ON THE CONTINUITY OF LATTICE AUTOMORPHISMS ON CONTINUOUS FUNCTION LATTICES

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1. Introduction

Let E be a compact Hausdorff space, $C(E)$ the lattice of all real-valued continuous functions on E, and let $T : f \rightarrow f^T$ be a lattice automorphism of $C(E)$.

I. Kaplansky has proved in [2] the following two results.

(I) If T is homeomorphic in the topology of uniform convergence, then T can be characterized in the following form:

$$f^T(x^t) = \Phi(f(x), x) \qquad (x \in E, f \in C(E))$$

where $x \rightarrow x^t$ is a homeomorphism of E, and $\Phi(\xi, x)$ ($\xi \in \mathbf{R}$, $x \in E$) is a continuous function on $\mathbf{R} \times E$, and for any fixed $x \in E$, $\Phi(\ , x)$ is a lattice automorphism of $\mathbf{R}$.

(II) If E satisfies a first axiom of countability, then all lattice automorphisms of $C(E)$ are homeomorphic in the topology of uniform convergence. However, generally speaking, lattice automorphisms are not necessarily continuous.

It may be natural to consider the following problem: What is the characteristic topological property of E in order that all lattice automorphisms of $C(E)$ be continuous in the topology of uniform convergence?

In view of this problem the following three classes of compact Hausdorff spaces are considered.

(1) E has property (K): All lattice automorphisms of $C(E)$ are continuous.
(2) E has property (K_0): All compact subspaces of E have property (K).
(3) E has property (K_1): A lattice automorphism T of $C(E)$ is continuous if and only if T^{-1} is continuous.

The above three classes obviously satisfy the relations

$$(K_0) \subset (K) \subset (K_1).$$

Our purpose in this paper is to give a complete topological characterization of properties (K_0) and (K_1).

THEOREM 1. *E has property (K_1) if and only if $E \neq \beta U$ for any dense open F_σ-subset $U \subset E$, $U \neq E$, where βU is a Stone-Čech compactification*[1] *of U.*

Received March 7, 1963.

[1] For the definition and the fundamental properties of Stone-Čech compactification the reader is referred to [3, Chapter 6].

THEOREM 2. *E has property* (K_0) *if and only if* $E \not\supset \beta N$. *N is a natural number space with discrete topology and* βN *is a Stone-Čech compactification of* N.

2. Some lemmas

Before the proof of theorems we shall begin with some lemmas.

LEMMA 1. (Kaplansky)

(i) *A lattice automorphism* T *of* $C(E)$ *induces uniquely a homeomorphism* t *of* E *such that for any* $x_0 \in E$ *and* $f, g \in C(E)$

$$(*) \qquad f(x_0) < g(x_0) \quad \text{implies} \quad f^T(x_0^t) \leqq g^T(x_0^t).$$

Furthermore,

(ii) *If* T *is continuous, then* $f(x_0) = g(x_0)$ *implies* $f^T(x_0^t) = g^T(x_0^t)$.

(iii) *If* T^{-1} *is continuous, then* $f(x_0) < g(x_0)$ *implies* $f^T(x_0^t) < g^T(x_0^t)$.

Proof. (i) was proved in [1] and [2].

(ii) Using the property $(*)$ and the ξ-continuity of $(\xi\mathbf{1})^T(x_0^t)$ for each $x_0 \in E$, we can see $f^T(x_0^t) = (f(x_0)\mathbf{1})^T(x_0^t) = (g(x_0)\mathbf{1})^T(x_0^t) = g^T(x_0^t)$ from $f(x_0) = g(x_0)$.

(iii) is evident from (ii) and the fact that the homeomorphism induced by T^{-1} is the inverse t^{-1} of t.

LEMMA 2. *A lattice automorphism* T *of* $C(E)$ *is continuous if and only if* $(\xi\mathbf{1})^T(x)$ *is a continuous function of* $\xi \in \mathbf{R}$ *for each fixed* $x \in E$.

Proof. The ξ-continuity of $(\xi\mathbf{1})^T(x)$ and the property $(*)$ imply $f^T(x_0^t) = (f(x_0)\mathbf{1})^T(x_0^t)$ for each $x_0 \in E$. Therefore if $\{f_n\}$ converges pointwise to g, then $\{f_n^T\}$ also converges pointwise to g^T, because for each $x_0 \in E$,

$$\lim_{n\to\infty} f_n^T(x_0^t) = \lim_{n\to\infty}(f_n(x_0)\mathbf{1})^T(x_0^t) = (g(x_0)\mathbf{1})^T(x_0^t) = g^T(x_0^t).$$

Moreover if $\{f_n\}$ converges uniformly to g, it follows that

$$g - \varepsilon_n \mathbf{1} \leqq f_n \leqq g + \varepsilon_n \mathbf{1} \qquad (n = 1, 2, \cdots)$$

for some sequence $\{\varepsilon_n\}$ of decreasing positive numbers. Since $\{(g - \varepsilon_n \mathbf{1})^T\}$ and $\{(g + \varepsilon_n \mathbf{1})^T\}$ are monotone and converge pointwise to g^T, they converge uniformly to g^T. Therefore the uniform convergence of $\{f_n^T\}$ follows from $(g - \varepsilon_n \mathbf{1})^T \leqq f_n^T \leqq (g + \varepsilon_n \mathbf{1})^T$ $(n = 1, 2, \cdots)$.

3. Proof of Theorem 1

1. $E \in (K_1) \Rightarrow E \neq \beta U$ *for any dense open* F_σ*-subset* $U \subset E$, $U \neq E$.

If we assume the existence of a dense open F_σ-subset $U_0 \subset E$, $U \neq E$, such that $\beta U_0 = E$, then we can construct a lattice automorphism T of $C(E)$ such that T is discontinuous and T^{-1} is continuous. The method is due to a slight generalization of Kaplansky's example in [2].

For U_0 we can find a nonnegative $f_0 \in C(E)$ such that the zero-set of f_0 coincides with the complement U_0^c of U_0: $\{x \mid f_0(x) = 0\} = U_0^c \neq \emptyset$. Using

f_0 we may define the following mapping T from $C(E)$ to $C(E)$. For all $f \in C(E)$ and $x \in U_0$ we put

$$\begin{aligned} f^T(x) &= f(x) && \text{if } f(x) \leqq 0, \\ &= f(x)/f_0(x) && \text{if } 0 \leqq f(x) \leqq f_0(x), \\ &= f(x) - f_0(x) + 1 && \text{if } f_0(x) \leqq f(x). \end{aligned}$$

Then we have $|f^T(x)| \leqq |f(x)| + 1$ $(x \in U_0)$, and obviously f^T is a bounded continuous function on U_0. Therefore, by the assumption $\beta U_0 = E$, f^T can be extended continuously to a function on E. That unique continuous extension of f^T may be denoted by the same notation f^T. The inverse mapping T^{-1} is defined by the following: For all $g \in C(E)$ and $x \in E$

$$\begin{aligned} g^{T^{-1}}(x) &= g(x) && \text{if } g(x) \leqq 0, \\ &= f_0(x)g(x) && \text{if } 0 \leqq g(x) \leqq 1, \\ &= g(x) + f_0(x) - 1 && \text{if } 1 \leqq g(x). \end{aligned}$$

It is almost obvious that T is a lattice automorphism such that $0^T = 0$. T is discontinuous at 0, because $\inf_{\xi>0}(\xi\mathbf{1})^T(x) = 1$ for $x \in U_0^c$. However $(\xi\mathbf{1})^{T^{-1}}(x)$ is a continuous function of $\xi \in \mathbf{R}$, and therefore T^{-1} is continuous by Lemma 2.

2. $E \in (\mathrm{K}_1) \Leftarrow E \neq \beta U$ *for any dense open F_σ-subset $U \subset E$, $U \neq E$.*

To prove this, we can, without loss of generality, assume the existence of a lattice automorphism T satisfying (1) and (2) below, and show that this assumption leads to a contradiction.

(1) T^{-1} is continuous,

(2) $0^T = 0$, and T is discontinuous at 0; furthermore,

$$\operatorname{Max}_{x \in E} \varphi_0(x) = 1, \quad \text{where} \quad \varphi_0(x) = \inf_{\xi>0}(\xi\mathbf{1})^T(x) \quad (x \in E).$$

Let $Z_0 = \{x \mid \varphi_0(x) \geqq 1\}$ and $Z_1 = \{x \mid \varphi_0(x) \geqq \frac{1}{4}\}$; then $Z_1 \supset Z_0 \neq \emptyset$, and obviously Z_0^c and Z_1^c are dense open F_σ-subsets.

For Z_0 and Z_1 we can find subsets F_1 and F_2 such that

(i) $Z_1^c \supset F_1, F_2$,

(ii) $\bar{F}_1 \cap \bar{F}_2 \cap Z_0^c = \emptyset$,

(iii) $\bar{F}_1 \cap \bar{F}_2 \neq \emptyset$.

To show this, we note that $E \neq \beta Z_0^c$ by hypothesis, so that we can find $\psi \in C(Z_0^c)$ such that $0 \leqq \psi(x) \leqq 1$ $(x \in Z_0^c)$ and $\bar{X}_1 \cap \bar{X}_2 \neq \emptyset$ in E for two sets $X_1 = \{x \mid \psi(x) = 1\}$ and $X_2 = \{x \mid \psi(x) = 0\}$ (see [3, Chapter 6]). If we put

$$F_1 = \{x \mid \psi(x) \geqq \tfrac{1}{2}\} \cap Z_1^c, \qquad F_2 = \{x \mid \psi(x) \leqq \tfrac{1}{3}\} \cap Z_1^c,$$

then (i) is obvious, (ii) follows from

$$\bar{F}_1 \cap Z_0^c \subset \{x \mid \psi(x) \geqq \tfrac{1}{2}\}, \qquad \bar{F}_2 \cap Z_0^c \subset \{x \mid \psi(x) \leqq \tfrac{1}{3}\},$$

and the denseness of Z_1^c implies $\bar{F}_i \supset \bar{X}_i$ $(i = 1, 2)$; therefore (iii) is obvious.

Next, if we put, for $n = 1, 2, \cdots$,

$$G_n = \{x \mid ((1/(n+1))\mathbf{1})^T(x) \leqq \tfrac{1}{2} \leqq ((1/n)\mathbf{1})^T(x)\},$$

$$H_n = \{x \mid ((1/(n+1))\mathbf{1})^T(x) \leqq \tfrac{1}{3} \leqq ((1/n)\mathbf{1})^T(x)\},$$

the sequences of compact sets $\{G_n\}$ and $\{H_n\}$ have the following properties:

(iv) $Z_0^c \supset G_n, H_n$ $(n = 1, 2, \cdots)$,

(v) $G_n \cap \overline{(\bigcup_{\nu \geqq n+2} G_\nu)} = \emptyset$, $H_n \cap \overline{(\bigcup_{\nu \geqq n+2} H_\nu)} = \emptyset$ $(n = 1, 2, \cdots)$,

(vi) $\overline{\bigcup_{n \geqq 1}(G_n \cap \bar{F}_1)} \supset \bar{F}_1 \cap \bar{F}_2$, $\overline{\bigcup_{n \geqq 1}(H_n \cap \bar{F}_2)} \supset \bar{F}_1 \cap \bar{F}_2$.

(iv) is obvious. (v) follows from the continuity of T^{-1}, that is, since $\overline{(\bigcup_{\nu \geqq n+2} G_\nu)} \subset \{x \mid \frac{1}{2} \leqq ((1/(n+2))\mathbf{1})^T(x)\}$, we have from Lemma 1(iii)

$$G_n \cap \overline{(\bigcup_{\nu \geqq n+2} G_\nu)}$$
$$\subset \{x \mid ((1/(n+1))\mathbf{1})^T(x) \leqq \tfrac{1}{2}\} \cap \{x \mid \tfrac{1}{2} \leqq ((1/(n+2))\mathbf{1})^T(x)\} = \emptyset$$

We show (vi): $Z_1^c \supset F_1$ implies $\bigcup_{n \geqq 1}(G_n \cap F_1) = \{x \mid \frac{1}{2} \leqq \mathbf{1}^T(x)\} \cap F_1$, and since $Z_0 \supset \bar{F}_1 \cap \bar{F}_2$, $\{x \mid \frac{1}{2} \leqq \mathbf{1}^T(x)\}$ is a neighbourhood of $\bar{F}_1 \cap \bar{F}_2$; therefore $\overline{\{x \mid \frac{1}{2} \leqq \mathbf{1}^T(x)\} \cap F_1} \supset \bar{F}_1 \cap \bar{F}_2$.

Without loss of generality, for a fixed point $p_0 \in \bar{F}_1 \cap \bar{F}_2$ we can assume

$$p_0 \in \overline{\bigcup_{n \geqq 1}(G_{2n} \cap \bar{F}_1)} \cap \overline{\bigcup_{n \geqq 1}(H_{2n} \cap \bar{F}_2)}.$$

If we put

$$C_n = G_{2n} \cap \bar{F}_1, \qquad D_n = H_{2n} \cap \bar{F}_2 \qquad (n = 1, 2, \cdots),$$
$$C = \bigcup_{n \geqq 1} C_n, \qquad D = \bigcup_{n \geqq 1} D_n,$$

then we have

(vii) $p_0 \in \bar{C} \cap \bar{D}$,

(viii) $\frac{1}{2} \leqq ((1/2n)\mathbf{1})^T(x)$ $(x \in C_n)$, $\frac{1}{3} \geqq ((1/(2n+1))\mathbf{1})^T(x)$ $(x \in D_n)$,

(ix) $C_n \cap \overline{(\bigcup_{\nu > n} C_\nu)} = \emptyset$, $D_n \cap \overline{(\bigcup_{\nu > n} D_\nu)} = \emptyset$ $(n = 1, 2, \cdots)$,

(x) $C_n \cap \bar{D} = \emptyset$, $D_n \cap \bar{C} = \emptyset$ $(n = 1, 2, \cdots)$.

(vii) and (viii) are evident from the construction of C and D. (ix) follows from (v). (x) is shown from (ii):

$$C_n \cap \bar{D} \subset C_n \cap \bar{F}_2 \subset Z_0^c \cap \bar{F}_1 \cap \bar{F}_2 = \emptyset.$$

Finally from properties (vii)–(x), we obtain a contradiction as follows: We can define a continuous function h on $(\overline{C \cup D})^{t^{-1}}$ such as

$$h(x) = 1/(2n-1) \quad \text{if} \quad x^t \in C_n,$$
$$= 1/(2n+2) \quad \text{if} \quad x^t, \in D_n,$$
$$= 0 \quad \text{if} \quad x^t \in \overline{C \cup D} - C \cup D.$$

Properties (ix) and (x) guarantee the definition of h and the continuity on $(\overline{C \cup D})^{t^{-1}}$. If $f_0 \in C(E)$ is one of the continuous extensions of h, then from

(viii) and Lemma 1 we have

$$f_0^T(x) \geqq \tfrac{1}{2} \quad (x \in C) \qquad \text{and} \qquad f_0^T(x) \leqq \tfrac{1}{3} \quad (x \in D).$$

Therefore from (vii) we have a contradiction: $f_0^T(p_0) \geqq \frac{1}{2}$ and $f_0^T(p_0) \leqq \frac{1}{3}$.

4. Proof of Theorem 2

1. $E \in (\mathrm{K}_0) \Leftarrow E \not\supset \beta N$.

The proof is done by the same idea as Theorem 1, but it is more simple than Theorem 1. It is sufficient to prove $E \in (\mathrm{K})$ under the assumption $E \not\supset \beta N$.

Let T be a discontinuous lattice automorphism of $C(E)$, and t an induced homeomorphism of E. Without loss of generality we can assume T is discontinuous at 0 and $\mathrm{Max}_{x\in E}\, \varphi_0(x) = \mathrm{Max}_{x \in E}(\inf_{\xi>0}(\xi\mathbf{1})^T(x)) = 1$.

By mathematical induction, we can find a sequence of $x_\nu \in E$ ($\nu = 1, 2, \cdots$) and the sequence n_ν ($\nu = 1, 2, \cdots$) of natural numbers such that

(i) $n_\nu < n_{\nu+1}$ $(\nu = 1, 2, \cdots)$,

(ii) $((1/n_\nu)\mathbf{1})^T(x_\nu) \geqq \frac{1}{2}$ $(\nu = 1, 2, \cdots)$,

(iii) $((1/n_{\nu+1})\mathbf{1})^T(x_\nu) \leqq \frac{1}{3}$ $(\nu = 1, 2, \cdots)$.

Assuming the existence of $x_i \in E$ ($i = 1, 2, \cdots, \nu$) and n_i ($i = 1, 2, \cdots, \nu + 1$), we can define $x_{\nu+1} \in E$ and $n_{\nu+2}$ as follows:

$$U_{\nu+1} = \{x \mid ((1/n_{\nu+1})\mathbf{1})^T(x) \geqq \tfrac{1}{2}\}$$

is a neighbourhood of $\{x \mid \varphi_0(x) = 1\}$ and $V = \{x \mid \varphi_0(x) < \frac{1}{4}\}$ is a dense open set; therefore $U_{\nu+1} \cap V \neq \emptyset$. $x_{\nu+1}$ is defined to be one of the points of $U_{\nu+1} \cap V$. Next, since $x_{\nu+1} \in V$, that is $\inf_{\xi>0}(\xi\mathbf{1})^T(x_{\nu+1}) < \frac{1}{4}$, we can find $n_{\nu+2} > n_{\nu+1}$ such that $((1/n_{\nu+2})\mathbf{1})^T(x_{\nu+1}) \leqq \frac{1}{3}$.

From (ii) and (iii) we see $x_\mu \notin U_\nu$ $(\mu < \nu)$ and $x_\mu \in U_\nu$ $(\mu \geqq \nu)$; therefore $x_\mu \neq x_\nu$ $(\mu \neq \nu)$ and $\{x_\nu \mid \nu = 1, 2, \cdots\} = A$ is a discrete subset of E: A is homeomorphic to N. By the assumption $E \not\supset \beta A$ we shall find infinite subsets B and C of A such that $A = B \cup C$, $B \cap C = \emptyset$, and $\bar{B} \cap \bar{C} \neq \emptyset$.

We shall define a continuous function h on $(\bar{A})^{t^{-1}}$ as follows:

$$\begin{aligned} h(x) &= 2/n_\nu && \text{if} \quad x^t = x_\nu \in B, \\ &= 1/2n_{\nu+1} && \text{if} \quad x^t = x_\nu \in C, \\ &= 0 && \text{if} \quad x^t \in \bar{A} - A. \end{aligned}$$

Let $f_0 \in C(E)$ be one of the continuous extensions of h; then from Lemma 1 and (ii) and (iii) we see that

$$f_0^T(x) \geqq \tfrac{1}{2} \quad (x \in B) \qquad \text{and} \qquad f_0^T(x) \leqq \tfrac{1}{3} \quad (x \in C).$$

Therefore we have a contradiction: $\frac{1}{2} \leqq f_0^T(x_0) \leqq \frac{1}{3}$ for a point $x_0 \in \bar{C} \cap \bar{B}$.

2. $E \in (\mathrm{K}_0) \Rightarrow E \not\supset \beta N$.

If $E \supset F = \beta N$, then we can construct a discontinuous lattice automorphism of $C(F)$ in the same manner as in the first part of the proof of Theorem 1. (It is Kaplansky's example of discontinuous lattice automorphisms in [2].)

References

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