GLOBAL SECTIONS OF TRANSFORMATION GROUPS

BY
ROBERT ELLIS[1]

Let (X, R) be a transformation group with phase space X and phase group R, the additive group of real numbers. Suppose further that (X, R) is minimal. Then what can be said about X? Various answers have been given to this question, see for example [4], [5], [6], [11], [12]. In [12] Schwartzman shows that if in addition X is compact, locally pathwise connected, and if (X, R) admits a global section, then X is the base of a covering space with discrete fibers. This allows him to say something about the homotopy groups of X. In particular he shows that $\pi_1(X) \neq 0$. Recently Chu and Geraghty [5] showed that if X is compact, locally pathwise connected, and if (X, R) is minimal but not totally minimal, then $\pi_1(X) \neq 0$.

The first part of this paper is devoted to generalizing the notion of global section. The above results are considered in a more general setting, and the relation between them is studied. They are generalized to the case where R is replaced by any topological group whose underlying space is R^n.

The second part of the paper is concerned with the following problem. Suppose X is a manifold which is minimal under R; need X be orientable? This question is answered in the negative by exhibiting an action of R on the cartesian product X of the torus with the Klein bottle such that (X, R) is minimal. The flow is constructed by first producing a homeomorphism f of $S^1 \times K$ (the circle cross the Klein bottle) such that $S^1 \times K$ is minimal under the resulting discrete flow, and then R is allowed to act on $(S^1 \times K \times I)/f$ in the standard way; here I is the unit interval and $(S^1 \times K \times I)/f$ is obtained from $S^1 \times K \times I$ by identifying $(z, 0)$ with $(f(z), 1)$ $(z \in S^1 \times K)$. Since f turns out to be isotopic to the identity, the resulting space is homeomorphic to the cartesian product of the torus with the Klein bottle. This flow may be lifted to a flow on the four-torus, T^4. From a result of Auslander and Hahn [1] this flow does not come from a one-parameter subgroup of T^4.

For the remainder of this paper R will denote the additive group of real numbers, and Z the additive group of integers. Let (X, Z) be a transformation group with phase group Z. Then the action of Z on X is completely determined by the homeomorphism f of X onto X, where $f(x) = x1$ $(x \in X)$. For this reason the transformation group (X, Z) will often be denoted (X, f).

For a general discussion of the notions used see [9].

DEFINITION 1. A *left* [*right*] *transformation group* is a pair (G, X) $[(X, G)]$ where X is a topological space and G is a topological group together with a continuous map $(g, x) \to gx$ $[(x, g) \to xg]$ $(x \in X, g \in G)$ from $G \times X \to X$

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$[X \times G \to X]$ such that

$$ex = x, \quad g_1(g_2 x) = (g_1 g_2)x \qquad [xe = x, \quad (xg_1)g_2 = x(g_1 g_2)] \quad (x \in X, g_1, g_2 \in G)$$

where e is the identity element of G.

DEFINITION 2. A *bitransformation group* (G, X, T) is a triple where (G, X) is a left transformation group, (X, T) a right transformation group, and $(gx)t = g(xt)$ $(g \in G,\ x \in X,\ t \in T)$; i.e., the elements of G commute with those of T. I shall not distinguish between the identity element of G and the identity element of T. Both will be denoted e.

When (G, X, T) is a bitransformation group, $[X/G, T)$ is a transformation group in a natural manner.

Let X be a topological space, T a topological group. Then $(X \times T, T)$ will denote the transformation group defined by the action $(x, t)r = (x, tr)$ $(x \in X,\ t, r \in T)$.

DEFINITION 3. Let (X, T) be a transformation group, K a subset of X, and G a topological group. Then K is a *global section of* (X, T) *with respect to* G if there exists an action of G on $K \times T$ such that (i) $(G, K \times T, T)$ is a bitransformation group, (ii) the map $\varphi : K \times T \to X$, where $\varphi(k, t) = kt$ $(k \in K,\ t \in T)$ induces an isomorphism of $((K \times T)/G, T)$ onto (X, T). (The letter φ will retain the above meaning throughout the remainder of the paper.)

The subset K of X is a *global section* of (X, T) if there exists a syndetic subgroup S of T such that K is a global cross section of (X, T) with respect to S.

THEOREM 1. *Let* (X, T) *be a transformation group,* K *a subset of* X*, and* G *a topological group. Then* K *is a global section of* (X, T) *with respect to* G *if and only if*

(1) $KT = X$,

(2) *there exist an action of* G *on* K *and a continuous function* f *from* $G \times K$ *into* T *such that*

- (i) $f(g_1 g_2, k) = f(g_1, g_2 k)f(g_2, k)$ $(g_1, g_2 \in G,\ k \in K)$,
- (ii) $(gk)f(g, k) = k$ $(g \in G,\ k \in K)$,
- (iii) *If* $kt \in K$ *for some* $k \in K$ *and* $t \in T$*, then* $f(g, k)t = e$ *for some* $g \in G$,
- (iv) *Given* N *a neighborhood of the identity of* T *and* $k \in K$ *there exists a neighborhood* V *of* k *such that* $lt \in V$ *for some* $l \in K$ *and* $t \in T$ *implies* $f(g, l)t \in N$ *for some* $g \in G$.

Proof. Assume that K is a global section of (X, T) with respect to G. Then by Definition 3, φ must map $K \times T$ onto X. Thus $KT = X$.

Set $g(k, t) = (A(g, k, t), B(g, k, t))$ $(g \in G,\ k \in K,\ t \in T)$. Then A and B are continuous functions from $G \times K \times T$ into K and T respectively. The relation $[g(k, t)]r = g(k, tr)$ $(g \in G,\ k \in K,\ t, r \in T)$ implies that

$$A(g, k, t) = A(g, k, tr) \quad \text{and} \quad B(g, k, t)r = B(g, k, tr) \quad (g \in G, k \in K, t, r \in T).$$

Set $gk = A(g, k, e)$ and $f(g, k) = B(g, k, e)$. Then $g(k, t) = (gk, f(g, k)t)$ $(g \in G, k \in K, t \in T)$.

The relations

$$e(k, e) = (k, e) \quad \text{and} \quad g_1[g_2(k, e)] = (g_1 g_2)(k, e) \qquad (g_1, g_2 \in G, k \in K)$$

imply that

$$ek = k, \qquad g_1(g_2 k) = (g_1 g_2)k, \qquad f(g_1, g_2 k)f(g_2, k) = f(g_1 g_2, k) \qquad (k \in K, g_1, g_2 \in G).$$

Thus we have defined an action of G on K and a continuous function f from $G \times K$ into T satisfying (i).

Let $k \in K$, $g \in G$. Then $(gk, f(g, k)) = g(k, e) \equiv (k, e) \pmod G$. Hence $(gk)f(g, k) = ke = k$ by Definition 3.

Let $kt \in K$ with $k \in K$ and $t \in T$. Then $kt = (kt)\cdot e$ implies by Definition 3 that $(k, t) \equiv (kt, e) \pmod G$. Hence there exists $g \in G$ with $f(k, g)t = e$.

Let N be a neighborhood of the identity of T, $k \in K$, and suppose (iv) not satisfied. Then there would be nets $(l_\alpha \mid \alpha \in I)$, $(t_\alpha \mid \alpha \in I)$ with $l_\alpha t_\alpha \to k$, $l_\alpha \in K$, $t_\alpha \in T$, and $f(g, l_\alpha)t_\alpha \notin N$ $(g \in G, \alpha \in I)$. Let F be the canonical map of $K \times T$ onto $(K \times T)/G$. Then by assumption $F(l_\alpha, t_\alpha) \to F(k, e)$. Since F is an open mapping, there exist $g \in G$ and $\alpha \in I$ with $g(l_\alpha, t_\alpha) \in K \times N$. But this implies that $f(g, l_\alpha)t_\alpha \in N$, a contradiction.

Now assume that conditions (1) and (2) are satisfied. Set $g(k, t) = (gk, f(g, k)t)$. Then one verifies directly that (G, K, T) is a bitransformation group.

Let $k_1 t_1 = k_2 t_2$ for some $k_1, k_2 \in K$ and $t_1, t_2 \in T$. Then $k_1 t_1 t_2^{-1} = k_2$. Hence $f(g, k_1)t_1 t_2^{-1} = e$ for some $g \in G$ by 2(iii). Then $gk_1 = (gk_1)f(g, k_1)t_1 t_2^{-1} = k_1 t_1 t_2^{-1} = k_2 t_2 t_2^{-1} = k_2$ by 2(ii). Hence $g(k_1, t_1) = (gk_1, f(g, k_1)t_1) = (k_2, t_2)$, i.e., $(k_1, t_1) \equiv (k_2, t_2) \pmod G$.

Conversely, let $g(k_1, t_1) = (k_2, t_2)$ for some $g \in G$, $k_1, k_2 \in K$, $t_1, t_2 \in T$. Then $gk_1 = k_2$ and $f(g, k_1)t_1 = t_2$. Hence $k_2 t_2 = gk, f(g, k_1)t_1 = k_1 t_1$.

Thus φ induces a continuous injective map F of $(K \times T)/G$ into X. Condition 1 implies that this induced map is onto. It remains to be shown that F^{-1} is continuous. Let $(x_\alpha \mid \alpha \in I)$ be a net of elements of X with $x_\alpha \to x \in X$, $k_\alpha t_\alpha = x_\alpha$, $kt = x$, with $k_\alpha, k \in K, t_\alpha, t \in T$, $(\alpha \in I)$. Then $k_\alpha t_\alpha t^{-1} \to k$ whence by 2(iv) (choosing a subnet if necessary) there exist $g_\alpha \in G$ $(\alpha \in I)$ with $f(g_\alpha, k_\alpha)t_\alpha t^{-1} \to e$. This together with the fact that $g_\alpha k_\alpha f(g_\alpha, k_\alpha)t_\alpha t^{-1} = k_\alpha t_\alpha t^{-1} \to k$ shows that $g_\alpha k_\alpha \to k$. Let H be the canonical map of $K \times T$ onto $(K \times T)/G$. Then

$$F^{-1}(x_\alpha) = H(k_\alpha, t_\alpha) = H(g_\alpha k_\alpha, f(g_\alpha, k_\alpha)t_\alpha) \to H(k, t) = F^{-1}(x).$$

The proof is completed.

Remark 1. When $T = R$ and $G = Z$, (1) shows that $f(n, k)$ is determined by the set $[f(1, l) \mid l \in K]$. Thus condition (2) could be stated in terms of a function with domain K rather than $Z \times K$.

Now let (X, R) be a transformation group with compact Hausdorff phase space X, and let K be a closed subset of X such that φ is a local homeomorphism onto. Then Schwartzman [12] shows that K is a section of (X, T) with respect to Z. In this case $nk = \psi^n(k)$ $(n \in Z, k \in K)$ where ψ is the homeomorphism of K into K which sends k into the first point at which K intersects the positive semiorbit of k, and $f(n, k)$ is the negative of the time of n^{th} return of the point k to K.

The most important application of Theorem 1 is to the case where G is a closed syndetic subgroup of T.

THEOREM 2. *Let (X, T) be a transformation group, K a closed subset of X, S a closed syndetic subgroup of T such that* (i) $KT = X$, (ii) $KS \subset K$, (iii) *if $kt \in K$ with $k \in K$ and $t \in T$, then $t \in S$. Then K is a global section of (X, T) with respect to S.*

Proof. Make (S, K) into a left transformation group by setting $sk = ks^{-1}$ $(s \in S, k \in K)$, and set $f(s, k) = s$ $(s \in S, k \in K)$. Then conditions (1), (2)(i), and (2)(ii) of Theorem 1 are immediately verified.

Let $kt \in K$ for some $k \in K$, $t \in T$. Then condition (iii) implies that $t \in S$. Hence condition 2(iii) of Theorem 1 is verified.

If condition 2(iv) of Theorem 1 did not hold, there would be nets $(l_\alpha \mid \alpha \in I)$ and $(t_\alpha \mid \alpha \in I)$ of elements of K and T respectively with $l_\alpha t_\alpha \to k \in K$ and $st_\alpha \notin N$ $(s \in S)$, where N is a neighborhood of the identity. Since S is syndetic, $t_\alpha = s_\alpha c_\alpha$ $(\alpha \in I)$ where $c_\alpha \in C$ $(\alpha \in I)$ and C is a compact subset of T. We may assume that $c_\alpha \to c \in C$. Set $k_\alpha = l_\alpha s_\alpha$. Then $k_\alpha \in K$ $(\alpha \in I)$ and $k_\alpha c_\alpha \to k$. Hence $k_\alpha \to kc^{-1}$, whence $kc^{-1} \in K$. Thus $c^{-1} \in S$. Then $c^{-1}s_\alpha^{-1}t_\alpha \to e$ and $c^{-1}s_\alpha \in S$ $(\alpha \in I)$, a contradiction.

Remark 2. If in addition to the assumptions of Theorem 2, the canonical map of T onto $T/S = [St \mid t \in T]$ admits a local cross section, X is a fiber bundle over T/S with fiber K, and $K \times T$ is a fiber bundle over X with fiber S.

Proof. In this case T is a principal fiber bundle over T/S with structure group S and $X = K \times_S T$ is the associated fiber bundle with fiber K.

Moreover, $K \times T$ is the pullback of the bundle $(T, T/S)$ by means of map $x \to St$ of X onto T/S, where $x = kt$ for some $k \in K$.

In the situation under consideration we have two exact sequences, namely

$$\to \pi_1(X) \to \pi_1(T/S) \to \pi_0(K) \to \pi_0(X),$$

$$\to \pi_1(K \times T) \to \pi_1(X) \to \pi_0(S) \to \pi_0(K \times T).$$

The results of Schwartzman and Chu-Geraghty mentioned in the introduction are obtained by putting enough conditions on the various spaces involved to deduce that $\pi_1(X) \neq 0$.

Notice further that if Q is the image in T of an open neighborhood of $\{S\}$ under the local cross section, then φ restricted to $K \times Q$ is a homeomorphism onto the open subset $KQ = KSQ$ of X. Thus all the local properties of X are transmitted to K and T/S.

The following corollaries to Theorem 2 illustrate the above remarks.

COROLLARY 1. *In addition to the assumption of Theorem 2 let K and T be connected, X locally arcwise connected, $T \to T/S$ admit a local cross section, and let S possess an open proper subgroup S_0. Then $\pi_1(X) \neq 0$. (Note that the conditions on S and T/S are satisfied if S is a discrete nontrivial subgroup of T.)*

Proof. Let S act on $S/S_0 = [S_0 s \mid s \in S]$ on the right. Then $(T/S_0, T/S)$ may be identified with the fiber bundle with fiber S/S_0 associated with the principal bundle $(T, T/S)$. Thus we have an exact sequence

$$\pi_1(T/S) \to \pi_0(S/S_0) \to \pi_0(T/S_0).$$

By the preceding remarks T/S is locally pathwise connected, whence so is T/S_0 since S/S_0 is discrete. Now T is connected. Hence T/S_0 is connected. This implies that $\pi_0(T/S_0) = 0$. Since S/S_0 is discrete and does not reduce to a single point, $\pi_0(S/S_0) \neq 0$. Hence $\pi_1(T/S) \neq 0$.

Now consider the exact sequence $\pi_1(X) \to \pi_1(T/S) \to \pi_0(K)$ associated with the fiber bundle $(X, T/S)$. Since K is connected and locally pathwise connected, $\pi_0(K) = 0$. Hence $\pi_1(X) \neq 0$.

DEFINITION 4. Let T be a topological group. Then T is *syndetically simple* if T is connected and every proper syndetic subgroup is disconnected.

COROLLARY 2. *Let (X, T) be minimal with locally pathwise connected phase space X and syndetically simple Lie group T. Suppose there exist K, a closed connected proper subset of X, and H a syndetic invariant proper subgroup of T such that K is invariant and minimal under H. Then $\pi_1(X) \neq 0$.*

Proof. Let $S = [t \mid Kt \subset K]$. Since $H \subset S$, S is a closed syndetic subset of T. Moreover S is a semigroup. By [9, 2.06] S is a subgroup of T. Let $T = SC$, where C is a compact subset of T. Then $X = \overline{KT} = \overline{KSC} = \overline{KC} = KC \subset KT$. Now let $k \in K$, $t \in T$ with $kt \in K$. Then $Kt = k\overline{H}t = \overline{kHt} = \overline{ktH} \subset \overline{KH} = \bar{K} = K$. Thus $t \in S$. Hence the hypotheses of Theorem 2 are satisfied.

Since T is a Lie group, $T \to T/S$ admits a local cross section. The exact sequence $\pi_1(T/S) \to \pi_0(S) \to \pi_0(T)$ of the bundle $(T, T/S)$ shows that $\pi_1(T/S) \neq 0$. Again as in Corollary 1, $\pi_0(K) = 0$. Then the exact sequence $\pi_1(X) \to \pi_1(T/S) \to \pi_0(K)$ of the bundle $(X, T/S)$ shows that $\pi_1(X) \neq 0$.

Remark 3. Let T be a connected Lie group whose only compact subgroup is the identity. Then T is syndetically simple. Note that in this case T is homeomorphic to R^n for some positive integer n.

Proof. Let S be a closed, connected syndetic subgroup of T. Then T/S is a compact manifold on which T operates transitively. The stability group at the point $\{S\}$ of T/S is S itself. Hence by [10] there exists a compact subgroup C of T which operates transitively on T/S. By hypothesis $C = e$. Hence $S = T$.

THEOREM 3. *Let (X, T) be a minimal set with compact Hausdorff locally pathwise connected phase space X and syndetically simple Lie phase group T; let H be a syndetic invariant subgroup of T, $x \in X$ with $\overline{xH} = K \neq X$. Then K and all its components are global sections of (X, T), and $\pi_1(X) \neq 0$.*

Proof. Let $S = [t \mid Kt \subset K]$. Then S is a closed, proper, syndetic subgroup of T, and K is a global section of (X, T) with respect to S as in Corollary 2 because K is minimal under H [9]. Now K is locally pathwise connected. Hence K has only finitely many components, because it is compact. Let L be a component of K and $G = [s \mid s \in S$ and $Ls \subset L]$. Then G is a closed syndetic subgroup of S. Thus G is a syndetic subgroup of T.

Let $l \in L$, $t \in T$ with $lt \in L$. Then $Kt = \overline{lH}t = \overline{lHt} = \overline{ltH} = \overline{LH} \subset K$. Hence $t \in S$. Since $Lt \cap L \neq \emptyset$ and L is a component of K, $Lt \subset L$; i.e., $t \in G$. The proof is now completed as in Corollary 2.

Remark 4. Let (X, T) be a minimal set with compact Hausdorff locally pathwise connected phase space X and syndetically simple Lie group T. Then we may paraphrase the conclusion of Theorem 3 by saying that a sufficient condition for the existence of a global section is that (X, T) not be totally minimal [9].

Let T be abelian and E the equicontinuous structure relation [8]. Then X/E is a topological group called the structure group of (X, T) [8]. Then (X, T) is totally minimal if and only if $X/E \neq [e]$.

Suppose further that $E = P$, the proximal relation [8]. Then $E \neq X \times X$ because $(x, xt) \notin P$ if $t \neq e$. Thus in this case $X/E \neq [e]$, and (X, T) admits a section. For conditions under which $E = P$ see [3].

DEFINITION 5. Theorem 1 shows that if K is a global section of (X, T) with respect to G, then G acts on K. Let (G, K) and (X, T) be transformation groups. Then (G, K) is a *global section* of (X, T) if there exists a subset L of X such that L is a global section of (X, T) with respect to G and (G, K) is isomorphic to (G, L) where (G, L) is the transformation group determined in Theorem 1.

THEOREM 4. *Let (G, K) be a transformation group with phase group G and phase space K; let T be a topological group and f a continuous function from $G \times K$ to T such that*

(1) $f(g_1 g_2, k) = f(g_1, g_2 k)f(g_2, k) \quad (g_1, g_2 \in G, k \in K)$,

(2) *if $g_\alpha k_\alpha \to k$ and $f(g_\alpha, k_\alpha) \to e$, then $k_\alpha \to k$ (where (g_α) and (k_α)* are *nets of elements in G and K respectively and $k \in K$).*

Then there exists a transformation group (X, T) of which (G, K) is a global section.

Proof. Set $g(k, t) = (gk, f(g, k)t)$ $(g \in G, k \in K, t \in T)$. Then condition (1) ensures that $(G, K \times T, T)$ is a bitransformation group.

Let F be the canonical map of $K \times T$ onto $(K \times T)/S$. Set $X = (K \times T)/S$ and $L = F(K \times e)$. If $F(k, e) = F(l, e)$ for some

$k, l \in K$, then $gk = l$ and $f(g, k) = e$ for some $g \in G$. Condition (2) then implies that $k = l$. Hence the map $k \to F(k, e)$ is a bijective continuous map of K onto L. Now let $F(k_\alpha, e) \to F(k, e)$ where (k_α) is a net of elements of K and $k \in K$. Then condition (2) implies that $k_\alpha \to k$. Hence K is homeomorphic to L.

Now set $gl = F(gk, e)$ and $h(g, l) = f(g, k)$ $(g \in G, l \in L)$, where $l = F(k, e)$. Then (G, K) is isomorphic to (G, L), and Theorem 1 shows that L is a global section of (X, T) with respect to G.

We will identify K and L.

Remark 5. As with Theorem 1 we are mainly interested in the case where G is a subgroup of T. In this case the function $f(g, k) = g$ $(g \in G, k \in K)$ defines a transformation group called the *canonical transformation group built on (G, K) and T.*

THEOREM 5. *Let (S, K) be a transformation group with Hausdorff phase space K and phase group S, which is a closed syndetic subgroup of the topological group T. Then the canonical transformation group (X, T) built on (S, K) and T has a Hausdorff phase space, X. If K is compact, then so is X.*

Proof. Let F be the canonical map of $K \times T$ onto X, (k_α), (t_α) nets of elements of K and T respectively such that

$$F(k_\alpha, t_\alpha) \to F(k, t) \quad \text{and} \quad F(k_\alpha, t_\alpha) \to F(l, r)$$

for some $k, l \in K$, $t, r \in T$. Then we may assume (taking subnets if necessary) that there exist nets (s_α), (p_α) of elements of S such that

$$s_\alpha(k_\alpha, t_\alpha) \to (k, t) \quad \text{and} \quad p_\alpha(k_\alpha, t_\alpha) \to (l, r).$$

Then $s_\alpha t_\alpha \to t$ and $p_\alpha t_\alpha \to r$ imply that $p_\alpha s_\alpha^{-1} \to rt^{-1}$. Then $rt^{-1} \in S$, and $rt^{-1}k = \lim p_\alpha s_\alpha^{-1} s_\alpha k_\alpha = \lim p_\alpha k_\alpha = l$ (since K is Hausdorff). Hence $rt^{-1}(k, t) = (l, r)$, whence $F(k, t) = F(l, r)$ and X is Hausdorff.

Let K be compact. Let M be a compact subset of T such that $T = SM$. Let $t \in T$ and $k \in K$. Then $t = sm$ for some $s \in S$, $m \in M$, and $f(k, t) = F(s^{-1}k, m)$. Thus $X = F(K \times T) = F(K \times M)$ is compact.

Remark 6. Under the conditions stated in Remark 1 or in [12, Theorem 1] the global section therein obtained is related to the canonical section built on K in the following manner.

Let $g(k, t) = (n + 1 - t)f(-n, k) + (t - n)f(-n - 1, k)$ where $k \in K$, $t \in R$ with $n \leqq t \leqq n + 1$ and

$$[f(-n - 1, k) - f(-n, k)]h(k, r) = (r + f(-n, k))(n + 1) - (r + f(-n - 1, k))n$$

$(k \in K, r \in R$ with $-f(-n - 1, k) \leqq r \leqq -f(-n, k))$. Then h is well defined because in the case under consideration the maps $n \to f(n, k)$ of Z into R are strictly decreasing functions $(k \in K)$ such that $f(n, k) \to \pm\infty$ as $n \to \mp\infty$ $(k \in K)$.

Then the map F of $K \times R$ into $K \times R$ such that $F(k, t) = (k, -g(k, t))$ $(k \in K, t \in R)$ is a homeomorphism onto, its inverse being the map

$$(k, t) \rightarrow (k, -h(k, t)) \qquad (k \in K, t \in R).$$

Let $(Z, K \times R)_1$ $[(Z, K \times R)_2]$ be the transformation group with phase space $K \times R$ and phase group Z where the action of Z is given by

$$n(k, t) = (nk, n + t) \qquad [n(k, t) = (nk, f(n, k) + t)] \quad (n \in Z, k \in K, t \in R).$$

Let $k \in K$, $t \in R$, $m \in Z$. Let $n \leqq t \leqq n + 1$ for some $n \in Z$. Then

$$f(mk, m + t) = (mk, -g(mk, m + t)).$$

Now

$$g(mk, m + t) = (n + 1 - t)f(-n - m, mk) + (t - n)f(-n - m - 1, mk)$$

because $n + m \leqq m + t \leqq n + m + 1$. Furthermore

$$f(-n - m, mk) = f(-n, mk) + f(-m, mk)$$

and

$$f(-n - m - 1, mk) = f(-n - 1, mk) + f(-m, mk)$$

by relation (1) of Theorem 1. Also $f(-m, mk) + f(m, k) = f(0, k) = 0$. Hence

$$F(mk, m + t) = (mk, f(m, k) - g(k, t)).$$

Thus F is an isomorphism of the transformation group $(Z, K \times R)_1$ onto $(Z, K \times R)_2$ and therefore induces a homeomorphism of the canonical transformation group built on (Z, K) and R onto the original transformation group (X, R). Hence, when only topological considerations are involved, we may assume that the transformation group (X, R) is the canonical one built on (Z, K).

In what follows we adhere to the notation of [2]; the coefficient group for cohomology is arbitrary. For additional information concerning the concepts involved see [6].

In the remainder of this paper all the phase spaces involved are assumed to be locally compact Hausdorff.

Definition 6. Let (G, Y, T) be a bitransformation group. Then (G, Y, T) is *locally n-coherent* if given $y \in Y$ there exists an open neighborhood V of y such that

$$(*) \qquad \begin{array}{ccc} & & H_c^n(V) \\ & \nearrow & \\ H_c^n(V \cap gVt) & & g^* \Big\uparrow t^* \\ & \searrow & \\ & & H_c^n(gVt) \end{array}$$

is commutative for all $g \in G$ and $t \in T$. A diagram such as $(*)$ will be denoted $(V \cap gVT, V, gVT)$.

THEOREM 6. *Let (G, Y, T) be a bitransformation group such that Y/G is locally compact Hausdorff,* dim $Y \leqq n$, *and the canonical map F of Y onto Y/G is a local homeomorphism. Then (G, Y, T) is locally n-coherent if and only if $(Y/G, T)$ is locally n-coherent.*

Proof. Assume $(Y/G, T)$ locally n-coherent. Let $y \in Y$. Pick N an open neighborhood of y such that F is a homeomorphism of N onto $F(N) = U$ where U is an n-coherent open subset of Y/G. Let $g \in G$, $t \in T$. Then F maps $N \cap gNt$ homeomorphically onto an open subset W of $U \cap Ut$. Consider the commutative diagram

$$\begin{array}{ccccc} H_c^n(W) \to H_c^n(Ut) & \xrightarrow{t^*} & H_c^n(U) \leftarrow H_c^n(W) \\ F^*\downarrow \qquad F^*\downarrow & & F^*\downarrow \qquad F^*\downarrow \\ H_c^n(N \cap gNt) \to H_c^n(gNt) & \xrightarrow{g^*,\, t^*} & H_c^n(N) \leftarrow H_c^n(N \cap gNt), \end{array}$$

and let $u \in H_c^n(N \cap gNt)$. Then there exists $v \in H_c^n(W)$ with $F^*v = u$. Now $(v \mid Ut)t^* = v \mid U$ since U is n-coherent. Thus

$$\begin{aligned} g^*(u \mid gNt)t^* &= g^*(F^*v \mid gNt)t^* = g^*(F^*(v \mid Ut))t^* \\ &= F^*((v \mid Ut)t^*) = F^*(v \mid U) = F^*v \mid N = u \mid N. \end{aligned}$$

Thus $(N \cap gNt, N, gNt)$ is commutative.

Now assume that (G, Y, T) is locally n-coherent. Let N and U be as above except that now N is n-coherent. I must show that this implies that U is n-coherent.

Let $t \in T$. For $g \in G$ set $W_g = F(N \cap gNt)$. Let $v \in H_c^n(W_g)$. Then $F^*v = u \in H_c^n(N \cap gNt)$. By assumption $g^*(u \mid gNt)t^* = u \mid N$. Then $F^*((v \mid Ut)t^*] = F^*(v \mid U)$, whence

$$(v \mid Ut)t^* = v \mid U.$$

Thus (W_g, U, Ut) is commutative for all $g \in G$. Since dim $Y \leqq n$, a simple inductive argument then shows that $(\bigcup[W_g \mid g \in A], U, Ut)$ is commutative for all finite subsets A of G. Since

$$H_c^n(\bigcup[W_g \mid g \in G]) = \text{ind lim } H_c^n(\bigcup[W_g \mid g \in A])$$

where ind lim is taken over the finite subsets of G, $(\bigcup[W_g \mid g \in G], U, Ut)$ is commutative. Now $\bigcup[W_g \mid g \in G] = U \cap Ut$. The proof is completed.

THEOREM 7. *Let (Z, K) be a transformation group with locally compact Hausdorff phase space K with* dim $K \leqq n - 1$; *let (X, R) be the canonical transformation group built on (Z, K) and R. Then X is locally compact Hausdorff,* dim $X \leqq n$, *and (X, R) is locally n-coherent if and only if (Z, K) is locally $(n - 1)$-coherent.*

Proof. Let F be the canonical map of $K \times R$ onto X. If we identify

K with $F(K)$, then Z and K satisfy the assumptions of Theorem 2. Hence F is a local homeomorphism, and X is locally compact Hausdorff by Theorem 5.

Let V be an open subset of K, J an open interval of R, $n \in Z$, and $r \in R$. Then

$$(V \times J) \cap n(V \times J)r = (V \cap nV) \times J \cap (n + r + J)$$

and

$$H_c^n(V \times J) = H_c^{n-1}(V) \otimes H_c^1(J),$$

with this latter isomorphism commuting with maps, show that $(Z, K \times R, R)$ is locally n-coherent if and only if (Z, K) is locally $(n - 1)$-coherent. Theorem 7 now follows from Theorem 6.

COROLLARY 1. *Let (X, R) be minimal and locally n-coherent where X is compact Hausdorff with* dim $X = n$; *let K be a closed global section of (X, R) with respect to Z, such that φ is a local homeomorphism. Then $H_c^{n-1}(X) \neq 0$.*

Proof. By [6], X is locally connected; hence so is K. Since each component of K yields a global section with the desired properties [12], we may assume K connected.

Let $(Z, K \times R, R)$ be the bitransformation group where $n(k, r) = (nk, n + r)$ $(n \in Z,\ k \in K,\ r \in R)$, and let $Y = (K \times R)/Z$. Then Y is homeomorphic to X (Remark 5), and $lr \in L$ with $l \in L$ and $r \in R$ implies $r \in Z$, where L is the image of $K \times 0$ under the canonical map.

Let U be an open connected subset of X. Then the canonical map

$$H_c^n(U) \to H_c^n(X)$$

is an isomorphism onto, and $H_c^n(X) \neq 0$ [6]. Let $J = (-\frac{1}{4}, \frac{1}{4})$. Then $K \times J$ is homeomorphic to LJ, an open connected subset of Y. Hence $0 \neq H_c^{n-1}(K) = H_c^{n-1}(L)$. Then $Y - L = L\cdot(0, 1)$ is connected. The exactness of the sequence

$$H_c^{n-1}(Y) \to H_c^{n-1}(L) \to H_c^n(Y - L) \xrightarrow{j} H_c^n(Y)$$

together with the facts that j is an isomorphism and $H_c^{n-1}(L) \neq 0$ shows that $H_c^{n-1}(Y) \neq 0$. The proof is completed.

COROLLARY 2. *Let (Z, K) be minimal, where K is a compact connected $(n - 1)$-dimensional manifold; let (X, R) be the canonical transformation group built on (Z, K) and R. Then X is a compact n-dimensional manifold, (X, R) is minimal, and X is orientable if and only if K is orientable and the map $k \to 1k$ $(k \in K)$ of K into K is orientation-preserving.*

Proof. Theorem 5 implies that X is compact Hausdorff. Since φ is a local homeomorphism, X is an n-dimensional manifold.

If we consider K as a subset of X, it is a global section of (X, R) with respect to Z. Let $x \in X$. Then $xr \in K$ for some $r \in R$. Hence $K \subset \overline{xR}$. Thus $X = KR \subset \overline{xR}$ and (X, R) is minimal.

The manifold X is orientable if and only if X is locally n-coherent [6]. By Theorem 7, X is locally n-coherent if and only if K is locally $(n - 1)$-coherent. Finally K is locally $(n - 1)$-coherent if and only if K is orientable and Z acts trivially on $H_c^{n-1}(K)$ [6]. The proof is completed.

The remainder of this article is devoted to the construction of a compact connected nonorientable manifold M and a homeomorphism f of M onto M such that (f, M) is minimal. This result together with Corollary 2 above allows one to construct a nonorientable manifold N together with an action of R on N such that (N, R) is minimal.

The following notations will be used throughout the remainder of the paper: (Z, K) is a minimal, distal [9] transformation group such that X is infinite and compact Hausdorff. This implies in particular that if $nx = x$ for some $n \in Z$ and $x \in X$, then $n = 0$.

G will denote the topological group whose underlying space is $R \times C$ and $(r, \alpha)(s, \beta) = (r + s, \alpha + e^{\pi i r}\beta)$ $(r, s \in R, \alpha, \beta \in C)$ where C is the additive group of complex numbers.

LEMMA 1. *Let $H = [(m, n + ir) \mid m, n \in Z, r \in R]$. Then H is a closed subgroup of G and $G/H = [Hg \mid g \in G]$ is homeomorphic to the Klein bottle.*

Proof. One verifies directly that H is a closed subgroup of G. Let F be the canonical map of G onto G/H, and let

$$D = [(x, y + i0) \mid x, y \in R, \quad 0 \leqq x, y \leqq 1].$$

Then D is a closed subset of G, and F maps D onto G/H. An examination of the equivalence relation induced by F and G/H on D shows that G/H is indeed homeomorphic to the Klein bottle.

LEMMA 2. *The transformation group $(G/H, G)$ is distal.*

Proof. Let $Hag_n \to Hc$ and $Hbg_n \to Hc$ where $a, b, c, g_n \in G$. Then there exist sequences (h_n), (l_n) in H with $h_n ag_n \to c$ and $l_n bg_n \to c$. Then $h_n ab^{-1}l_n^{-1} \to e$. Let

$$h_n = (p_n, q_n + ir_n)l_n^{-1} = (j_n, k_n + is_n), \qquad ab^{-1} = (t, u + iv)$$

where $p_n, q_n, j_n, k_n \in Z$, $s_n, r_n, t, u, v \in R$. Then

$$h_n ab^{-1}l_n$$
$$= (p_n + t + j_n, q_n + ir_n + (u + iv) \exp(\pi i p_n) + (k_n + is_n) \exp(\pi_i(p_n + t))).$$

Since $p_n + r + j_n \to 0$ and $p_n, j_n \in Z$, $t \in Z$. Hence $\exp(\pi i(p_n + t)) = \pm 1$, and since

$$q_n + u \exp(\pi i p_n) + k_n \exp(\pi i(p_n + t)) \to 0 \quad \text{with} \quad q_n,$$

$k_n \exp(\pi i(p_n + t)) \in Z$, $u \in Z$. Thus $ab^{-1} \in H$ and $Ha = Hb$. The proof is completed.

LEMMA 3. *Let f be a continuous function from X into G, let $\bar{f}$ be the map of $X \times G/H$ into $X \times G/H$ such that $\bar{f}(x, Ha) = (1 \cdot x, Haf(x))$ $(x \in X, a \in G)$. Then $\bar{f}$ is a homeomorphism onto, and*

$$\bar{f}^n(x, Ha) = (nx, Haf(x) \cdot f(1x) \cdots f((n-1)x))\bar{f}^{-n}(x, Ha)$$
$$= ((-n)x, Ha(f(-1x))^{-1}(f(-2x))^{-1} \cdots (f(-nx))^{-1})$$

$(x \in X, a \in G, n \in Z$ with $n \geqq 1)$.

Proof. Let $\bar{f}(x, Ha) = \bar{f}(y, Hb)$ with $x, y \in X$, $a, b \in G$. Then $1x = 1y$, whence $x = y$. Also $Haf(x) = Hbf(y)$ whence $Ha = Hb$. Thus $\bar{f}$ is injective.

Let $y \in X$, $b \in G$. Then set $x = (-1)y$ and $a = b(f(x))^{-1}$. Then $\bar{f}(x, Ha) = (y, Hb)$. Thus f is surjective. Since

$$\bar{f}^{-1}(x, Ha) = ((-1)x, Haf(-1x)^{-1}) \qquad (x \in X, a \in G),$$

f is a homeomorphism.

The remainder of Lemma 3 follows directly by induction.

Let $C(X, G)$ denote the space of continuous functions from X to G provided with the topology of uniform convergence. Let $f \in C(X, G)$. Then $(\bar{f}, X \times G/H)$ will denote the transformation group determined by the homeomorphism $\bar{f}$.

LEMMA 4. *Let $f \in C(X, G)$. Then the transformation group $(\bar{f}, X \times G/H)$ is distal.*

Proof. Let $\bar{f}^{n_\alpha}(x, Ha) \to (z, Hc)$ and $\bar{f}^{n_\alpha}(y, Hb) \to (z, Hc)$ for some $x, y, z \in X$ and $a, b, c \in G$. Then $n_\alpha x \to z$ and $n_\alpha y \to z$. Since (Z, X) is distal, $x = y$. Then $HaF^{n_\alpha}(x) \to Hc$ and $HbF^{n_\alpha}(x) \to Hc$ where $F^{n_\alpha}(x)$ is the appropriate element of G given by Lemma 3. Hence $Ha = Hb$ by Lemma 2. The proof is completed.

For the remainder of the paper x_0 will denote a fixed element of X. Let $f \in C(X, G)$. Then $0(f)$ will denote the set $[\bar{f}^n(x_0, H) \mid n \in Z]$.

LEMMA 5. *Let U be open in X, V open in G/H, and let*

$$A(U, V) = [f \mid f \in C(X, G) \text{ with } 0(f) \cap U \times V \neq \emptyset].$$

Then $A(U, V)$ is an open subset of $C(X, G)$.

Proof. Let $f \in A(U, V)$. Then there exists $n \in Z$ with $\bar{f}^n(x_0, H) \in U \times V$; i.e, $nx_0 \in U$ and $HF^n(x_0) \in V$ where $F^n(x_0)$ is given by Lemma 3. If $p \in C(X, G)$ is close to f, then $P^n(x_0)$, the corresponding product for p, is close to $F^n(x_0)$ for fixed n. Hence $p \in A(U, V)$ if p is close to f.

The following lemmas are aimed at proving that $A(U, V)$ is dense in $C(X, G)$.

Let $a, b, \in R$ with $a > 0 < b$. Then

$I(r, a) = [t \mid t \in R, |r - t| < a]$,

$$\mathrm{Sq}(\alpha, a) = [\gamma \mid \gamma \in C,\ \gamma = \gamma_1 + i\gamma_2 \text{ and } |\alpha_j - \gamma_j| < a,\ j = 1, 2]$$
where $\alpha = \alpha_1 + i\alpha_2$,

$$S(\alpha, a) = [\gamma \mid \gamma \in C,\ |\alpha - \gamma| < a],$$

$$D(r, \alpha; a, b) = I(r, a) \times \mathrm{Sq}(\alpha, b),$$

$$L(r, \alpha; a, b) = I(r, a) \times S(\alpha, b).$$

The proof of the following lemma is straightforward and is omitted.

LEMMA 6. 1. $I(r, a) + I(s, b) \supset I(r + s, a + b)$,
2. $\mathrm{Sq}(\alpha, a) + \mathrm{Sq}(\beta, b) \supset Sq(\alpha + \beta, a + b)$,
3. $\mathrm{Sq}(\alpha, a) \supset S(\alpha, a)$,
4. $S(\alpha, a) \supset \mathrm{Sq}(\alpha, a/\sqrt{2})$,
5. $D(r, \alpha; a, b) \supset L(r, \alpha; a, b) \supset D(r, \alpha; a, b/\sqrt{2})$ *where* $r, s, a, b \in R$, $\alpha, \beta \in C$, *and* $a > 0 < b$.

LEMMA 7. *Let* $t, s, a, b \in R$, $\alpha \in C$, $a, b > 0$. *Then*
1. $(t, 0)L(s, \alpha; a, b) = L(t + s, \alpha e^{\pi i t}; a, b)$, *and*
2. $L(s, \alpha; a, b)(t, 0) = L(t + s, \alpha; a, b)$.

Proof. 1. Let $r \in R$, $\beta \in C$, and let $(r, \beta) \in L(s, \alpha; a, b)$. Then $|r - s| < a$ and $|\alpha - \beta| < b$. Also $(t, 0)(r, \beta) = (t + r, e^{\pi i t}\beta)$. Hence $|t + s - (t + r)| < a$ and $|\alpha e^{\pi i t} - e^{\pi i t}\beta| = |\alpha - \beta| < b$.
Now suppose $(r, \beta) \in L(t + s, \alpha e^{\pi i t}; a, b)$. Then $|(r - t) - s| < a$, and $|\beta e^{-\pi i t} - \alpha| < b$. Hence $(r - t, \beta e^{-\pi i t}) \in L(s, \alpha; a, b)$ and

$$(t, 0)(r - t, \beta e^{-\pi i t}) = (r, \beta).$$

Statement 2 is proved similarly.

LEMMA 8. *Let* $t, s, a, b\ c \in R$ *with* $a, b, c > 0$, *and let* $\alpha, \beta \in C$. *Then*
1. $D(0, \beta; a, c) \cdot D(t, \alpha; a, b) \supset D(t, \alpha + \beta; a, b + c)$.
2. $L(s, \beta; a, b) \cdot L(t, \alpha; a, b) \supset L(t + s, \alpha e^{\pi i s} + \beta; a, \sqrt{2}\, b)$.

Proof. 1. Let $r \in R$, $\gamma \in C$ with $(r, \gamma) \in D(t, \alpha + \beta; a, b + c)$. Then $r \in I(t, a)$ and $\gamma \in \mathrm{Sq}(\alpha + \beta; b + c)$. By Lemma 6,

$$\gamma = \gamma_1 + \gamma_2 \quad \text{with} \quad \gamma_1 \in \mathrm{Sq}(\alpha, b),\ \gamma_2 \in \mathrm{Sq}(\beta, c).$$

Then $(0, \gamma_2) \in D(0, \beta; a, c)$ and $(r, \gamma_1) \in D(t, \alpha; a, b)$, and

$$(0, \gamma_2)(r, \gamma_1) = (r, \gamma_1 + \gamma_2).$$

2. Set $L_1 = L(s, \beta; a, b)$, $L_2 = (t, \alpha; a, b)$. Then

$$L_1 = (s, 0) \cdot L(0, \beta e^{-\pi i s}; a, b)$$

by Lemma 7. Thus $L_1 \supset (s, 0) \cdot D(0, \beta e^{-\pi i s}; a, b/\sqrt{2})$ by Lemma 6. Hence

$$\begin{aligned} L_1 \cdot L_2 &\supset (s, 0)D(0, \beta e^{-\pi i s}; a, b/\sqrt{2})D(t, \alpha; a, b/\sqrt{2}) \\ &\supset (s, 0)D(t, \alpha + \beta e^{-\pi i s}; a, \sqrt{2}\, b) \supset (s, 0)L(t, \alpha + \beta e^{-\pi i s}; a, \sqrt{2}\, b) \\ &= L(s + t, \alpha e^{\pi i s} + \beta; a, \sqrt{2}\, b) \end{aligned}$$

by 1 above and Lemma 6.

Statement 2 and induction on n yield the following.

COROLLARY 1. *Let* $L_j = L(t_j, \alpha_j; a, b), j = 1, \cdots, 2^n, L = L_1 \cdot L_2 \cdots L_{2^n}$. *Then* $L \supset L(\sum t_j, \alpha; a, (\sqrt{2})^n b)$ *where* $t_j, a, b \in R$ *with* $a > 0 < b$ *and* $\alpha_j, \alpha \in C, j = 1, \cdots, 2^n$.

LEMMA 9. *Let* $b > 1, a > 0, t \in R, \alpha \in C$. *Then* $(n + t, \gamma) \in HL(t, \alpha; a, b)$ *for all* $n \in Z, \gamma \in C$.

Proof. Let $v + iu = \gamma - e^{-\pi i n}\alpha$ and $v = m + y$ with $m \in Z$ and $|y| < 1$. Set $\delta = e^{-\pi i n}y + \alpha$. Then $(t, \delta) \in L(t, \alpha; a, b)$, $(n, m + iu) \in H$, and

$$\begin{aligned}(n, m + iu)(t, \delta) &= (n + t, m + iu + e^{\pi i n}\delta) \\ &= (n + t, m + y + iu + e^{\pi i n}\alpha) = (n + t, \gamma).\end{aligned}$$

LEMMA 10. *Let* $L_j = L(t_j, \alpha_j; a, b)$, $t_j, a, b \in R$, $a > 0 < b$, $\alpha_j \in C$, $j = 1, \cdots$; *let* $na > 1$ *and* $[\sqrt{2})^k b > 1$. *Then* $G = H \cdot L_1 \cdots L_p$ *if* $p > n + 2^k$.

Proof. Let $(r, \gamma) \in G$. Set $t = \sum_{j=1}^{p-n-1} t_j$. Then

$$L_1 \cdots L_{p-n-1} \supset L(t, \alpha; a, c)$$

where $\alpha \in C$ and $c > 1$ by the corollary to Lemma 8.

Since $na > 1$, there exists $(s, \delta) \in L_{p-n} \cdots L_p$ such that $s + t = r + m$ with $m \in Z$ (use 1 of Lemma 6). Let $\beta = \gamma - e^{\pi i(-m)}\delta$. Then

$$(t - m, \beta) \in HL(t, \alpha; a, c)$$

by Lemma 9. Finally $(r, \gamma) = (t - m, \beta)(s, \delta) \in HL_1 \cdots L_p$.

LEMMA 11. *Let* U *be open in* X, V *open in* G/H,

$$A(U, V) = [f \mid 0(f) \cap U \times V \neq \emptyset].$$

Then $A(U, V)$ *is dense in* $C(X, G)$.

Proof. Let $f \in C(X, G]$ and $\varepsilon > 0$. We must find $u \in C(X, G)$ with $u \in A(U, V)$ and $d(u(x), f(x)) < \varepsilon$ $(x \in X)$.

Since X is compact, so is $f(X)$. Hence there exist finitely many $t_j \in R$, $\alpha_j \in C$, $j = 1, \cdots, n$ and $a > 0 < b$ such that $\bigcup_{j=1}^n L(t_j, \alpha_j; a, b) \supset f(X)$ and such that the diameter of $L(t_j, \alpha_j; a, b) < \varepsilon/2$.

Let $ma > 1$ and $(\sqrt{2})^k b > 1$. Since (Z, X) is minimal, there exists $p \in Z$ with $p - 1 > m + 2^k$ and $px_0 \in U$.

Let L_j be that one of the above L's which contains $f(jx_0)$, $j = 0, \cdots, p - 1$. Then $HL_0 \cdots L_{p-1} = G$ by Lemma 10. Since G acts transitively on G/H, there exists $g \in G$ with $Hg \in V$. Let $l_j \in L_j$, $j = 0, \cdots, p - 1$ with $l_0 \cdots l_{p-1} \in Hg$. Then since the points jx_0, $j = 0, \cdots, p - 1$ are all distinct, there exists $u \in C(X, G)$ with $d(u(x), f(x)) < \varepsilon$ $(x \in X)$ and $u(jx) = l_j$, $j = 0, \cdots, p - 1$. Then

$$\bar{u}^p(x_0, H) = (px_0, Hu(x)u(1x) \cdots u((p - 1)x)) = (px_0, Hg) \in U \times V;$$

i.e., $u \in A(U, V)$.

Now suppose further that X is second countable. Then there exists $f \in \bigcap [A(U, V) \mid$ where U runs over a base for X and V runs over a base for $G]$. This implies that the orbit of (x_0, H) under the group generated by $\bar{f}$ is dense. Since $(\bar{f}, X \times G/H)$ is distal, every orbit is dense [7], i.e., $(\bar{f}, X \times G/H)$ is minimal.

If we take X to be the circle group and $1 \cdot x$ to be a rotation of x through one radian $(x \in X)$, X satisfies all the conditions imposed. In this way we have produced a compact connected nonorientable manifold $X \times G/H$ and a homeomorphism $\bar{f}$ of $X \times G/H$ onto itself such that $(\bar{f}, X \times G/H)$ is minimal and distal.

Remark 7. If X is taken to be the circle and $1 \cdot x$ to be a rotation of x through one radian $(x \in X)$, the resulting phase space of the canonical transformation group built on $(\bar{f}, X \times G/H)$ is homeomorphic to the cartesian product of the torus with the Klein bottle.

Proof. It suffices to show that $\bar{f}$ is isotopic to the identity. To this end identify X with the set of complex numbers of modulus 1. Then the map $x \to 1 \cdot x$ is just $x \to e^{i} \cdot x$ $(x \in X)$ (complex multiplication).

Let $F(x, Ha, t) = (e^{it}x, Ha(tfx))$ $(x \in X, a \in G, t \in [0, 1])$. Then F is the desired isotopy.

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The Institute for Advanced Study
Princeton, New Jersey
University of Pennsylvania
Philadelphia, Pennsylvania