d)\right] \\ &= \bar{v}_2^u(U^l)\, \bar{v}_2^u(U^k) \exp\left[\frac{2\pi i}{24}(lm + k\omega)\right] W(K_s) \\ &= \exp(2\pi i\, q_s)\, W(K_s), \end{aligned}$$

where we have made use of (5) applied to the multiplier system v_2^u. The summation condition on $W(K_s')$ is $M \equiv K_s' \pmod{24}$, or

$$(16) \qquad a \equiv \alpha_s', \quad d \equiv \delta_s' \pmod{24}, \qquad ad \equiv 1 + \beta_s' c \pmod{24c}.$$

Since $(\alpha_s, \gamma_s) = 1$ we can choose an integer l_s so that $(\alpha_s + l_s\gamma_s, 24) = 1$. Then we choose an integer k_s so that $k_s(\alpha_s + l\gamma_s) \equiv -\beta_s - l_s\delta_s \pmod{24}$. In short we choose k_s and l_s such that $\beta_s' \equiv 0 \pmod{24}$. Then the conditions (16) become

$$a \equiv \alpha_s', \quad d \equiv \delta_s' \pmod{24}, \qquad ad \equiv 1 \pmod{24c}.$$

Now $(\delta_s', 24) = 1$, $\alpha_s'\delta_s' \equiv a\delta_s' \pmod{24}$, and $d \equiv \delta_s' \pmod{24}$ together imply that $a \equiv \alpha_s' \pmod{24}$. The same reasoning shows that if $a \equiv \alpha_s' \pmod{24}$, then $d \equiv \delta'_s \pmod{24}$. Therefore

$$W(K'_s) = \sum_{d=1}^{24c} \bar{v}_2^u(M) \exp\left[\frac{2\pi i}{24c}(ma + \omega d)\right],$$

where the sum is restricted by the conditions $ad \equiv 1 \pmod{24c}$, $d \equiv \delta'_s \pmod{24}$.

From the fact that

$$\frac{1}{24}\sum_{t=1}^{24} \exp(2\pi irt/24) = 1 \quad \text{if} \quad r \equiv 0 \pmod{24},$$
$$= 0 \quad \text{otherwise}$$

we conclude that

$$W(K'_s) = \frac{1}{24}\sum_{t=1}^{24} \exp(-2\pi i\delta'_s t/24) \cdot \sum_{d=1}^{24c}{}^{*} \bar{v}_2^u(M) \exp\left[\frac{2\pi i}{24c}\{ma + (\omega + tc)d\}\right]$$

where the sum $\sum^*$ is restricted only by the conditions $ad \equiv 1 \pmod{24c}$, $(d, 24c) = 1$. Denoting this inner sum on d by $K(t)$, we conclude from (14) and (15) that

$$(17) \quad 24W = \frac{1}{24} \sum_{1 \leqq s \leqq \nu}{}' \bar{v}_1(K_s) \exp(-2\pi i q_s) \sum_{t=1}^{24} \exp(-2\pi i\delta'_s t/24) K(t).$$

4. Proof of (1)

Recall that $u = 0$ or 1. If $u = 0$, $K(t)$ is the classical Kloosterman sum for which we have the famous estimate of Salié and Weil [7], [8].

$$K(t) = O((\omega + tc, c)^{1/2} \cdot c^{1/2+\varepsilon}) = O(c^{1/2+\varepsilon})$$

for any $\varepsilon > 0$, where the constant involved depends only on ω. Since $\omega = 24(\mu + \kappa)$, (1) follows from (17) and this concludes the case $u = 0$.

If $u = 1$, the proof is more complicated. In this case we make use of an explicit expression for v_2 [2]. It is

$$v_2(M) = \left(\frac{d}{c}\right) \exp[-\pi i\{(a + d)c - bd(c^2 - 1) - 3c\}/12], \qquad \text{if } c \text{ is odd},$$

$$= \left(\frac{c}{d}\right) \exp[-\pi i\{(a + d)c - bd(c^2 - 1) + 3d - 3 - 3cd\}/12], \quad \text{if } c \text{ is even},$$

where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$, with $c > 0$, $d > 0$. Putting this into the definition of $K(t)$, with $u = 1$, we get *for odd c*,

$$K(t) = \sum_{d=1}^{24c}{}^{*} \left(\frac{d}{c}\right) \exp\left[\frac{\pi i}{12}\{(a + d)c - bd(c^2 - 1) - 3c\}\right]$$
$$\cdot \exp\left[\frac{2\pi i}{24c}\{ma + (\omega + tc)d\}\right]$$
$$= e^{-\pi ic/4} \sum_{d=1}^{24c}{}^{*} \left(\frac{d}{c}\right) \exp\left[\frac{2\pi i}{24c}\{(m + c^2)a + (\omega + tc + c^2 - bc(c^2 - 1))d\}\right].$$

But since $ad \equiv 1 \pmod{24c}$, $bc \equiv 0 \pmod{24c}$, and this becomes

$$K(t) = e^{-\pi ic/4} \sum_{d=1}^{24c}{}^{*} \left(\frac{d}{c}\right) \exp\left[\frac{2\pi i}{24c}\{(m + c^2)a + (\omega + tc + c^2)d\}\right]$$
$$= e^{-\pi ic/4} K_c(\omega + tc + c^2, m + c^2; 24c),$$

in the notation of Malishev. By (12), we conclude that

$$K(t) = O((\omega, c)^{1/2} c^{1/2+\varepsilon}) = O(c^{1/2+\varepsilon}),$$

for any $\varepsilon > 0$, where the constant depends only on ω. Hence for odd c, (1) follows from (17) as before.

If *c is even*,

$$K(t) = \sum_{d=1}^{24c}{}^{*} \left(\frac{c}{d}\right) \exp\left[\frac{\pi i}{12}\{(a + d)c - bd(c^2 - 1) + 3d - 3 - 3cd\}\right]$$
$$\cdot \exp\left[\frac{2\pi i}{24c}\{ma + (\omega + tc)d\}\right]$$
$$= e^{-\pi i/4} \sum_{d=1}^{24c}{}^{*} \left(\frac{c}{d}\right) \exp\left[\frac{2\pi i}{24c}\{(m + c^2)a + (\omega + tc - 2c^2 + 3c)d\}\right].$$

Write $c = 2^z c_1$, c_1 odd and $z \geqq 1$. Then by quadratic reciprocity

$$\left(\frac{c}{d}\right) = \left(\frac{2}{d}\right)^z \left(\frac{c_1}{d}\right) = (-1)^{(d^2-1)z/8} (-1)^{(c_1-1)(d-1)/4} \left(\frac{d}{c_1}\right),$$

and we obtain

$$K(t) = e^{-\pi ic_1/4} \sum_{d=1}^{24c}{}^{*} \left(\frac{d}{c_1}\right) (-1)^{(d^2-1)z/8} \exp\left\{\frac{\pi i}{4}(c_1 d - d)\right\}$$
$$\cdot \exp\left[\frac{2\pi i}{24c}\{(m + c^2)a + (\omega + tc - 2c^2 + 3c)d\}\right].$$

A simple calculation shows that

$$(-1)^{(d^2-1)/8} = (-1)^{(d-1)/4} \quad \text{if} \quad d \equiv 1 \pmod 4,$$
$$= (-1)^{(d+1)/4} \quad \text{if} \quad d \equiv 3 \pmod 4.$$

Hence

$$K(t) = e^{-\pi iz/4} e^{-\pi ic_1/4} \sum_{\substack{d=1 \\ d \equiv 1 \pmod 4}}^{24c}{}^{*} \left(\frac{d}{c_1}\right)$$
$$\cdot \exp\left[\frac{2\pi i}{24c}\{(m + c^2)a + (\omega + tc - 2c^2 + 3c(z + c_1 - 2))d\}\right]$$
$$+ e^{\pi iz/4} e^{-\pi ic_1/4} \sum_{\substack{d=1 \\ d \equiv 3 \pmod 4}}^{24c}{}^{*} \left(\frac{d}{c_1}\right) \tag{18}$$
$$\cdot \exp\left[\frac{2\pi i}{24c}\{(m + c^2)a + (\omega + tc - 2c^2 + 3c(z + c_1 - 2))d\}\right].$$

The same device used to remove the condition $d \equiv \delta_s'$ (mod 24) in the sum $W(K_s')$ can now be used to remove the congruence condition on each of the sums appearing on the right side of (18). If we then apply (12), with q replaced by $24c$ and c replaced by c_1, to each of the resulting sums we obtain

$$K(t) = O(\omega, c)^{1/2}c^{1/2+\varepsilon}) = O(c^{1/2+\varepsilon}), \qquad \text{for any } \varepsilon > 0,$$

where the constant depends only on ω. Again, (1) follows from (17), and we are done.

5. Conclusion

As Rademacher remarked in [5, p. 69], it is not possible to obtain (1) for sums W connected with all choices of the parameters n, μ, and v. In fact, he observed that if (1) did hold in all cases, we could conclude from an application of the circle method that the cusp forms $\eta^{\beta}(\tau)$ $(0 < \beta < 1)$ of dimensions $-\beta/2$, vanish identically, contrary to well-known fact. This shows that for each dimension r $(-\frac{1}{2} < r < 0)$ there exist μ and n such that (1) does not hold for $W(c, n, \mu, v_2^{2r})$. By observing that $\bar{W}(c, n, \mu, v) = W(c, -n, -\mu, \bar{v})$ we see that *for each r in the range $0 < |r| < \frac{1}{2}$ there exist μ, n, and v, connected with the dimension r, such that* (1) *does not hold for $W(c, n, \mu, v)$*. Since, as is readily seen from (3), a multiplier system for the dimension r is also one for the dimensions $r + 2j$ $(j = 0, \pm 1, \pm 2, \cdots)$, the same holds true in all dimensions r given by $0 < |r - 2j| < \frac{1}{2}$ $(j = 0, \pm 1, \pm 2, \cdots)$. We conjecture that for each real r not equal to a half-integer there exist μ, n, and a multiplier system v for the dimension r such that (1) is false for $W(c, n, \mu, v)$. No proof has yet been found.

Another application of the circle method and the estimate (1) together show that if a_n $(n \geqq 1)$ are the Fourier coefficients of a cusp form of dimension r $(r < 0)$, then as $n \to +\infty$,

$$a_n = O(n^{-r/2-1/4+\varepsilon}), \qquad \text{for any } \varepsilon > 0.$$

REFERENCES

1. D. H. LEHMER, *On the series for the partition function*, Trans. Amer. Math. Soc., vol. 43 (1938), pp. 271–295.
2. J. H. VAN LINT, *On the multiplier system of the Riemann-Dedekind function η*, Indag. Math., vol. 20 (1958), pp. 522–527.
3. A. V. MALISHEV, *Generalized Kloosterman sums and their estimatione*, Vestnik Leningrad Univ., 1960, no. 13, pp. 59–75. (Russian).
4. H. PETERSSON, *Über Modulfunktionen und Partionenprobleme*, Abh. Deutsch. Akad. Wiss. Berlin Kl. Math. Allg. Nat., 1954, no. 2, 59 pp.
5. H. RADEMACHER, *Fourier expansions of modular forms and problems of partition*, Bull. Amer. Math. Soc., vol. 46 (1940), pp. 59–73.
6. H. RADEMACHER AND H. S. ZUCKERMAN, *On the Fourier coefficients of certain modular forms of positive dimension*, Ann. of Math. (2), vol. 39 (1938), pp. 433–462.

7. H. Salié, *Über die Kloosterman Summen* $S(u, v; q)$, Math. Zeitschr., vol. 34 (1931), pp. 91–109.
8. A. Weil, *On some exponential sums*, Proc. Nat. Acad. Sci. U. S. A., vol. 34 (1948), pp. 204–207.

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