ON L-SERIES WITH REAL CHARACTERS

BY

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1. Introduction

Let d be the discriminant of an imaginary quadratic field. Thus there exists a square-free negative integer D with

$$\begin{aligned} d &= D \quad \text{if} \quad D \equiv 1 \pmod 4 \\ &= 4D \quad \text{if} \quad D \equiv 2, 3 \pmod 4. \end{aligned}$$

Such integers d are frequently called fundamental discriminants.

Let

$$\chi_d = \chi_d(n) = \left(\frac{d}{n}\right)$$

be the Kronecker symbol and suppose that

$$L(s, \chi_d) = \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s}$$

is the Dirichlet series associated with the real nonprincipal primitive character $\chi_d \bmod |d|$.

The behaviour of $L(s, \chi_d)$ for real s between 0 and 1 has important implications in the study of the class number $h(d)$ of quadratic fields of discriminant d. In particular the existence or nonexistence of roots of $L(s, \chi_d)$ in the interval $0 < s < 1$ has far-reaching consequences.

A conjecture, in milder form due to Hecke, states that if $0 < s < 1$, then $L(s, \chi_d) \neq 0$. This conjecture is still unsettled.

The object of this note is to examine the mean value of $L(s, \chi_d)$ summed over fundamental discriminants. In particular our object is to prove the following

THEOREM. *If d is a fundamental discriminant and $\chi_d(n)$ the associated Kronecker symbol, then for $\frac{1}{2} < s \leqq 1$, we have*

$$\sum_{0<-d\leqq N} L(s, \chi_d) = N \frac{\zeta(2s)}{\zeta(2)} \prod_p \left(1 - \frac{1}{(p+1)p^{2s}}\right) + O\left(\frac{N^{(2/3)(2-s)} \log N}{2s - 1}\right),$$

where the summation is over fundamental discriminants and the constant implied by the O is absolute.

This leads immediately to the following

COROLLARY. *For any given s in the interval $\frac{1}{2} < s \leqq 1$, there exists $N_0 = N(s)$*

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such that for all $N > N_0$,

$$\textstyle\sum_{0<-d\leqq N} L(s, \chi_d) > 0.$$

This result contributes nothing to the conjecture of Hecke but perhaps sheds a modicum of light upon it.

The editor has kindly pointed out to the author that for the range $\frac{3}{4} < s < 1$, the corollary is inherent in a theorem due to Chowla and Erdös [1]. They proved that if $g(a, x)$ is the number of discriminants d with $0 < -d \leqq x$ for which $L(s, \chi_d) < a$, then for $\frac{3}{4} < s < 1$,

$$\lim_{x\to\infty} \frac{g(a, x)}{x/2} = g(a)$$

exists, $g(0) = 0$, $g(\infty) = 1$, and $g(a)$ is a continuous and strictly increasing function of a.

The proof of our theorem is straightforward and is based on a lemma which in its essential features is due to C. L. Siegel [2].

2. Proof of a lemma

LEMMA. *Let*

$$T(n, N) = \sum_{0<-d\leqq N} \chi_d(n) = \sum_{0<-d\leqq N} \left(\frac{d}{n}\right). \tag{1}$$

Then

(a) *if n is not a square,*

$$T(n, N) = O(N^{1/2}n^{1/4}\log^{1/2}n); \tag{2}$$

(b) *if n is a square, $n = m^2$, then*

$$T(m^2, N) = \frac{N}{2\zeta(2)}\, g(m) + O(\sqrt{m}\,\sqrt{N}) \tag{3}$$

where

$$g(m) = \textstyle\prod_{p\mid m}(1 + 1/p)^{-1} \tag{4}$$

and the constants implied by the O are absolute.

Proof. We have

$$\begin{aligned} T(n, N) &= \sum_{\substack{0<-d\leqq N \\ d\equiv 1 \pmod 4}} \chi_d(n) + \sum_{\substack{0<-d\leqq N \\ d/4\equiv 2 \pmod 4}} \chi_d(n) + \sum_{\substack{0<-d\leqq N \\ d/4\equiv 3 \pmod 4}} \chi_d(n) \\ &= T_1(n, N) + T_2(n, N) + T_3(n, N). \end{aligned} \tag{5}$$

We consider these sums separately but concentrate on the easiest of them, viz. T_1. The others are treated in the same way. Indeed we have

$$(6)\qquad \begin{aligned} T_1(n, N) &= \sum_{\substack{0<-d\leqq N \\ d\equiv 1 \pmod 4}} \left(\frac{d}{n}\right)\mu^2(d) = \sum_{\substack{0<-l^2k\leqq N \\ k\equiv 1 \pmod 4 \\ (l,2n)=1}} \left(\frac{k}{n}\right)\mu(l) \\ &= \sum_{\substack{0<l\leqq\sqrt{N} \\ (l,2n)=1}} \mu(l) \sum_{\substack{0<-k\leqq N/l^2 \\ k\equiv 1 \pmod 4}} \left(\frac{k}{n}\right). \end{aligned}$$

Let

$$(7)\qquad P(n, r, M) = \sum_{\substack{0<-k\leqq M \\ k\equiv r \pmod 4}} \left(\frac{k}{n}\right).$$

Then if $\chi^{(1)}(k)$ and $\chi^{(3)}(k)$ are the two characters mod 4, $\chi^{(1)}(k)$ being the principal one, we have

$$(8)\qquad P(n, 1, M) = \frac{1}{2}\sum_{0<-k\leqq M} \left(\frac{k}{n}\right)(\chi^{(1)}(k) + \chi^{(3)}(k)).$$

Case (a). If n is odd and not a square, then $\left(\frac{k}{n}\right)$ is a character mod n which is not principal and it is then easily seen that

$$\left(\frac{k}{n}\right)\chi^{(1)}(k) \quad \text{and} \quad \left(\frac{k}{n}\right)\chi^{(3)}(k)$$

are nonprincipal characters mod $4n$. According to Pólya's theorem [3], as generalized by Landau [4], if χ is a character modulo k which is not principal and

$$S(a, b) = \textstyle\sum_{a\leqq m\leqq b} \chi(m),$$

then

$$(9)\qquad S(a, b) = O(k^{1/2}\log k)$$

where the constant implied by the O is absolute. It follows then from (8) and (9), that

$$(10)\qquad P(n, 1, M) = O(\min(n^{1/2}\log n, M)).$$

Thus by (6), (7), and (10),

$$(11)\qquad \begin{aligned} T_1(n, N) &= O(\textstyle\sum_{0<l\leqq\sqrt{N}} \min(n^{1/2}\log n, N/l^2)) \\ &= O(N^{1/2}n^{1/4}\log^{1/2} n). \end{aligned}$$

If n is even and not a square, a similar argument applies and need only be used on $T_1(n, N)$, since when n is even $T_2 = T_3 = 0$.

Thus if n is not a square, we get from (5) and (11),

$$T(n, N) = O(N^{1/2}n^{1/4}\log^{1/2} n),$$

thus proving the first assertion of the lemma.

Case (b). Suppose now that n is a square, $n = m^2$, and assume in addition that m is odd. Then

$$(12) \qquad P(m^2, 1, M) = \frac{1}{2} \sum_{0<-k\leqq M} \left(\frac{k}{m^2}\right) (\chi^{(1)}(k) + \chi^{(3)}(k)).$$

On the other hand,

$$\left(\frac{k}{m^2}\right) \chi^{(1)}(k)$$

is the principal character mod $4m$ whereas

$$\left(\frac{k}{m^2}\right) \chi^{(3)}(k)$$

is nonprincipal. Thus

$$P(m^2, 1, N/l^2) = \frac{\phi(4m)}{8m} \frac{N}{l^2} + O(\min(m, N/l^2)).$$

Similar arguments hold for the corresponding sums in T_2 and T_3. Combining these we get, if m is odd,

$$(13) \qquad \begin{aligned} T(m^2, N) &= \frac{\phi(m)}{4m} N \sum_{\substack{0<l\leqq\sqrt{N} \\ (l,2m)=1}} \frac{\mu(l)}{l^2} + \frac{\phi(m)}{8m} N \sum_{\substack{0<l\leqq\sqrt{N/2} \\ (l,2m)=1}} \frac{\mu(l)}{l^2} \\ &\qquad + O\Big(\sum_{0<l\leqq\sqrt{N}} \min(m, N/l^2)\Big) \\ &= \frac{3\phi(m)}{8m} \sum_{\substack{l=1 \\ (l,2m)=1}}^{\infty} \frac{\mu(l)}{l^2} + O(\sqrt{N}) + O(\sqrt{m}\sqrt{N}). \end{aligned}$$

However

$$(14) \qquad \begin{aligned} \sum_{(l,2m)=1} \frac{\mu(l)}{l^2} &= \prod_{p\nmid 2m} \left(1 + \frac{\mu(p)}{p^2}\right) = \prod_{p} \left(1 + \frac{\mu(p)}{p^2}\right) \prod_{p\mid 2m} \left(1 - \frac{1}{p^2}\right)^{-1} \\ &= \frac{1}{\zeta(2)} \frac{4m}{3\phi(m)} g(m). \end{aligned}$$

By the same token, if m is even,

$$(15) \qquad T(m^2, N) = \frac{\phi(m)}{2m} N \sum_{\substack{l=1 \\ (l,m)=1}}^{\infty} \frac{\mu(l)}{l^2} + O(\sqrt{m}\sqrt{N}).$$

The sum on the right is

$$\sum_{\substack{l=1 \\ (l,m)=1}}^{\infty} \frac{\mu(l)}{l^2} = \frac{1}{\zeta(2)} \frac{m}{\phi(m)} g(m).$$

Combining (13), (14), (15), and (16) we get the second assertion of the lemma.

3. Proof of the theorem

From the definition, we infer that

$$\sum_{0<-d\leqq N} L(s,\chi_d) = \sum_{0<-d\leqq N}\sum_{n\leqq M}\frac{\chi_d(n)}{n^s} + \sum_{0<-d\leqq N}\sum_{n>M}\frac{\chi_d(n)}{n^s} = A(s,N,M) + R(s,N,M), \tag{1}$$

where M will be chosen later as a function of N. On the other hand if we write

$$S_d(x) = \sum_{n\leqq x}\chi_d(n)$$

then from §2, (9),

$$\begin{aligned} R(s,N,M) &= O\left(\sum_{0<-d\leqq N}\left| -\frac{S_d(M)}{M^s} + s\int_M^\infty \frac{S_d(x)}{x^{s+1}}\,dx\right|\right)\\ &= O(M^{-s}\textstyle\sum_{0<-d\leqq N}|d|^{1/2}\log|d|)\\ &= O(M^{-s}N^{3/2}\log N). \end{aligned} \tag{2}$$

For $A(s, N, M)$, we have by §2, (1),

$$\begin{aligned} A(s,N,M) &= \sum_{n\leqq M}\frac{1}{n^s}T(n,N)\\ &= \sum_{\substack{n\leqq M\\ n=m^2}}\frac{1}{n^s}T(n,N) + \sum_{\substack{n\leqq M\\ n\neq m^2}}\frac{1}{n^s}T(n,N)\\ &= A_1(s,N,M) + A_2(s,N,M). \end{aligned} \tag{3}$$

By the lemma, §2, (2),

$$\begin{aligned} A_2(s,N,M) &= O\Big(\sum_{\substack{n\leqq M\\ n\neq m^2}}\frac{1}{n^s}N^{1/2}n^{1/4}\log^{1/2} n\Big)\\ &= O(N^{1/2}M^{-s+5/4}\log^{1/2} M). \end{aligned} \tag{4}$$

Again by the same lemma, §2, (3),

$$\begin{aligned} A_1(s,N,M) &= \sum_{m\leqq\sqrt{M}}\frac{1}{m^{2s}}T(m^2,N)\\ &= \frac{N}{2\zeta(2)}\sum_{m\leqq\sqrt{M}}\frac{g(m)}{m^{2s}} + O\left(N^{1/2}\sum_{m\leqq\sqrt{M}}\frac{1}{m^{2s-1/2}}\right)\\ &= \frac{N}{2\zeta(2)}\sum_{m=1}^{\infty}\frac{g(m)}{m^{2s}} + O\left(N\sum_{m>\sqrt{M}}\frac{g(m)}{m^{2s}}\right)\\ &\qquad + O\left(N^{1/2}\sum_{m\leqq\sqrt{M}}\frac{1}{m^{2s-1/2}}\right)\\ &= \frac{N}{2\zeta(2)}\sum_{m=1}^{\infty}\frac{g(m)}{m^{2s}} + O\left(N\frac{M^{-s+1/2}}{2s-1}\right) + O(N^{1/2}M^{1/4}). \end{aligned} \tag{5}$$

Therefore by (1), (2), (3), (4), and (5), we get

$$\sum_{0<-d\leqq N} L(s,\chi_d) = \frac{N}{2\zeta(2)}\sum_{n=1}^{\infty}\frac{g(m)}{m^{2s}} + O\left((M^{-s}N^{1/2})\left(M^{5/4}\log^{1/2}M + \frac{N^{1/2}M^{1/2}}{2s-1} + M^{s+1/4}\right)\right). \tag{6}$$

If we put $M = N^{2/3}$, we infer from (6),

$$\sum_{0<-d\leqq N} L(s,\chi_d) = \frac{N}{2\zeta(2)}\sum_{m=1}^{\infty}\frac{g(m)}{m^{2s}} + O\left(\frac{N^{(2/3)(2-s)}\log^{1/2}N}{2s-1}\right). \tag{7}$$

However we can sum the series on the right. Indeed since $g(m)$ is multiplicative and since

$$g(p^\alpha) = g(p) = 1 - 1/(p+1)$$

for a prime p, we deduce that

$$\sum_{m=1}^{\infty}\frac{g(m)}{m^{2s}} = \prod_p\left(\sum_{k=0}^{\infty} g(p^k)p^{-2ks}\right) = \prod_p\left(1 + \frac{g(p)p^{-2s}}{1-p^{-2s}}\right)$$
$$= \zeta(2s)\prod_p\left(1 - \frac{1}{(p+1)p^{2s}}\right).$$

The theorem is therefore proved.

It may be remarked that the argument can be applied to the case of a real quadratic field with virtually no change in the details.

References

1. S. Chowla and P. Erdös, *A theorem on the distribution of the values of L-functions*, J. Indian Math. Soc. (new series), vol. 15 (1951), pp. 11–18.
2. C. L. Siegel, *The average measure of quadratic forms with given determinant and signature*, Ann. of Math. (2), vol. 45 (1944), pp. 667–685.
3. G. Pólya, *Über die Verteilung der quadratischen Reste und Nichtreste*, Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen (Mathematisch-physikalische Klasse), 1918, pp. 21–29.
4. E. Landau, *Abschätzung von Charaktersummen, Einheiten und Klassenzahlen*, Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen (Mathematisch-physikalische Klasse), 1918, pp. 79–97.

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