A NONLINEAR INTEGRAL OPERATION[1]

BY

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In an earlier paper [4], there was developed a fundamental correspondence between certain additive and multiplicative integration processes, where the integration is directed along intervals in some linearly ordered system, and where the functions involved have their values in a complete normed ring. That development led to an analysis of *linear* integral equations of the form

$$(1) \qquad u(x) = P + (R)\int_x^c V\cdot u,$$

where the right integral is the limit, through successive refinement of subdivisions, of sums of the form $\sum_1^n V(t_{i-1}, t_i)u(t_i)$ [4, p. 155]. This analysis, of the linear case, required only locally bounded variation of the functions involved, thus obviating additional continuity hypotheses of earlier treatments [9], [2], [3], [1] of similar systems. These interrelated treatments are summarized in [4, Sec. 10] (also in [5]).

Now we present an extension to a *nonlinear* situation of the aforementioned fundamental correspondence. This extension leads us to a characterization of solutions u of equations of the form (1), where the linearity hypothesis on the values $V(x, y)$ (of the function V) is replaced by a Lipschitz-type condition. Such a condition, in connection with integrals of the type contemplated here, seems first to have been investigated by J. W. Neuberger [7, p. 542 ff.]. Our results overlap those of Neuberger only in case the underlying system is a linear continuum and all of the functions involved are subjected to additional hypotheses of continuity (see [7, Theorems F and G]).

Throughout this paper, it is to be supposed that S denotes some nondegenerate set, with linear ($\leq$) ordering $\mathcal{O}$; $\{G, +, \|\ \|\}$ denotes a complete normed Abelian group, with zero element 0; and H denotes the class of all functions from G to G to which $\{0, 0\}$ belongs, with identity function 1. As in [4], we let $\mathcal{O}\mathcal{A}^+$ denote the class of all $\mathcal{O}$-additive functions from $S \times S$ to the set of nonnegative real numbers, and $\mathcal{O}\mathcal{M}^+$ denote the class of all $\mathcal{O}$-multiplicative functions from $S \times S$ to the set of real numbers not less than 1. It should be recalled [4, Theorems 2.2 and 4.3] that there is a reversible function $\mathcal{E}^+$ to which the ordered pair $\{\alpha, \mu\}$ belongs only in case one of the following holds:

(a) α is in $\mathcal{O}\mathcal{A}^+$ and $\mu(x, y) = {}_x\prod^y[1 + \alpha]$ for all $\{x, y\}$ in $S \times S$.
(b) μ is in $\mathcal{O}\mathcal{M}^+$ and $\alpha(x, y) = {}_x\sum^y[\mu - 1]$ for all $\{x, y\}$ in $S \times S$.
(c) $\{\alpha, \mu\}$ is in $\mathcal{O}\mathcal{A}^+ \times \mathcal{O}\mathcal{M}^+$ and, for each $\{x, y\}$ in $S \times S$,

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$$\mu(x, y) = 1 + (R) \int_x^y \alpha \cdot \mu(\ , y).$$

The integral equation (1) is to be considered in connection with the class $\mathcal{OA}$, consisting of all functions V from $S \times S$ to H such that

(i) V is $\mathcal{O}$-additive in the sense that, for each $\{x, z\}$ in $S \times S$ and P in G, if $\{x, y, z\}$ is an $\mathcal{O}$-subdivision of $\{x, z\}$ then

$$V(x, y)P + V(y, z)P = V(x, z)P,$$

and (ii) there is a member α of $\mathcal{OA}^+$ such that if $\{x, y\}$ is in $S \times S$ and $\{P, Q\}$ is in $G \times G$ then

$$\| V(x, y)P - V(x, y)Q \| \leq \alpha(x, y) \| P - Q \|.$$

It is found that there is a reversible function $\mathcal{E}$, from the class $\mathcal{OA}$ onto a corresponding class $\mathcal{OM}$, such that a member $\{V, W\}$ of $\mathcal{E}$ provides a solution of (1) in the form $u(x) = W(x, c)P$. Thus, analysis of $\mathcal{E}$ furnishes information about global behavior of solutions of the nonlinear integral equation. The linear case persists, of course, with those members of $\mathcal{OA}$ having their values in the ring of endomorphisms of the group $\{G, +\}$. As in [4, Sec. 5], it is shown that the theory of the seemingly more general equation

$$u(x) = P_1 + (R) \int_x^c V_1 \cdot u + V_2(x, c)P_2 \tag{2}$$

is subsumed in this treatment. There is also a representation theory for canonically ordered semigroups [4, p. 164], and there is an example related to the defining differential equation of the tangent function.

1. Extension of the fundamental correspondence

$\mathcal{OM}$ denotes the class of all functions W from $S \times S$ to H such that

(i) W is $\mathcal{O}$-multiplicative in the sense that, for each $\{x, z\}$ in $S \times S$ and P in G, if $\{x, y, z\}$ is an $\mathcal{O}$-subdivision of $\{x, z\}$ then

$$W(x, y)W(y, z)P = W(x, z)P,$$

and (ii) there is a member μ of $\mathcal{OM}^+$ such that if $\{x, y\}$ is in $S \times S$ and $\{P, Q\}$ is in $G \times G$ then

$$\|[W(x, y) - 1]P - [W(x, y) - 1]Q \| \leq [\mu(x, y) - 1] \| P - Q \|.$$

Remark. It can be shown (but the sequel does not require it) that, as in the purely linear case [4, Lemma 3.2], the latter condition (ii) is equivalent to the condition that, for each $[x, y\}$ in $S \times S$, there exists a number b such that if $\{t_j\}_0^n$ is an $\mathcal{O}$-subdivision of $\{x, y\}$ and $\{P, Q\}$ is in $G \times G$ then

$$\sum_1^n \|[W(t_{j-1}, t_j) - 1]P - [W(t_{j-1}, t_j) - 1]Q \| \leq b \| P - Q \|.$$

A similar result holds with respect to $\mathcal{OA}$ (compare [4, Lemma 3.1]).

Three lemmas, involving sequences of functions from G to G, will be convenient for the sequel. As in [4], we adopt the notational convention that $\prod_1^n A_i$ always means the "left-to-right" continued product $A_1 \cdots A_n$ and that each of $\prod_1^0 A_i$ and $\prod_{n+1}^n A_i$ denotes 1.

LEMMA 1.1. *If $\{A_i\}_1^n$ is a sequence with values in H and $\{a_i\}_1^n$ is a numerical sequence such that, for each $\{P, Q\}$ in $G \times G$ and $i = 1, \cdots, n$,*

$$\|(A_i - 1)P - (A_i - 1)Q\| \leq (a_i - 1)\| P - Q \|,$$

then, for each $\{P, Q\}$ in $G \times G$,

(i) $\|(\prod_1^n A_i - 1)P - (\prod_1^n A_i - 1)Q\| \leq (\prod_1^n a_i - 1)\| P - Q \|,$

and

(ii) $\|(\prod_1^n A_i - 1)P - \sum_1^n [A_i - 1]P\| \leq (\prod_1^n a_i - 1 - \sum_1^n [a_i - 1])\| P \|.$

Proof. Noting that, for each P in G,

$$(\textstyle\prod_1^n A_i - 1)P = (\prod_1^n A_i)P - P = \sum_{j=1}^n (A_j - 1)(\prod_{j+1}^n A_i)P,$$

we see that if each of P and Q is in G then

$$\begin{aligned}
&\|(\textstyle\prod_1^n A_i - 1)P - (\prod_1^n A_i - 1)Q\| \\
&\qquad \leq \textstyle\sum_{j=1}^n \|(A_j - 1)(\prod_{j+1}^n A_i)P - (A_j - 1)(\prod_{j+1}^n A_i)Q\| \\
&\qquad \leq \textstyle\sum_{j=1}^n (a_j - 1)\|(\prod_{j+1}^n A_i)P - (\prod_{j+1}^n A_i)Q\| \\
&\qquad \leq \textstyle\sum_{j=1}^n (a_j - 1)(\prod_{j+1}^n a_i)\| P - Q \| = (\prod_1^n a_i - 1)\| P - Q \|
\end{aligned}$$

which establishes assertion (i). Using (i) we have, for each P in G,

$$\begin{aligned}
&\|(\textstyle\prod_1^n A_i - 1)P - \sum_1^n [A_i - 1]P\| \\
&\quad \leq \textstyle\sum_{j=1}^n \|(A_j - 1)(\prod_{j+1}^n A_i)P - (A_j - 1)P\| \\
&\quad \leq \textstyle\sum_{j=1}^n (a_j - 1)\|(\prod_{j+1}^n A_i)P - P\| \\
&\quad \leq \textstyle\prod_{j=1}^n (a_j - 1)(\prod_{j+1}^n a_i - 1)\| P \| = (\prod_1^n a_i - 1 - \sum_1^n [a_i - 1])\| P \|.
\end{aligned}$$

LEMMA 1.2. *If each of $\{A_i\}_1^n$ and $\{B_i\}_1^n$ is a sequence with values in H, and each of $\{a_i\}_1^n$ and $\{b_i\}_1^n$ is a numerical sequence such that for each $\{P, Q\}$ in $G \times G$ and $i = 1, \cdots, n$,*

$$\| A_i P - A_i Q \| \leq a_i \| P - Q \|, \qquad \| B_i P \| \leq b_i \| P \|,$$

and $\| B_i P - A_i P \| \leq (b_i - a_i)\| P \|$, then for each P in G

$$\|(\textstyle\prod_1^n B_i)P - (\prod_1^n A_i)P\| \leq (\prod_1^n b_i - \prod_1^n a_i)\| P \|.$$

Proof. If $\{P, Q\}$ is in $G \times G$ then

$$\begin{aligned}
\| B_1 Q - A_1 P \| &= \|(B_1 - A_1)Q + A_1 Q - A_1 P \| \\
&\leqq (b_1 - a_1)\| Q \| + a_1 \| Q - P \|;
\end{aligned}$$

hence a simple induction yields, for each $\{P, Q\}$ in $G \times G$,

$$\| (\prod_1^n B_i)Q - (\prod_1^n A_i)P \| \leq (\prod_1^n b_i - \prod_1^n a_i)\| Q \| + (\prod_1^n a_i)\| Q - P \|$$

LEMMA 1.3. *If each of $\{A_i\}_1^n$ and $\{B_i\}_1^n$ is a sequence with values in H and each of $\{a_i\}_1^n$, $\{b_i\}_1^n$ and $\{c_i\}_1^n$ is numerical sequence such that, for each $\{P, Q\}$ in $G \times G$ and $i = 1, \cdots, n$,*

$$\| A_i P - A_i Q \| \leq a_i \| P - Q \|, \qquad \| B_i P \| \leq b_i \| P \|,$$

and $\| A_i B_i P - P \| \leq c_i \| P \|$, then for each P in G

$$\| (\prod_1^n A_i)(\prod_1^n B_{n+1-i})P - P \| \leq \sum_{j=1}^n (\prod_1^{j-1} a_i) c_j (\prod_{n+2-j}^n b_{n+1-i}) \| P \|.$$

Proof. For each P in G, we have

$$\begin{aligned} &\| (\prod_1^n A_i)(\prod_1^n B_{n+1-i})P - P \| \\ &= \| \sum_{j=1}^n (\prod_1^j A_i)(\prod_{n+1-j}^n B_{n+1-i})P - \sum_{j=1}^n (\prod_1^{j-1} A_i)(\prod_{n+2-j}^n B_{n+1-i})P \| \\ &\leq \sum_{j=1}^n (\prod_1^{j-1} a_i) \| A_j B_j (\prod_{n+2-j}^n B_{n+1-i})P - (\prod_{n+2-j}^n B_{n+1-i})P \| \\ &\leq \sum_{j=1}^n (\prod_1^{j-1} a_i) c_j (\prod_{n+2-j}^n b_{n+1-i}) \| P \|. \end{aligned}$$

Concerning the *continuously continued sum* and *continuously continued product* indicated in the following theorem, this definition is used (compare [4, p. 150]): if g is a function from $S \times S$ to G and h is a function from $S \times S$ to H and $\{x, y\}$ is in $S \times S$ and P is in G, then

(i) ${}_x\sum^y g$ denotes a member P_1 of G with the property that, for each positive number c, there is an $\mathcal{O}$-subdivision s of $\{x, y\}$ such that if $\{t_j\}_0^n$ is a refinement of s then $\| P_1 - \sum_t g \| < c$, where $\sum_t g$ denotes the continued sum (in G)

$$\sum_1^n g(t_{j-1}, t_j) = g(t_0, t_1) + \cdots + g(t_{n-1}, t_n),$$

and (ii) ${}_x\prod^y [h]P$ denotes a member P_2 of G with the property that, for each positive number c, there is an $\mathcal{O}$-subdivision s of $\{x, y\}$ such that if $\{t_j\}_0^n$ is a refinement of s then $\| P_2 - \prod_t [h]P \| < c$, where $\prod_t [h]P$ denotes the image of P under the continued product (functional composite)

$$\prod_1^n h(t_{i-1}, t_i) = h(t_0, t_1) \cdots h(t_{n-1}, t_n).$$

THEOREM 1.1. *There is a reversible function $\mathcal{E}$ from $\mathcal{O}\mathcal{A}$ onto $\mathcal{O}\mathcal{M}$ such that the following are equivalent:*

(1) *$\{V, W\}$ belongs to $\mathcal{E}$.*

(2) *W is in $\mathcal{O}\mathcal{M}$ and V is the function defined by the condition that, for each $\{x, y\}$ in $S \times S$ and P in G, $V(x, y)P = {}_x\sum^y [W - 1]P$.*

(3) *V is in $\mathcal{O}\mathcal{A}$ and W is the function defined by the condition that, for each $\{x, y\}$ in $S \times S$ and P in G, $W(x, y)P = {}_x\prod^y [1 + V]P$.*

(4) *$\{V, W\}$ is in $\mathcal{O}\mathcal{A} \times \mathcal{O}\mathcal{M}$ and there is a member $\{\alpha, \mu\}$ of $\mathcal{E}^+$ such that, for each $\{x, y\}$ in $S \times S$ and P in G,*

$$\| W(x, y)P - P - V(x, y)P \| \leq [\mu(x, y) - 1 - \alpha(x, y)] \| P \|.$$

Proof. Suppose W is in $\mathfrak{OM}$, μ is a member of $\mathfrak{OM}^+$ such that

$$\| [W(x,y) - 1]P - [W(x,y) - 1]Q \| \leq [\mu(x,y) - 1] \| P - Q \|$$

for each $\{x, y\}$ in $S \times S$ and $\{P, Q\}$ in $G \times G$, and $\mu = \mathcal{E}^+(\alpha)$. If $\{x, y\}$ is in $S \times S$ and $\{s_i\}_0^n$ is an $\mathfrak{O}$-subdivision of $\{x, y\}$ and P is in G then, by Lemma 1.1 (ii),

$$\begin{aligned}\| [W(x, y) - 1]P - \textstyle\sum_s [W - 1]P \| \\ &= \| [\textstyle\prod_1^n W(s_{i-1}, s_i) - 1]P - \textstyle\sum_1^n [W(s_{i-1}, s_i) - 1)]P \| \\ &\leq \{\textstyle\prod_1^n \mu(s_{i-1}, s_i) - 1 - \textstyle\sum_1^n [\mu(s_{i-1}, s_i) - 1)\} \| P \|;\end{aligned}$$

it follows directly that if s is an $\mathfrak{O}$-subdivision of $\{x, y\}$ and t is a refinement of s and P is in G then

$$\| \textstyle\sum_s [W - 1]P - \textstyle\sum_t [W - 1]P \| \leq \{ \textstyle\sum_s [\mu - 1] - \textstyle\sum_t [\mu - 1]\} \| P \|.$$

Hence, by the completeness of $\{G, +, \| \ \|\}$, there is a function V defined by the condition that $V(x, y)P = {}_x\sum^y [W - 1]P$ for each $\{x, y\}$ in $S \times S$ and P in G. Clearly V is $\mathfrak{O}$-additive and each value of V contains the ordered pair $\{0, 0\}$ in $G \times G$. If $\{x, y\}$ is in $S \times S$ and $\{P, Q\}$ is in $G \times G$ then, since

$$\| \textstyle\sum_s [W - 1]P - \textstyle\sum_s [W - 1]Q \| \leq \textstyle\sum_s [\mu - 1] \| P - Q \|$$

for each $\mathfrak{O}$-subdivision s of $\{x, y\}$, it follows that

$$\| V(x, y)P - V(x, y)Q \| \leq \alpha(x, y) \| P - Q \|,$$

so that V belongs to the class $\mathfrak{OA}$. Moreover, statement (4) holds. Thus we see that (2) implies (4); the converse implication is evident.

Suppose that V is in $\mathfrak{OA}$, α is a member of $\mathfrak{OA}^+$ such that

$$\| V(x, y)P - V(x, y)Q \| \leq \alpha(x, y) \| P - Q \|$$

for each $\{x, y\}$ in $S \times S$ and $\{P, Q\}$ in $G \times G$, and $\mu = \mathcal{E}^+(\alpha)$. If $\{x, y\}$ is in $S \times S$ and $\{s_i\}_0^n$ is an $\mathfrak{O}$-subdivision of $\{x, y\}$ and P is in G then, again by Lemma 1.1(ii),

$$\begin{aligned}\| \textstyle\prod_s [1 + V]P - [1 + V(x, y)]P \| \\ &= \| (\textstyle\prod_1^n [1 + V(s_{i-1}, s_i)] - 1)P - \textstyle\sum_1^n V(s_{i-1}, s_i)P \| \\ &\leq \{\textstyle\prod_1^n [1 + \alpha(s_{i-1}, s_i)] - 1 - \textstyle\sum_1^n \alpha(s_{i-1}, s_i)\} \| P \|;\end{aligned}$$

now, it follows from Lemma 1.2 that if s is an $\mathfrak{O}$-subdivision of $\{x, y\}$ and t is a refinement of s and P is in G then

$$\| \textstyle\prod_t [1 + V]P - \textstyle\prod_s [1 + V]P \| \leq \{\textstyle\prod_t [1 + \alpha] - \textstyle\prod_s [1 + \alpha]\} \| P \|.$$

Hence, by the completeness of $\{G, +, \| \ \|\}$, there is a function W defined by the condition that $W(x, y)P = {}_x\prod^y [1 + V]P$ for each $\{x, y\}$ in $S \times S$ and P in G. Clearly W is $\mathfrak{O}$-multiplicative and each value of W contains the ordered pair $\{0, 0\}$ in $G \times G$. If $\{x, y\}$ is in $S \times S$ and $\{P, Q\}$ is in $G \times G$

then, since by Lemma 1.1(i)

$$\| (\textstyle\prod_s [1 + V] - 1)P - (\prod_s[1 + V] - 1)Q \| \leq (\prod_s [1 + \alpha] - 1)\| P - Q\|$$

for each $\mathcal{O}$-subdivision s of $\{x, y\}$, it follows that

$$\| [W(x, y) - 1]P - [W(x, y) - 1]Q \| \leq [\mu(x, y) - 1] \| P - Q \|$$

so that W belongs to the class $\mathcal{OM}$. Moreover, statement (4) holds. Thus we see that (3) implies (4), which we know is equivalent to (2).

Suppose, now, that (2) is true. As we have seen in the first part of this proof, there exists a member $\{\alpha, \mu\}$ of $\mathcal{E}^+$ such that, for each P and Q in G and $\{x, y\}$ in $S \times S$, both the following hold:

$$\| [1 + V(x, y)]P - [1 + V(x, y)]Q \| \leq [1 + \alpha(x, y)] \| P - Q \|,$$

$$\| W(x, y)P - [1 + V(x, y)]P \| \leq \{\mu(x, y) - [1 + \alpha(x, y)]\}\| P \|.$$

Hence, by Lemma 1.2, if P is in G and $\{x, y\}$ is in $S \times S$ and s is an $\mathcal{O}$-subdivision of $\{x, y\}$ then

$$\| W(x, y)P - \textstyle\prod_s [1 + V]P \| \leq \{\mu(x, y) - \prod_s [1 + \alpha]\}\| P \|,$$

and this implies (3). Theorem 1.1 is now proved.

There emerges from the preceding argument a fact which we record, for convenient reference, as follows:

COROLLARY 1.1. *If $\{V, W\}$ is in $\mathcal{E}$ then there is a member $\{\alpha, \mu\}$ of $\mathcal{E}^+$ such that, for each $\{x, y\}$ in $S \times S$ and $\{P, Q\}$ in $G \times G$,*

(1) $\| V(x, y)P - V(x, y)Q \| \leq \alpha(x, y)\| P - Q \|,$

(2) $\| [W(x, y) - 1]P - [W(x, y) - 1]Q \| \leq [\mu(x, y) - 1] \| P - Q \|$

and (3) $\| W(x, y)P - P - V(x, y)P \| \leq [\mu(x, y) - 1 - \alpha(x, y)] \| P \|.$

COROLLARY 1.2. *If $\{V, W\}$ is in $\mathcal{E}$ and $\{\alpha, \mu\}$ is a member of $\mathcal{E}^+$ as in Corollary 1.1 and $\{c, P\}$ is in $S \times G$ then, for each x in S and each $\mathcal{O}$-subdivision $\{t_j\}_0^n$ of $\{x, c\}$,*

$$\| W(x, c)P - P - \textstyle\sum_1^n V(t_{j-1}, t_j)W(t_j, c)P \|$$
$$\leq \{\mu(x, c) - 1 - \textstyle\sum_1^n \alpha(t_{j-1}, t_j)\mu(t_j, c)\}\| P \|.$$

Proof. Supposing $\{V, W\}$, $\{\alpha, \mu\}$, $\{c, P\}$, x, and t to be as indicated, we see that

$$W(x, c)P - P - \textstyle\sum_1^n V(t_{j-1}, t_j)W(t_j, c)P$$
$$= \textstyle\sum_1^n [W(t_{j-1}, c)P - W(t_j, c)P] - \sum_1^n V(t_{j-1}, t_j)W(t_j, c)P$$
$$= \textstyle\sum_1^n [W(t_{j-1}, t_j)W(t_j, c)P - W(t_j, c)P - V(t_{j-1}, t_j)W(t_j, c)P],$$

and therefore, by Corollary 1.1(3),

$$\| W(x, c)P - P - \textstyle\sum_1^n V(t_{j-1}, t_j)W(t_j, c)P \|$$
$$\leq \textstyle\sum_1^n [\mu(t_{j-1}, t_j) - 1 - \alpha(t_{j-1}, t_j)] \| W(t_j, c)P \|$$
$$\leq \textstyle\sum_1^n [\mu(t_{j-1}, t_j) - 1 - \alpha(t_{j-1}, t_j)]\mu(t_j, c)\| P \|.$$

THEOREM 1.2. *If $\{V, W\}$ is in $\mathcal{E}$ then the following are equivalent*:

(1) *For each $\{x, y\}$ in $S \times S$ and P in G, $W(x, y)W(y, x)P = P$.*

(2) *There is a function γ from $S \times S$ to the nonnegative real numbers such that, for each $\{x, y\}$ in $S \times S$ and P in G, ${}_x\sum^y \gamma = 0$ and*

$$\| [1 + V(x, y)][1 + V(y, x)]P - P \| \leq \gamma(x, y)\| P \|.$$

Proof. Suppose $\{V, W\}$ is in $\mathcal{E}$ and $\{\alpha, \mu\}$ is a member of $\mathcal{E}^+$ as in Corollary 1.1. If (2) is true and $\{x, y\}$ is in $S \times S$ and P is in G and $\{t_j\}_0^n$ is an $\mathcal{O}$-subdivision of $\{x, y\}$ then, by Lemma 1.3,

$$\| \textstyle\prod_1^n[1 + V(t_{i-1}, t_i)]\prod_1^n [1 + V(t_{n+1-i}, t_{n-i})]P - P \|$$
$$\leq \textstyle\sum_{j=1}^n \mu(x, t_{j-1})\gamma(t_{j-1}, t_j)\mu(t_{j-1}, y)\| P \|$$
$$\leq \mu(x, y)\mu(y, x)[\textstyle\sum_t \gamma] \| P \|,$$

so that (1) is true. Suppose, now, that (1) is true and let γ be the function from $S \times S$ defined by

$$\gamma(x, y) = [\mu(x, y) - 1 - \alpha(x, y)]\mu(y, x) + \mu(x, y)[\mu(y, x) - 1 - \alpha(y, x)].$$

Clearly, if $\{x, y\}$ is in $S \times S$ then ${}_x\sum^y \gamma = 0$ and, for each P in G,

$$[1 + V(x,y)][1 + V(y,x)]P - P$$
$$= [1 + V(x,y) - W(x,y)][1 + V(y,x)]P$$
$$+ W(x, y)[1 + V(y, x)]P - W(x, y)W(y, x)P$$

which has norm not exceeding

$$[\mu(x, y) - 1 - \alpha(x, y)] \| [1 + V(y, x)]P \|$$
$$+ \mu(x, y)\| [1 + V(y, x)]P - W(y, x)P \|,$$

and this in turn does not exceed $\gamma(x, y)\| P \|$, so that (2) is true.

2. The first integral equation

$\mathcal{OB}$ denotes the class of all functions U from S to H such that dU belongs to the class $\mathcal{OA}$ (cf. [4, p. 155], and also [7, p. 542]).

LEMMA 2.1. *If c is in S and W is in $\mathcal{OM}$, then $W(\ \ , c)$ is in $\mathcal{OB}$ and there is a member β of $\mathcal{OA}^+$ such that, for each $\{x, y\}$ in $S \times S$ and P in G,*

$$\| W(c, y)P - W(c, x)P \| \leq \beta(x, y)\| P \|.$$

Remark. For the purely linear case [4, Corollary 3.2, p. 154], the latter conclusion implies that $W(c, \ \ )$ is also a member of $\mathcal{OB}$.

Indication of proof. Supposing c to be in S and W in $\mathcal{OM}$, let μ be a member of $\mathcal{OM}^+$ such that, for each $\{x, y\}$ in $S \times S$ and $\{P, Q\}$ in $G \times G$,

$$\| [W(x, y) - 1]P - [W(x, y) - 1]Q \| \le [\mu(x, y) - 1] \| P - Q \|.$$

There are members α and β of $\mathcal{OA}^+$ defined as follows:

(i) if $\{x, y, c\}$ is an $\mathcal{O}$-subdivision of $\{x, c\}$ then

$$\alpha(x, y) = \mu(x, c) - \mu(y, c) \quad \text{and} \quad \beta(x, y) = \mu(c, y) - \mu(c, x),$$

(ii) if $\{x, c, y\}$ is an $\mathcal{O}$-subdivision of $\{x, y\}$ then

$$\alpha(x, y) = \mu(x, c) - 1 + \mu(y, c) - 1$$

and

$$\beta(x, y) = \mu(c, x) - 1 + \mu(c, y) - 1,$$

and (iii) if $\{c, x, y\}$ is an $\mathcal{O}$-subdivision of $\{c, y\}$ then

$$\alpha(x, y) = \mu(y, c) - \mu(x, c) \quad \text{and} \quad \beta(x, y) = \mu(c, y) - \mu(c, x).$$

If $\{P, Q\}$ is in $G \times G$ then, for case (i),

$$\begin{aligned} &\| [W(y, c) - W(x, c)]P - [W(y, c) - W(x, c)Q \| \\ &\qquad = \| [1 - W(x, y)]W(y, c)P - [1 - W(x, y)]W(y, c)Q \| \\ &\qquad \le [\mu(x, y) - 1] \| W(y, c)P - W(y, c)Q \| \le \alpha(x, y) \| P - Q \|, \end{aligned}$$

whereas

$$\| W(c, y)P - W(c, x)P \| \le \mu(c, y) \| P - W(y, x)P \| \le \beta(x, y) \| P \|;$$

cases (ii) and (iii) are readily checked by similar calculation.

If u is a function from S to G, V is a function from $S \times S$ to H, and g is the function defined on $S \times S$ by $g(x, y) = V(x, y)u(y)$, then

$$(R) \int_x^y V \cdot u = {}_x\sum^y g$$

for each $\{x, y\}$ in $S \times S$ (see definition preceding Theorem 1.1; also, compare [4, p. 155]). For each $\{c, P\}$ in $S \times G$,

(i) $\mathfrak{F}(c, P)$ denotes the class of all functions u from S to G such that $u(c) = P$ and there is a member β of $\mathcal{OA}^+$ such that $\| u(y) - u(x) \| \le \beta(x, y)$ for each $\{x, y\}$ in $S \times S$ (i.e., u is of bounded variation on each $\mathcal{O}$-interval of S),

and (ii) $\mathfrak{F}'(c, P)$ denotes the class of all functions u from S to G such that u is the limit, uniformly on each $\mathcal{O}$-interval of S, of an infinite sequence with values in $\mathfrak{F}(c, P)$.

Remark. In case $\{S, \mathcal{O}\}$ is the real line with its usual ordering then, for each $\{c, P\}$ in $S \times G$, $\mathfrak{F}'(c, P)$ is the class of all functions u from S to G such that, for each x in S, each of the limits $u(x-)$ and $u(x+)$ exists (i.e., u is quasi-continuous) and $u(c) = P$.

LEMMA 2.2. *If* $\{c, P\}$ *is in* $S \times G$ *and* u *is in* $\mathfrak{F}'(c, P)$ *then, for each* V *in* $\mathfrak{O}\mathfrak{A}$, *there is a member* w *of* $\mathfrak{F}(c, 0)$ *such that*

$$w(x) = (R) \int_x^c V \cdot u \qquad \textit{for each } x \textit{ in } S.$$

Indication of proof. Suppose $\{c, P\}$ is in $S \times G$, V is in $\mathfrak{O}\mathfrak{A}$, and α is a member of $\mathfrak{O}\mathfrak{A}^+$ such that, for each $\{x, y\}$ in $S \times S$ and $\{P, Q\}$ in $G \times G$,

$$\| V(x, y)P - V(x, y)Q \| \leq \alpha(x, y) \| P - Q \|.$$

If u is in $\mathfrak{F}(c, P)$ and β is a member of $\mathfrak{O}\mathfrak{A}^+$ such that $\| du \| \leq \beta$, then, for each $\{x, y\}$ is $S \times S$ such that $\{x, y, c\}$ is an $\mathfrak{O}$-subdivision of $\{x, c\}$ and each $\mathfrak{O}$-subdivision $\{t_i\}_0^n$ of $\{x, y\}$,

$$\begin{aligned} &\| (R) \textstyle\sum_t V \cdot u - V(x, y)u(y) \| \\ &\qquad = \| \textstyle\sum_1^n V(t_{i-1}, t_i)u(t_i) - \sum_1^n V(t_{i-1}, t_i)u(y) \| \\ &\qquad \leq \textstyle\sum_1^n \| V(t_{i-1}, t_i)u(t_i) - V(t_{i-1}, t_i)u(y) \| \\ &\qquad \leq \textstyle\sum_1^n \alpha(t_{i-1}, t_i)\beta(t_i, y) = (R) \sum_t \alpha \cdot \beta(\ , c) - \alpha(x, y)\beta(y, c); \end{aligned}$$

it follows that, if s is an $\mathfrak{O}$-subdivision of $\{x, c\}$ and t is a refinement of s, then (compare [4, Lemma 4.3, p. 156])

$$\| (R) \textstyle\sum_t V \cdot u - (R) \sum_s V \cdot u \| \leq (R) \sum_t \alpha \cdot \beta(\ , c) - (R) \sum_s \alpha \cdot \beta(\ , c);$$

since $(R)\int^c \alpha \cdot \beta(\ , c)$ exists then, by the completeness of $\{G, +, \| \ \|\}$, the right integral $(R)\int_x V \cdot u$ also exists.

If, now, u_1 is in $\mathfrak{F}(c, P)$ and u_2 is in $\mathfrak{F}'(c, P)$ and x is in S and m is a number such that $\| u_1(y) - u_2(y) \| \leq m$ for all y in S such that $\{x, y, c\}$ is an $\mathfrak{O}$-subdivision of $\{x, c\}$ then, for each $\mathfrak{O}$-subdivision t of $\{x, c\}$,

$$\| (R) \textstyle\sum_t V \cdot u_1 - (R) \sum_t V \cdot u_2 \| \leq (R) \sum_t \| V \cdot u_1 - V \cdot u_2 \| \leq \alpha(x, y)m.$$

Therefore, the integral $(R)\int_x^c V \cdot u$ exists for each u in $\mathfrak{F}'(c, P)$ and each x in S; the function w so defined is clearly a member of $\mathfrak{F}(c, 0)$.

LEMMA 2.3. *If* $\{c, P\}$ *is in* $S \times G$ *and* u_1 *and* u_2 *are in* $\mathfrak{F}'(c, P)$ *then*
(i) *there is a member* β *of* $\mathfrak{O}\mathfrak{A}^+$ *such that, for each* x *in* S,

$$\| u_1(x) - u_2(x) \| \leq \beta(x, c),$$

and (ii) *for each such* β, *if* V *is in* $\mathfrak{O}\mathfrak{A}$ *and* α *is in* $\mathfrak{O}\mathfrak{A}^+$ *and*

$$\| V(x, y)Q_1 - V(x, y)Q_2 \| \leq \alpha(x, y) \| Q_1 - Q_2 \|$$

for each $\{x, y\}$ *in* $S \times S$ *and* $\{Q_1, Q_2\}$ *in* $G \times G$ *then, for each* x *in* S

$$\left\| (R) \int_x^c V \cdot u_1 - (R) \int_x^c V \cdot u_2 \right\| \leq (R) \int_x^c \alpha \cdot \beta(\ , c).$$

Indication of proof. For each x in S, let

$$\beta(x, c) = \beta(c, x) = \text{L.U.B.} \, \| u_1(y) - u_2(y) \|$$

for all y in S such that $\{x, y, c\}$ is an $\mathcal{O}$-subdivision of $\{x, c\}$, and extend this definition so that β is a member of $\mathcal{O}\mathcal{A}^+$. Assertion (ii) is an immediate consequence of relevant estimates on approximating sums.

THEOREM 2. *If* $\{V, W\}$ *is in* $\mathcal{E}$ *and* $\{c, P\}$ *is in* $S \times G$ *and* u *is a function from* S *to* G, *the following are equivalent*:

(1) $u(x) = W(x, c)P$ *for each* x *in* S.

(2) u *is in* $\mathcal{F}(c, P)$ *and* $u(x) = P + (R)\int_x^c V \cdot u$ *for each* x *in* S.

(3) *If* h_0 *is in* $\mathcal{F}'(c, P)$ *and* h_n $(n = 1, 2, \cdots)$ *is defined by*

$$h_n(x) = P + (R)\int_x^c V \cdot h_{n-1} \qquad \text{for each } x \text{ in } S,$$

then the sequence h *has limit* u, *uniformly on each* $\mathcal{O}$-*interval of* S.

Indication of proof. By Corollary 1.2 and Lemma 2.1, (1) implies (2). On the other hand, by Lemma 2.2, (2) is equivalent to

(2′) u is in $\mathcal{F}'(c, P)$ and $u(x) = P + (R)\int_x^c V \cdot u$ for each x in S.

Now, letting K be the function from $\mathcal{F}'(c, P)$ into $\mathcal{F}'(c, P)$ defined by

$$K[w](x) = P + (R)\int_x^c V \cdot w,$$

the last assertion of Lemma 2.3 has the form

$$\| K[u_1](x) - K[u_2](x) \| \leqq (R)\int_x^c \alpha \cdot \beta(\ , c);$$

moreover, $\mathcal{F}'(c, P)$ is closed with respect to the topology of uniform convergence on each $\mathcal{O}$-interval of S. Therefore [6] there is only one member w of $\mathcal{F}'(c, P)$ such that $K[w] = w$ and it is the limit, uniformly on each $\mathcal{O}$-interval of S, of each infinite sequence h as indicated in (3).

COROLLARY 2.1. *If* V *is in* $\mathcal{O}\mathcal{A}$ *and* W *is a function from* $S \times S$ *each value of which is a function from* G *into* G *then, in order that* W *be the member* $\mathcal{E}(V)$ *of* $\mathcal{O}\mathcal{M}$, *it is necessary and sufficient that, for each* $\{y, P\}$ *in* $S \times G$, $W(\ , y)P$ *be a member of* $\mathcal{F}(y, P)$ *and, for each* x *in* S,

$$W(x, y)P = P + (R)\int_x^y V \cdot W(\ , y)P.$$

COROLLARY 2.2. *If* $\{V, W\}$ *belongs to* $\mathcal{E}$ *then there is an infinite sequence* L *such that*

(i) *each value of* L *is a function from* $S \times S$ *to* H,

(ii) *if* $\{x, y\}$ *is in* $S \times S$ *then* $L_0(x, y) = 1$ *and for each* P *in* G

$$L_n(x, y)P = P + (R)\int_x^y V \cdot L_{n-1}(\ , y)P \qquad (n = 1, 2, \cdots),$$

and (iii) *for each* $\{a, y\}$ *in* $S \times S$ *and* P *in* G, *if* x *is in* S *then*

$$L_n(x, y)\, P \rightarrow W(x, y)P \quad as \quad n \rightarrow \infty,$$

and the convergence is uniform over the set of all x *such that* $\{a, x, y\}$ *is an* $\mathcal{O}$*-subdivision of* $\{a, y\}$.

COROLLARY 2.3. *If* $\{V, W\}$ *is in* $\mathcal{E}$ *and* $\{\alpha, \mu\}$ *is a member of* $\mathcal{E}^+$ *as in Corollary* 1.1, *and* c *is in* S *and* P *and* Q *are members of* G, *and* u_1 *and* u_2 *are members of* $\mathfrak{F}(c, P)$ *and* $\mathfrak{F}(c, Q)$ *respectively, such that for each* x *in* S

$$u_1(x) = P + (R)\int_x^c V\cdot u_1 \quad and \quad u_2(x) = Q + (R)\int_x^c V\cdot u_2\,,$$

then, for each x *in* S, $\| u_1(x) - u_2(x) \| \leq \mu(x, c)\| P - Q \|$.

COROLLARY 2.4. *Suppose* $\{V, W\}$ *is in* $\mathcal{E}$ *and* $\{\alpha, \mu\}$ *is a member of* $\mathcal{OA}^+$ *as in Corollary* 1.1, $\{c, d\}$ *is in* $S \times S$ *and* Q *is in* G, *and, for each* x *in* S, β_x *is a member of* $\mathcal{OA}^+$ *such that*

$$\begin{aligned} \beta_x(c, d) &= \mu(x, c) - \mu(x, d) && or \\ &= \mu(x, c) - 1 + \mu(x, d) - 1 && or \\ &= \mu(x, d) - \mu(x, c), \end{aligned}$$

according as $\{c, d, x\}$ *is an* $\mathcal{O}$*-subdivision of* $\{c, x\}$, *or* $\{c, x, d\}$ *is an* $\mathcal{O}$*-subdivision of* $\{c, d\}$, *or* $\{x, c, d\}$ *is an* $\mathcal{O}$*-subdivision of* $\{x, d\}$. *If* u_2 *and* u_3 *are members* of $\mathfrak{F}(c, Q)$ *and* $\mathfrak{F}(d, Q)$, *respectively, such that for each* x *in* S

$$u_2(x) = Q + (R)\int_x^c V\cdot u_2 \quad and \quad u_3(x) = Q + (R)\int_x^d V\cdot u_3\,,$$

then, for each x *in* S, $\| u_2(x) - u_3(x) \| \leq \beta_x(c, d)\| Q \|$.

Indication of proof. See the argument for Lemma 2.1.

COROLLARY 2.5. *Suppose* $\{V_1, W_1\}$ *and* $\{V_2, W_2\}$ *are members of* $\mathcal{E}$, *with corresponding members* $\{\alpha_1, \mu_1\}$ *and* $\{\alpha_2, \mu_2\}$ *of* $\mathcal{E}^+$ *as in Corollary* 1.1, *and* β *is a member of* $\mathcal{OA}^+$ *such that for each* $\{x, y\}$ *in* $S \times S$ *and* P *in* G

$$\| V_1(x, y)P - V_2(x, y)P \| \leq \beta(x, y)\| P \|.$$

If $\{c, P\}$ *is in* $S \times G$ *and* u_1 *and* u_2 *are members of* $\mathfrak{F}(c, P)$ *such that for each* x *in* S

$$u_1(x) = P + (R)\int_x^c V_1\cdot u_1 \quad and \quad u_2(x) = P + (R)\int_x^c V_2\cdot u_2\,,$$

then, for each x *in* S,

$$\| u_1(x) - u_2(x) \| \leq (L, R)\int_x^c \mu_2(x, \;)\cdot\beta\cdot\mu_1(\;, c)\;\| P \|\,.$$

Indication of proof. If $\{t_i\}_0^n$ is an $\mathcal{O}$-subdivision of $\{x, c\}$ and P is in G then, by an apparent extension of Lemma 1.2, we have

$$\begin{aligned}
&\| \textstyle\prod_t [1 + V_1]P - \prod_t [1 + V_2]P \| \\
&\quad = \| (\textstyle\prod_1^n [1 + V_1(t_{i-1}, t_i)])P - (\prod_1^n [1 + V_2(t_{i-1}, t_i)])P \| \\
&\quad \leq \textstyle\sum_{j=1}^n (\prod_1^{j-1} [1 + \alpha_2(t_{i-1}, t_i)])\beta(t_{j-1}, t_j)(\prod_{j+1}^n [1 + \alpha_1(t_{i-1}, t_i)])\| P\| \\
&\quad \leq \textstyle\sum_{j=1}^n \mu_2(x, t_{j-1})\beta(t_{j-1}, t_j)\mu_1(t_j, c)\| P \| \\
&\quad = (L, R) \textstyle\sum_t \mu_2(x, \ )\cdot\beta\cdot\mu_1(\ , c)\| P \|.
\end{aligned}$$

Remark. In case $\beta = \alpha_1 - \alpha_2$ then the indicated left-right integral is simply $\mu_1(x, c) - \mu_2(x, c)$; in any case, we have the estimate

$$(L, R) \int_x^c \mu_2(x, \ )\cdot\beta\cdot\mu_1(\ , c) \leq \mu_1(x, c)\mu_2(x, c)\beta(x, c).$$

3. The second integral equation

As indicated in the introduction, the theory of the second equation mentioned there is subsumed in the results of the preceding section. To see this, we proceed as in our treatment of the nonhomogeneous equations in the purely linear case [4, pp. 158–159].

We consider the set $G \times G$, with addition and norm defined by

$$\{P_1, P_2\} + \{Q_1, Q_2\} = \{P_1 + Q_1, P_2 + Q_2\}, \quad \| \{P_1, P_2\} \| = \| P_1 \| + \| P_2 \|.$$

For this complete normed group, we let $\mathcal{OA}''$ and $\mathcal{OM}''$ be the functional classes corresponding, respectively, to the classes $\mathcal{OA}$ and $\mathcal{OM}$ for the group $\{G, +, \| \ \|\}$, and let $\mathcal{E}''$ be the corresponding mapping from $\mathcal{OA}''$ onto $\mathcal{OM}''$ as given by the apparent analogue of Theorem 1.1.

If each of V_1 and V_2 is in $\mathcal{OA}$ then there is a member V of $\mathcal{OA}''$ determined by the condition that, for $\{x, y\}$ in $S \times S$ and P in $G \times G$,

$$V(x, y)P = \{V_1(x, y)P_1 + V_2(x, y)P_2, 0\};$$

moreover, if c is in S and u_1 is in $\mathcal{F}(c, P_1)$ and u_2 is in $\mathcal{F}(c, P_2)$ then

$$(R) \int_x^c V\cdot u = \left\{(R) \int_x^c V_1\cdot u_1 + (R) \int_x^c V_2\cdot u_2, 0\right\} \quad \text{for } x \text{ in } S,$$

where u is the function $\{u_1, u_2\}$ from S to $G \times G$; hence, with these identifications, we see that the condition that, for each x in S,

$$u_1(x) = P_1 + (R)\int_x^c V_1\cdot u_1 + V_2(x, c)P_2 \quad \text{and} \quad u_2(x) = P_2$$

is equivalent to the condition that, for each x in S,

$$u(x) = P + (R) \int_x^c V\cdot u.$$

These considerations show that the following theorem is a reinterpretation of Theorem 2; we desist from stating the corresponding reinterpretations of the corollaries.

THEOREM 3. *Suppose c is in S, P is in $G \times G$, and each of V_1 and V_2 is in $\mathfrak{O}\mathfrak{A}$; let V be the member of $\mathfrak{O}\mathfrak{A}''$ determined by the condition that*

$$V(x, y)Q = \{V_1(x, y)Q_1 + V_2(x, y)Q_2, 0\}$$

for each $\{x, y\}$ in $S \times S$ and Q in $G \times G$, and let W be $\mathcal{E}''(V)$. If ω is a function from S to G, the following are equivalent:

(1) $\{\omega(x), P_2\} = W(x, c)P$ *for each x in S.*

(2) *ω is a member of $\mathfrak{F}(c, P_1)$ such that, for each x in S,*

$$\omega(x) = P_1 + (R) \int_x^c V_1 \cdot \omega + V_2(x, c)P_2 .$$

(3) *ω is the limit, uniformly on each $\mathfrak{O}$-interval of S, of each infinite sequence h such that h_0 is in $\mathfrak{F}'(c, P_1)$ and, for each positive integer n, h_n is the function from S to G defined by*

$$h_n(x) = P_1 + (R) \int_x^c V_1 \cdot h_{n-1} + V_2(x, c)P_2 \qquad \text{for each } x \text{ in } S.$$

4. Representations of semigroups

In this section, $\{S, \sigma\}$ denotes a canonically ordered semigroup [4, p. 164 ff.], $\mathfrak{O}$ is supposed to be the linear ordering of S determined by σ, δ is the function determined by the requirement [4, Theorem 8.1]

$$\sigma(x, \delta(x, z)) = z \qquad \text{for each } \{x, z\} \text{ in } \mathfrak{O},$$

and e is the member of S with the property [4, Theorem 8.2] that if x is in S then $\sigma(e, x) = \sigma(x, e) = x$.

Remark. There is an example, noticed by A. C. Mewborn (June 1962, oral communication), which supplements earlier examples and which can *not* be embedded in a group: S is the set to which x belongs only in case x is a complex number and either $\operatorname{Re} x = 0$ and $\operatorname{Im} x \geq 0$ or $\operatorname{Re} x > 0$, and for each $\{x, y\}$ in $S \times S$

$$\sigma_3(x, y) = x + y \quad \text{or} \quad \sigma_3(x, y) = (\operatorname{Re} x) + y$$

according as $\operatorname{Re} y = 0$ or $\operatorname{Re} y > 0$, respectively. In this case, as with σ_1 and σ_2 [4, p. 166], where for each $\{x, y\}$ in $S \times S$

$$\sigma_1(x, y) = x + y \quad \text{and} \quad \sigma_2(x, y) = x + (1 + \operatorname{Re} x)y,$$

the linear ordering $\mathfrak{O}$ is the subset of $S \times S$ to which $\{x, z\}$ belongs only in case either $\operatorname{Re} x = \operatorname{Re} z$ and $\operatorname{Im} x \leq \operatorname{Im} z$ or $\operatorname{Re} x < \operatorname{Re} z$.

Returning to the present development, we easily find the following variation of Theorem 8.3 of [4], and this leads (*via* Theorem 1.1) to the next theorem (compare [4, Remark 3, p. 168]).

LEMMA 4.1. *If each of x, y, and c is in S then*
(i) *if* $\{s_i\}_0^n$ *is an* $\mathcal{O}$*-subdivision of* $\{x, y\}$, $\{\sigma(c, s_i)\}_0^n$ *is then an* $\mathcal{O}$*-subdivision of* $\{\sigma(c, x), \sigma(c, y)\}$,
and (ii) *if* $\{t_i\}_0^n$ *is an* $\mathcal{O}$*-subdivision of* $\{\sigma(c, x), \sigma(c, y)\}$, $\{\delta(c, t_i)\}_0^n$ *is then an* $\mathcal{O}$*-subdivision of* $\{x, y\}$.

THEOREM 4.1. *If* $\{V, W\}$ *belongs to* $\mathcal{E}$, *the following are equivalent*:
(1) $V(\sigma(c, x), \sigma(c, y)) = V(x, y)$ *for all x, y, and c in S.*
(2) $W(\sigma(c, x), \sigma(c, y)) = W(x, y)$ *for all x, y, and c in S.*

$\mathcal{C}_\sigma$ denotes the class of all functions U from S to H such that there is a member μ of $\mathcal{OM}^+$ such that, if $\{x, y\}$ is in $\mathcal{O}$ and $\{P, Q\}$ is in $G \times G$,

$$\| [U(\delta(x, y)) - 1]P - [U(\delta(x, y)) - 1] Q \| \leq [\mu(x, y) - 1] P - Q \|.$$

THEOREM 4.2. *If the member W of* $\mathcal{OM}$ *is invariant with respect to* σ *as in Theorem* 4.1(2), *then*
(1) $W(e,\ )$ *is a member* U_1 of $\mathcal{C}_\sigma$ *such that* $U_1(a)U_1(b) = U_1(\sigma(a, b))$ *for each* $\{a, b\}$ *in* $S \times S$,
and (2) $W(\ , e)$ *is a member* U_2 of $\mathcal{C}_\sigma$ *such that* $U_2(a)U_2(b) = U_2(\sigma(b, a))$ *for each* $\{a, b\}$ *in* $S \times S$.
Conversely, each such ordered pair $\{U_1, U_2\}$ determines such a member W *of* $\mathcal{OM}$ *as follows:*

$$W(x, y) = U_1(\delta(x, y)) \quad \text{if} \quad \{x, y\} \text{ is in } \mathcal{O},$$
$$= U_2(\delta(y, x)) \quad \text{if} \quad \{y, x\} \text{ is in } \mathcal{O}.$$

Indication of proof. In connection with the primary implication, it is useful to note that if $\{a, b\}$ is in $S \times S$ then

$$b = \delta(a, \sigma(a, b)) \quad \text{and} \quad a = \delta(b, \sigma(b, a));$$

with respect to the converse implication, it may be noted that if each of μ_1 and μ_2 is a member of $\mathcal{OM}^+$ then there is a member μ of $\mathcal{OM}^+$ such that $\mu(x, y)$ is $\mu_1(x, y)$ or $\mu_2(y, x)$, according as $\{x, y\}$ is in $\mathcal{O}$ or $\{y, x\}$ is in $\mathcal{O}$. Verification then presents no difficulties.

$\mathcal{M}_\sigma^1$ and $\mathcal{M}_\sigma^2$ denote, respectively, the class of all functions U_1 as in Theorem 4.2(1) and the class of all functions U_2 as in Theorem 4.2(2); $\mathcal{A}_\sigma$ denotes the class of all functions F from S to H such that dF belongs to the class $\mathcal{OA}$ and $F(a) + F(b) = F(\sigma(a, b))$ for each $\{a, b\}$ in $S \times S$ (compare [4, p. 166]).

THEOREM 4.3. *There is a reversible function* $\mathcal{E}_\sigma^1$ *from* $\mathcal{A}_\sigma$ *onto* $\mathcal{M}_\sigma^1$ *such that the following are equivalent*:
(1) $\{F, U\}$ *belongs to* $\mathcal{E}_\sigma^1$.
(2) U *is in* $\mathcal{M}_\sigma^1$ *and F is the function from S to H defined by the condition that* $F(x)P = {}_e\sum^x [U(\delta) - 1]P$ *for each* $\{x, P\}$ *in* $S \times G$.
(3) F *is in* $\mathcal{A}_\sigma$ *and U is the function from S to H defined by the condition that* $U(x)P = {}_e\prod^x [1 + dF]P$ *for each* $\{x, P\}$ *in* $S \times G$.

Remark. In the purely linear case, if $\{F, U\}$ is in $\mathcal{E}_\sigma^1$ then U is a solution of a certain left-integral equation (see [4, Theorem 9.3]).

THEOREM 4.4. *There is a reversible function $\mathcal{E}_\sigma^2$ from $\mathfrak{A}_\sigma$ onto $\mathfrak{M}_\sigma^2$ such that the following are equivalent*:

(1) *$\{F, U\}$ belongs to $\mathcal{E}_\sigma^2$.*

(2) *U is in $\mathfrak{M}_\sigma^2$ and F is the function from S to H defined by the condition that $F(x)P = {}_e\sum^x [1 - U(\delta)]P$ for each $\{x, P\}$ in $S \times G$.*

(3) *F is in $\mathfrak{A}_\sigma$ and U is the function from S to H defined by the condition that $U(x)P = {}_x\prod^e [1 + dF]P$ for each $\{x, P\}$ in $S \times G$.*

(4) *F is in $\mathfrak{A}_\sigma$ and U is a function from S each value of which is a function from G to G such that, for each P in G, $U(\ )P$ is a member of $\mathfrak{F}(e, P)$ and, for each x in S,*

$$U(x)P = P + (R)\int_x^e dF \cdot U(\ )P.$$

Indication of proofs. The fully additive members of $\mathcal{O}\mathfrak{A}$ which are invariant with respect to σ, as in Theorem 4.1(1), are of the form dF for F in $\mathfrak{A}_\sigma$. The mappings $\mathcal{E}_\sigma^1$ and $\mathcal{E}_\sigma^2$ arise as follows (see Theorem 4.2): if F is in $\mathfrak{A}_\sigma$ and $W = \mathcal{E}(dF)$ then $\mathcal{E}_\sigma^1(F) = W(e,\ )$ and $\mathcal{E}_\sigma^2(F) = W(\ , e)$. The various assertions are consequences of results from Sections 1 and 2.

THEOREM 4.5. *If each of F_1 and F_2 is in $\mathfrak{A}_\sigma$, and $U_1 = \mathcal{E}_\sigma^1(F_1)$, and $U_2 = \mathcal{E}_\sigma^2(F_2)$, then the following hold*:

(1) *In order that $U_1(x)U_2(x) = 1$ for each x in S, it is necessary and sufficient that there exist a function γ from $\mathcal{O}$ to the nonnegative real numbers such that, for each $\{x, y\}$ in $\mathcal{O}$ and P in G, ${}_e\sum^x \gamma = 0$ and*

$$\| [1 + dF_1(x, y)][1 - dF_2(x, y)]P - P \| \leq \gamma(x, y)\| P \|.$$

(2) *In order that $U_2(x)U_1(x) = 1$ for each x in S, it is necessary and sufficient that there exist a function γ from $\mathcal{O}$ to the nonnegative real numbers such that, for each $\{x, y\}$ in $\mathcal{O}$ and P in G, ${}_e\sum^x \gamma = 0$ and*

$$\| [1 - dF_2(x, y)][1 + dF_1(x, y)]P - P \| \leq \gamma(x, y)\| P \|.$$

Indication of proof. There is a member V of $\mathcal{O}\mathfrak{A}$ such that

(i) $V(x, y) = dF_1(x, y)$ if $\{x, y\}$ is in $\mathcal{O}$,
(ii) $V(x, y) = dF_2(x, y)$ if $\{y, x\}$ is in $\mathcal{O}$,
and (iii) V is invariant with respect to σ as in Theorem 4.1(1).

Now, if $W = \mathcal{E}(V)$ then $U_1 = W(e,\ )$ and $U_2 = W(\ , e)$. The result follows from an argument which parallels that given for Theorem 1.2.

Remark. The result of Corollary 2.2 is available here, to give approximation theorems for members of $\mathfrak{M}_\sigma^1$ and $\mathfrak{M}_\sigma^2$, in accordance with the result described in Theorem 4.2.

5. An example

Let k be a positive number. Let T_0 be the function defined on the real line as follows:

$$T_0(x) = 1/(1 - x) \quad \text{if} \quad x \leq 1 - 1/k,$$
$$= k + k^2(x - 1 + 1/k) \quad \text{if} \quad 1 - 1/k \leq x.$$

Let T_1 be the function defined on the real line as follows:

$$T_1(x) = -k + (1 + k^2)(x + \tan^{-1} k) \quad \text{if} \quad x \leq -\tan^{-1} k,$$
$$= \tan x \quad \text{if} \quad -\tan^{-1} k \leq x \leq \tan^{-1} k,$$
$$= k + (1 + k^2)(x - \tan^{-1} k) \quad \text{if} \quad \tan^{-1} k \leq x.$$

Note that T_0 and T_1 are, respectively, increasing functions from the real line onto the positive numbers and onto the real line, and are solutions of the nonlinear differential equations:

$$u' = \inf (k^2, u^2) \quad \text{and} \quad u' = \inf (k^2, u^2) + 1.$$

Let A_0 and A_1 denote the inverses of T_0 and T_1, respectively.

Now, let $\{S, \sigma\}$ be a semigroup such that S is a set of nonnegative real numbers which contains all nonnegative rational numbers and, for each $\{x, y\}$ in $S \times S$, $\sigma(x, y) = x + y$, so that the linear ordering $\mathcal{O}$ determined by σ is the intersection of $S \times S$ with the usual $(\leq)$ ordering of the real line. Let the normed group $\{G, +, \| \ \|\}$ be as follows: G is the set of all ordered pairs $\{r, m\}$ such that r is a real number and $m = 0$ or $m = 1$; addition is ordinary addition in the first component and modulo-2 addition in the second; for each $\{r, m\}$ in G, $\| \{r, m\} \| = | r | + m$. Thus, $\{G, +\}$ is the direct sum of the additive group of real numbers and a group of order two.

It can be shown that a function F, from S to H, belongs to $\mathfrak{A}_\sigma$ only in case there exists a positive number b and a function h from G to the set of real numbers such that $h(0, 0) = 0$ and

(i) $F(x)\{r, m\} = \{xh(r, m), 0\}$ for each x in S and $\{r, m\}$ in G,

and (ii) for each $\{r, m\}$ and $\{s, n\}$ in G,

$$| h(r, m) - h(s, n) | \leq b\| \{r, m\} - \{s, n\} \|.$$

Moreover, for such an F (and b and h), we have the formulas

$$[1 + F(x)](1 - F(x)]\{r, n\} = [1 + F(x)]\{r - xh(r, n), n\}$$
$$= \{r - xh(r, n) + xh(r - xh(r, n), n), n\},$$

so that

$$\| [1 + F(x)][1 - F(x)]\{r, n\} - \{r, n\} \| = | xh(r - xh(r, n), n) - xh(r, n) |$$
$$\leq xb\| \{r - xh(r, n), n\} - \{r, n\} \|$$
$$= xb| xh(r, n) | \leq x^2b^2\| \{r, n\} \|,$$

and, similarly,

$$\| [1 - F(x)][1 + F(x)]\{r, n\} - \{r, n\} \| \leq x^2 b^2 \| \{r, n\} \| = x^2 b^2 (| r | + n);$$

therefore the conditions contemplated in Theorem 4.5 (with $F_1 = F_2$) are automatically satisfied in the present setting.

Now, for the example, let $h(r, n) = -[\inf (k^2, r^2) + n]$; the function F, as in (i) of the preceding paragraph, belongs to $\mathfrak{A}_\sigma$ (indeed, we may take b as $\sup (2k, 1)$). The function $U_2 = \mathcal{E}_\sigma^2(F)$ (as in Theorem 4.4) is readily found to be given by the following formulas:

$$U_2(x)\{r, 0\} = \{-T_0(A_0(-r) - x), 0\} \quad \text{or} \quad \{T_0(A_0(r) + x), 0\}$$

according as $r < 0$ or $r > 0$, with $U_2(x)\{0, 0\} = \{0, 0\}$, and

$$U_2(x)\{r, 1\} = \{T_1(A_1(r) + x), 1\}.$$

The function $U_1 = \mathcal{E}_\sigma^1(F)$ (as in Theorem 4.3) is found by Theorem 4.5 and the last observation of the preceding paragraph; for instance,

$$U_1(x)\{r, 1\} = \{T_1(A_1(r) - x), 1\}.$$

Finally, the function W, determined on $S \times S$ by

$$\begin{aligned} W(x, y) &= U_1(y - x) \quad \text{if} \quad 0 \leq x \leq y, \\ &= U_2(x - y) \quad \text{if} \quad 0 \leq y \leq x, \end{aligned}$$

is seen, by Theorem 4.2, to be the member $\mathcal{E}(dF)$ of the class $\mathcal{OM}$.

Remark. It should be pointed out that, in this example, there is exhibited a method of analyzing the integral equation (1) for the case that the function V satisfies all the conditions of membership in the class $\mathcal{OA}$ *except that* $V(x, y)0 \neq 0$ for some $\{x, y\}$ in $S \times S$. Namely, one considers the direct sum G' of the group G and the group of order two mentioned here, and in this context defines for each $\{x, y\}$ in $S \times S$

$$V'(x, y)\{P, m\} = \{V(x, y)P - V(x, y)0 + mV(x, y)0, 0\};$$

if there exists a member β of $\mathcal{OA}^+$ such that, if $\{x, y\}$ is in $S \times S$,

$$\| V(x, y)0 \| \leq \beta(x, y),$$

then it is easy to see that V' does belong to the corresponding class $\mathcal{OA}'$ for this enlarged group; subsequent procedures can parallel those of Section 3. Thus, one sees some point in the generality obtained by imposing only a group structure, rather than a vector space structure, on G in the initial development.

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