QUANTIFIER THEORY ON QUASI-ORTHOMODULAR LATTICES

BY

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1. Introduction

In [7], we introduced the notion of a *quantifier* on an orthomodular lattice, and subject to mild assumptions of completeness, explicitly determined all quantifiers on an atomic orthomodular lattice [7, Theorem 7, p. 1245]. The definition we gave of a quantifier can be extended to an arbitrary lattice L with 0 by agreeing that a mapping $\varphi : L \to L$ shall be called a *quantifier* on L in case it satisfies:

(Q1) $0\varphi = 0$.

(Q2) $e \leq e\varphi$ for all $e \in L$.

(Q3) $(e \wedge f\varphi)\varphi = e\varphi \wedge f\varphi$ for all $e, f \in L$.

Aside from the connection developed in [7, p. 1241], with P. Jordan's skew lattices, these mappings have a way of cropping up in a variety of situations. We present herewith a few examples to illustrate this point.

(i) In a lattice L with 0 and 1 there are always two quantifiers: the *discrete* quantifier—the identity map; the *indiscrete* quantifier φ defined by $0\varphi = 0$, $e\varphi = 1$ for $e \neq 0$.

(ii) The *column operator* used by Halmos in his treatment of spectral multiplicity [6, p. 89] is a quantifier.

(iii) In a Loomis dimension lattice the mapping $a \to | a |$ (see [12, p. 13]) is a quantifier.

(iv) If L is a lattice with 0 and 1 having the property that for each $a \in L$, $a\gamma = \wedge \{z \in L : z \text{ central}, z \geq a\}$ exists and is central, then γ turns out to be a quantifier on L. We shall call γ the *central cover quantifier* and L a *central cover lattice*.

(v) Let L be a pseudo-complemented distributive lattice [1, pp. 147–148]. If a^* denotes the pseudo-complement of a, then $a \to a^{**}$ [1, Theorem 16, p. 148] is a quantifier.

(vi) The closure operator of a topological space is a quantifier if and only if every open set is also closed (see [5, p. 43]).

(vii) Let S be a commutative semigroup with 0, and let L denote the lattice of ideals of S. For an ideal I of S, define the radical of I by

$$R(I) = \{x \in S : x^n \in I \quad \text{for some positive integer} \quad n\}.$$

Received April 28, 1964.

[1] The results in this paper are a slight generalization of a portion of the author's doctoral dissertation (Wayne State University, 1963) written under the direction of Professor David J. Foulis. The research was supported by an NSF cooperative graduate fellowship, an NSF summer fellowship and an NSF Grant.

Then $R(I) \in L$, $I \subset R(I)$, and $R(I \cap J) = R(I) \cap R(J)$. If S has no proper nilpotent elements, so that $R((0)) = (0)$, then R is a quantifier on L.

(viii) There are also various quantifiers on Boolean algebras. See [5] and [13].

Our purpose in this paper will be to develop the characterization in Theorem 3.12 of all quantifiers on a class of lattices which includes all complete orthomodular and complemented modular lattices. Following this, we shall make a careful study of center valued quantifiers on an orthomodular lattice, finally obtaining the result that every type I Loomis dimension lattice satisfying Loomis' axiom (B′) can be coordinatized by a Baer *-semigroup S in such a way that the abstract dimension relation ($\sim$) is induced in a natural manner by *-equivalence in S.

This paper is to be regarded as a continuation of [7]; for this reason we shall feel free on occasion to use the terminology and notation thereof without specifically re-introducing it here.

2. Some preliminary results

There are a few things that one can say about quantifiers on an arbitrary lattice with 0 and 1. The proofs are essentially those given in [7] for Theorem 2 and Lemma 4, p. 1243; they will therefore be omitted.

LEMMA 2.1. *Let φ be a quantifier on a lattice L with* 0. *Then:*

(i) *If L has a largest element* 1, $1\varphi = 1$.

(ii) $\varphi = \varphi^2$.

(iii) $e \leq f \Rightarrow e\varphi \leq f\varphi$.

(iv) $e\varphi \vee f\varphi \leq (e \vee f)\varphi$.

(v) $e\varphi \wedge f\varphi \geq (e \wedge f)\varphi$.

(vi) $(L)\varphi$ = *the set of fixed points of φ is closed under the formation of arbitrary infima whenever they exist in L.*

(vii) *For each $e \in L$, $e\varphi = \wedge \{x \in L : x \geq e, x = x\varphi\}$. Hence φ is completely determined by its set of fixed points.*

LEMMA 2.2. *Let L be a central cover lattice. A mapping $\varphi : L \rightarrow L$ is a center valued quantifier on L if and only if there exists a (uniquely determined) quantifier α on the center of L such that $\varphi = \gamma \circ \alpha$.*

The following lemma will prove useful in the next section. Since its proof is completely routine, it will be left for the reader.

LEMMA 2.3. (i) *Let A be a subset of a complemented lattice L. If $e < 1$ implies the existence of an element $a \in A$ such that $e \leq a < 1$, then $0 = \wedge \{a : a \in A\}$. If in addition every interval $L(e, 1)$ is complemented, then for $e < 1$, $e = \wedge \{a \in A, a \geq e\}$.*

(ii) *The dual holds.*

We close this section by introducing some terminology that will be needed in the sequel.

DEFINITION 2.4. Let φ be a quantifier on a lattice L with 0 and 1. An element $e \in L$ will be called *invariant* if $e = e\varphi$, *faithful* if $e\varphi = 1$ and *simple* if $a \leq e \Rightarrow a = a\varphi \wedge e$. We shall let F denote the set of faithful elements and $J = \{a \in L : \varphi|_{L(0,a)} = I_a\}$, where I_a denotes the identity map on $L(0, a)$.

3. Quasi-orthomodular lattices

A pair (b, c) of elements of a lattice is called a *modular pair*, written $M(b, c)$, if

$$a \leq c \Rightarrow (a \vee b) \wedge c = a \vee (b \wedge c);$$

it is called a *dual modular pair*, and denoted $M^*(b, c)$, if

$$a \geq c \Rightarrow (a \wedge b) \vee c = a \wedge (b \vee c).$$

In a lattice with 0 and 1, let us agree that the expression $a \oplus b = 1$ shall mean that a and b are complements with $M(a, b)$ and $M^*(b, a)$ true. A *quasi-orthomodular lattice* is a lattice L with 0 and 1 having the property that each $e \in L$ admits (not necessarily unique) complements f, g with $e \oplus f = g \oplus e = 1$; if each $e \in L$ has at least one complement h (called a *quasi-orthocomplement* of e) such that $e \oplus h = h \oplus e = 1$, then L is called a *symmetric quasi-orthomodular lattice*. Finally, by a *locally orthomodular lattice* we shall mean a lattice L with 0 and 1 satisfying the following condition: For each $e, f \in L$ there exist quasi-orthocomplements $e^{\#}$ of e, $f^{\#}$ of f such that $e^{\#} \wedge f^{\#}$ is a quasi-ortho-complement of $e \vee f$.

The theory of quasi-orthomodular lattices was developed in some detail in [8] and [9], so that it will not be necessary to repeat it here. We merely point out that every complemented modular lattice and every orthomodular lattice is locally orthomodular, while the lattice of principal right ideals generated by idempotents of a Baer ring is a symmetric quasi-orthomodular lattice. It is also worth mentioning the obvious fact that L locally orthomodular implies L is a symmetric quasi-orthomodular lattice, which in turn implies that L is quasi-orthomodular.

For the remainder of this section L will denote a relatively complemented lattice with 0 *and* 1, *and φ a quantifier on L.*

LEMMA 3.1. *If $e = e\varphi$ and $f \oplus e = 1$, then $f = f\varphi$.*

Proof. By axiom (Q3), $e \wedge f = 0$ implies that $0 = 0\varphi = (f \wedge e)\varphi = f\varphi \wedge e$. Using $M^*(e, f)$, we have $f\varphi = f\varphi \wedge (e \vee f) = (f\varphi \wedge e) \vee f = f$.

LEMMA 3.2. *Any lower bound of F is invariant.*

Proof. Let a be a lower bound for F, and let b be a complement of a in the interval $L(0, a\varphi)$. We then let c be a complement of $b\varphi$ in the interval

$L(b, 1)$. Since $b \leq c$ implies that $b\varphi \leq c\varphi$, we see that $c\varphi \geq b\varphi \vee c = 1$, so that $a \leq c$. Then $a\varphi = a \vee b \leq c$, whence $b\varphi \leq a\varphi \leq c$, from which it follows that $c = 1$ and $b = b\varphi$. But now $a \wedge b = 0$ implies that $b = b \wedge a\varphi = (b \wedge a)\varphi = 0\varphi = 0$, and $a = a\varphi$.

We are now in a position to characterize the elements of J.

THEOREM 3.3. *Let $a \in L$. The following are equivalent:*
(i) $a \in J$.
(ii) *a is simple and invariant.*
(iii) *a is a lower bound for F.*

Proof. (i) $\Rightarrow$ (ii). If $a \in J$, then a is evidently invariant; moreover,

$$e \leq a \Rightarrow e = e\varphi = e\varphi \wedge a,$$

so that a is also simple.

(ii) $\Rightarrow$ (iii). Let $x \in F$. Then $x \wedge a \leq a$, so that $x \wedge a = (x \wedge a)\varphi = x\varphi \wedge a = 1 \wedge a = a$. This shows that $a \leq x$.

(iii) $\Rightarrow$ (i). This follows from Lemma 3.2.

The next theorem, although not needed for the present development, does provide a characterization of simple elements that will prove useful in the sequel.

THEOREM 3.4. *Given $e \in L$, let $\tilde{\varphi}$ denote the restriction of φ to $L(0, e)$. The following conditions are equivalent:*
(i) *e is simple.*
(ii) *$\tilde{\varphi}$ preserves finite infima.*
(iii) *$\tilde{\varphi}$ is one-one.*
(iv) *$\tilde{\varphi}$ maps disjoint elements into disjoint elements.*
(v) *$a \leq e$ and $a\varphi = e\varphi \Rightarrow a = e$.*
(vi) *$\tilde{\varphi}$ is an isomorphism of $L(0, e)$ onto $\{g\varphi : g\varphi \leq e\varphi\}$.*

Proof. (i) $\Rightarrow$ (vi). Let e be simple. If $g\varphi \leq e\varphi$, then $g\varphi \wedge e \leq e$ and $(g\varphi \wedge e)\varphi = g\varphi \wedge e\varphi = g\varphi$, so that $\tilde{\varphi}$ maps $L(0, e)$ onto $\{g\varphi : g\varphi \leq e\varphi\}$. Given $a, b \in L(0, e)$, we know that $a \leq b \Rightarrow a\varphi \leq b\varphi$. If conversely, $a\varphi \leq b\varphi$, then $a = a\varphi \wedge e \leq b\varphi \wedge e = b$. It is immediate that $\tilde{\varphi}$ is an isomorphism.

(vi) $\Rightarrow$ (ii). This is obvious.

(ii) $\Rightarrow$ (iii). Let $a, b \in L(0, e)$ with $a\varphi = b\varphi$. If d is a complement of $a \wedge b$ in a, we have $a \wedge b \wedge d = 0$, and by (ii), $0 = 0\varphi = (a \wedge b \wedge d)\varphi = a\varphi \wedge d\varphi$. This shows that $d = d\varphi = 0$ and $a = a \wedge b$. In an analogous way one produces $b = a \wedge b$.

(iii) $\Rightarrow$ (iv). If $a, b \leq e$ with $a \wedge b = 0$, we are to show that $a\varphi \wedge b\varphi = 0$. Note that $a \wedge b\varphi \leq e$, $a\varphi \wedge b \leq e$, and $(a \wedge b\varphi)\varphi = (a\varphi \wedge b)\varphi = a\varphi \wedge b\varphi$. Hence $a \wedge b\varphi = a\varphi \wedge b$, and $0 = a \wedge b = a \wedge b \wedge b\varphi = a\varphi \wedge b$. But then $a\varphi \wedge b\varphi = (a\varphi \wedge b)\varphi = 0\varphi = 0$.

(iv) $\Rightarrow$ (v). Let $a \leq e$ with $a\varphi = e\varphi$. If b is a complement of a in e, we have $a \wedge b = 0 \Rightarrow a\varphi \wedge b\varphi = 0$. But $b \leq e$ implies $b\varphi \leq e\varphi = a\varphi$, whence $b = b\varphi = 0$ and $a = e$.

(v) $\Rightarrow$ (i). For a fixed $a \leq e$, let b be a complement of a in $a\varphi \wedge e$, and c a complement of $b\varphi \wedge e$ in the interval $L(b, e)$. Then $b \leq c$, $b\varphi \leq c\varphi$, so

$$e = (b\varphi \wedge e) \vee c \leq b\varphi \vee c \leq c\varphi \leq e\varphi.$$

Hence $c\varphi = e\varphi$, and by (v), $c = e$. But then $b\varphi \wedge e = b$, and $0 = a \wedge b = a \wedge b\varphi \wedge e = a \wedge b\varphi = a\varphi \wedge b\varphi = a\varphi \wedge b = b$. It follows that $a = a\varphi \wedge e$ and e is simple.

Corollary 1. *If φ is center valued, then e simple implies that $L(0, e)$ is a Boolean algebra.*

Proof. If φ is center valued, one can easily show that $\{g\varphi : g\varphi \leq e\varphi\}$ is a Boolean algebra. The corollary then follows from the fact that $\tilde{\varphi}$ is an isomorphism of $L(0, e)$ onto $\{g\varphi : g\varphi \leq e\varphi\}$.

Corollary 2. *If a quantifier on a relatively complemented lattice with* 0 *and* 1 *preserves finite infima, then it is the discrete quantifier.*

Proof. By the above theorem, 1 is simple. Since 1 is also invariant, we have by Theorem 3.3 that $1 \in J$. Hence the quantifier is the identity map.

In sharp contrast to Corollary 2, we point out that examples (v) and (vii) of §1 are non-discrete quantifiers that preserve finite infima.

It will now be assumed that L is a symmetric quasi-orthomodular lattice. Our plan of attack will be to establish certain distributivity properties of faithful elements, to decide just which elements can be expressed as the infimum of a family of faithful elements and finally subject to mild assumptions of completeness, to explicitly determine all quantifiers on a locally orthomodular lattice.

Lemma 3.5. *Let e be invariant and $f \oplus e = 1$. If x is faithful, then*

$$x = (x \wedge e) \vee (x \wedge f).$$

Proof. Let $y = (x \wedge e) \vee (x \wedge f)$. By [9, Lemma 2, p. 3] we can find a complement g of y in the interval $L(0, x)$ such that $M(g, y)$holds. Then

$$x \wedge f = (x \wedge f) \vee (g \wedge y) = [(x \wedge f) \vee g] \wedge y,$$

so that

$$0 = (x \wedge f) \wedge (x \wedge e) = \{[(x \wedge f) \vee g] \wedge y\} \wedge (x \wedge e) = [(x \wedge f) \vee g] \wedge e.$$

Applying φ to this equation, we see that

$$\begin{aligned} 0 = 0\varphi &= \{[(x \wedge f) \vee g] \wedge e\}\varphi \\ &= [(x \wedge f) \vee g]\varphi \wedge e \\ &\geq [(x \wedge f)\varphi \vee g\varphi] \wedge e \\ &= (f \vee g\varphi) \wedge e \geq 0, \end{aligned}$$

whence $(f \vee g\varphi) \wedge e = 0$. But then $f \wedge g\varphi \geq f$ implies

$$f \vee g\varphi = (f \vee g\varphi) \wedge (e \vee f) = [(f \vee g\varphi) \wedge e] \vee f = f, \quad g \leq g\varphi \leq f,$$

and finally $g = g \wedge f = g \wedge f \wedge x = 0$. Therefore $x = (x \wedge e) \vee (x \wedge f)$.

LEMMA 3.6. *Let e be invariant and $f \oplus e = 1$. If (x_α) is a family of faithful elements and $x = \bigwedge_\alpha x_\alpha$ exists, then $(x \vee f) \wedge e = x \wedge e$.*

Proof. By Lemma 3.5, $x_\alpha = (x_\alpha \wedge f) \vee (x_\alpha \wedge e)$, so that

$$x_\alpha \vee f = (x_\alpha \wedge e) \vee f,$$

and

$$(x_\alpha \vee f) \wedge e = [(x_\alpha \wedge e) \vee f] \wedge e = x_\alpha \wedge e.$$

It follows that $(x \vee f) \wedge e \leq (x_\alpha \vee f) \wedge e = x_\alpha \wedge e$ for all α, and consequently,

$$(x \vee f) \wedge e \leq \bigwedge_\alpha (x_\alpha \wedge e) = x \wedge e.$$

Therefore $(x \vee f) \wedge e = x \wedge e$, as desired.

LEMMA 3.7. *If $e < 1$ is an upper bound for J, then there is an element $x \in F$ such that $e \leq x < 1$.*

Proof. Case 1. If $e < e\varphi$, let f be a quasi-orthocomplement of $e\varphi$ and take $x = e \vee f$. Then $e = e \vee (f \wedge e\varphi) = (e \vee f) \wedge e\varphi = x \wedge e\varphi$, from which it follows that $e \leq x < 1$; furthermore, $1 \geq x\varphi = (e \vee f)\varphi \geq e\varphi \vee f = 1$, so that x is indeed faithful.

Case 2. If $e = e\varphi$, let f be a quasi-orthocomplement of e and note that f is invariant. Since $e \neq 1$ and e is an upper bound for J, we know that $f \notin J$. By Theorem 3.3, there must exist an element $g < f$ such that $g\varphi = f$. Taking $x = e \vee g$, we have $x\varphi \geq e \vee g\varphi = e \vee f = 1$, and since $g = g \vee (e \wedge f) = (g \vee e) \wedge f = x \wedge f$, we must have $x \neq 1$.

LEMMA 3.8. $e = \bigwedge\{x \in F : x \geq e\}$ *if and only if e is an upper bound for J.*

Proof. By Theorem 3.3, each $x \in F$ is an upper bound of J, so that if $e = \bigwedge\{x \in F : x \geq e\}$, the same must be true of e. The converse assertion follows from Lemmas 3.7 and 2.3.

THEOREM 3.9. *If $J = (0)$, then φ is a center valued quantifier.*

Proof. By Lemma 3.8, $a = \bigwedge\{x \in F : x \geq a\}$ for each $a \in L$. Let e be invariant and f a quasi-orthocomplement of e. If g is a complement of e, then by Lemma 3.6, $(g \vee f) \wedge e = g \wedge e = 0$. But then

$$g \vee f = (g \vee f) \wedge (e \vee f) = [(g \vee f) \wedge e] \vee f = f, g \leq f,$$

and

$$g = g \vee (e \wedge f) = (g \vee e) \wedge f = 1 \wedge f = f.$$

Thus e has a unique complement, and by [9, Theorem 16, p. 12] e is central.

LEMMA 3.10. *If either* $b = \bigvee \{a : a \in J\}$ *or* $d = \bigwedge \{x : x \in F\}$ *exist, then they both exist and are equal; furthermore,* $J = L(0, b)$.

Proof. If b exists then by Theorem 3.3 it is a lower bound for F, and therefore in J. Hence $J = L(0, b)$. But b is also an upper bound for J, so that by Lemma 3.8,

$$b = \bigwedge \{x : x \in F, x \geq b\} = \bigwedge \{x : x \in F\}.$$

If d exists, then it is both a lower bound for F and an upper bound for J. The conclusion is that $d \in J$ and $d = \bigvee \{a : a \in J\}$.

It will now be assumed that L is a locally orthomodular lattice.

LEMMA 3.11. *If* $b = \bigvee \{a : a \in J\}$ *exists, then it is central.*

Proof. Let d be a complement of b and e a quasi-orthocomplement of $d\varphi$. Since b is invariant, $d\varphi$ is also a complement of b. By Lemma 3.10, $b = \bigwedge \{x : x \in F\}$, and consequently, by Lemma 3.6, $(b \vee e) \wedge d\varphi = b \wedge d\varphi = 0$. It follows as in the proof of Theorem 3.9 that $b = e$; hence d is invariant. Thus (i) every complement of b is a quasi-orthocomplement of b; (ii) if d is a quasi-orthocomplement of b, then b is the *unique* quasi-orthocomplement of d. Suppose now that c, d are quasi-orthocomplements of b. Since L is locally orthomodular, there must exist quasi-orthocomplements $c^{\#}$ of c, $d^{\#}$ of d such that $c^{\#} \wedge d^{\#}$ is a quasi-orthocomplement of $c \vee d$. By (ii), we must have $c^{\#} = d^{\#} = b$, whence $(c \vee d) \wedge b = 0$. But then

$$c \vee d = (c \vee d) \wedge (c \vee b) = [(c \vee d) \wedge b] \vee c = c,$$

and similarly, $c \vee d = d$. It follows that b has a unique complement, and therefore that b is central.

If $b = \bigvee \{a : a \in J\}$ and b' is the unique complement of b, there are no nonzero elements $a \leq b'$ such that $\varphi|_{L(0,a)} = I_a$. Thus J as computed in $L(0, b')$ is (0). Since b is central, it is immediate that $L(0, b')$ is a symmetric quasi-orthomodular lattice. An application of Theorem 3.9 now produces the fact that $\varphi|_{L(0,b')}$ is a center valued quantifier on that lattice. For any mapping $\psi : L(0, b') \to L(0, b')$, it will prove convenient to define the mapping $I_b \times \psi : L \to L$ by the formula $e(I_b \times \psi) = (e \wedge b) \vee (e \wedge b')\psi$. The situation is summarized in the next theorem.

THEOREM 3.12. *Let L be a locally orthomodular lattice.*

(i) *If either* $b = \bigvee \{a: a \in J\}$ *or* $\bigwedge \{x: x \in F\}$ *exist, they both exist and are equal; furthermore, b is central and* $\varphi = I_b \times \tilde{\varphi}$ *with* $\tilde{\varphi}$ *a center valued quantifier on* $L(0, b')$. *If* $L(0, b')$ *is a central cover lattice, then there is a (uniquely determined) quantifier* α *on the center of* $L(0, b')$ *such that* $\varphi = I_b \times (\gamma \circ \alpha)$.

(ii) *If L is complete, then* φ *is a quantifier on L if and only if there exists a central element b such that* $\varphi = I_b \times (\gamma \circ \alpha)$, *where* α *is a (uniquely determined) quantifier on the center of* $L(0, b')$.

Proof. (i). This follows readily from Lemma 2.2 and the remarks preceding the theorem.

(ii) If L is complete, then by [9, Theorem 15, p. 12] it is a central cover lattice. The remaining assertions are now obvious.

COROLLARY. *Let L be a complete, irreducible, locally orthomodular lattice. Then L admits only the discrete and the indiscrete quantifiers.*

In retrospect, we have achieved a deep insight into the structure of quantifiers on a locally orthomodular lattice, provided only that the supremum of the invariant simple elements exists.

4. The induced homomorphism

Here we shall deal with a quantifier φ on a locally orthomodular lattice where it is not assumed that the supremum of the invariant simple elements exists. One still has that J is an ideal closed under the formation of arbitrary suprema whenever they exist, and that the restriction of φ to J is the identity map. We would like to produce a center valued quantifier on a homomorphic image of L by "dividing out" the ideal J. A sufficient property to guarantee the existence of a congruence relation Θ whose kernel is J turns out to be the following:

(α) If e and f are complements in L, there exist quasi-orthocomplements $e^{\#}$ of e, $f^{\#}$ of f such that (i) $e^{\#} \vee f^{\#} = 1$; (ii) $g \leq e^{\#}$ implies the existence of a quasi-orthocomplement $g^{\#}$ of g with $g^{\#} \geq e$; (iii) each $h \leq f$ has a quasi-orthocomplement $h^{\#} \geq f^{\#}$.

The above property, although complicated in its statement, is enjoyed by every complemented modular lattice and every orthomodular lattice.

LEMMA 4.1. *Let φ be a quantifier on a locally orthomodular lattice L. Then:*
(i) *$(L)\varphi$ is a sublattice of L.*
(ii) *φ preserves finite suprema.*

Proof. (i). In view of Lemma 2.1 (vi), we need only show that the supremum of a pair of invariant elements is itself invariant. Accordingly, let e and f be invariant. Since L is locally orthomodular, there exist quasi-orthocomplements $e^{\#}$ of e, $f^{\#}$ of f such that $e^{\#} \wedge f^{\#}$ is a quasi-orthocomplement of $e \vee f$. By Lemma 2.1 (vi), $e^{\#} \wedge f^{\#}$ is invariant. The invariance of $e \vee f$ now comes from Lemma 3.1.

(ii) Let $a, b \in L$. Then $a \vee b \leq a\varphi \vee b\varphi \leq (a \vee b)\varphi$. Applying φ, we have

$$(a \vee b)\varphi \leq (a\varphi \vee b\varphi)\varphi = a\varphi \vee b\varphi \leq (a \vee b)\varphi.$$

LEMMA 4.2. *Let φ be a quantifier on a locally orthomodular lattice L having property (α). Then $e \in J$, e perspective to f implies $f \in J$.*

Proof. Let x be a common complement of e and f. If $x \leq y < y\varphi$ and w were a quasi-orthocomplement of $y\varphi$, we would have that $w \vee y \neq 1$, $w \vee y$

is faithful, and by Theorem 3.3, $w \vee y \geq e$. But then $w \vee y \geq e \vee y = 1$, a contradiction. We conclude that the restriction of φ to $L(x, 1)$ is the identity map. By virtue of property (α), we next produce elements $f^\#$, $x^\#$ such that (i) $f^\#$ is a quasi-orthocomplement of f; (ii) $x^\#$ is a quasi-orthocomplement of x; (iii) $x^\# \vee f^\# = 1$; (iv) $g \leq x^\#$ implies the existence of a quasi-orthocomplement $g^\#$ of g with $g^\# \geq x$; (v) $h \leq f$ implies the existence of a quasi-orthocomplement $h^\#$ of h such that $h^\# \geq f^\#$. It follows from Lemma 3.1 that $x^\# \in J$, and consequently that the restriction of φ to $L(f^\#, 1)$ is the identity map. A second application of Lemma 3.1 now produces the fact that $f \in J$.

THEOREM 4.3. *Let L be a locally orthomodular lattice with property (α), and let φ be a quantifier on L. There exists a unique congruence relation Θ of L whose kernel is J; furthermore, φ induces a center valued quantifier $\tilde{\varphi}$ on L/Θ by means of the formula $(e/\Theta)\tilde{\varphi} = e\varphi/\Theta$. The original quantifier φ can be recaptured from $\tilde{\varphi}$ since $e/\Theta = (e/\Theta)\tilde{\varphi}$ if and only if $e = e\varphi$.*

Proof. (i) By Lemma 4.2 and [10, Corollary 2, p. 16] there exists a uniquely determined congruence relation θ whose kernel is J. If $a \equiv b(\theta)$, then by [10, Theorem 3.2, p. 10] $a \vee t = b \vee t$ for suitable $t \in J$. Since t is invariant, it follows that $a\varphi \vee t = (a \vee t)\varphi = (b \vee t)\varphi = b\varphi \vee t$, and consequently that $a\varphi \equiv b\varphi(\theta)$. This shows that the mapping

$$e/\theta \rightarrow (e/\theta)\tilde{\varphi} = e\varphi/\theta$$

is well defined. Since $e \rightarrow e/\theta$ is a homomorphism, $\tilde{\varphi}$ is automatically a quantifier on the locally orthomodular lattice L/θ.

(iii) If $a/\theta = (a/\theta)\tilde{\varphi}$, then $a \equiv a\varphi(\theta)$, whence $a\varphi = a \vee t$ for some $t \in J$. Since J is an ideal and L is relatively complemented, we can even take t disjoint from a. But then

$$t = t\varphi, \qquad t \wedge a = 0 \Rightarrow 0 = 0\varphi = (t \wedge a)\varphi = t \wedge a\varphi = t;$$

hence $a = a\varphi$. We conclude that $a/\theta = (a/\theta)\tilde{\varphi}$ if and only if $a = a\varphi$, so that φ can indeed be recaptured from $\tilde{\varphi}$.

(ii) The only remaining item is the proof that $\tilde{\varphi}$ is center valued on L/θ. If the restriction of $\tilde{\varphi}$ to the interval $L/\theta(0/\theta, a/\theta)$ is the identity map, then $b \leq a$ implies $b/\theta \leq a/\theta$, $b/\theta = (b/\theta)\tilde{\varphi}$, and therefore that $b = b\varphi$. Thus $\varphi|_{L(0,a)} = I_a$, so that $a \in J$ and $a/\theta = 0/\theta$. By Theorem 3.9, $\tilde{\varphi}$ is center valued on L/θ.

5. Baer *-semigroups

In order to make this paper more self-contained and readable, we introduce here the notation and terminology that will be used in the last two sections. The results we shall now sketch are due basically to D. J. Foulis. For a more complete treatment, see Foulis [2], [3] and [4].

An *involution semigroup* is a multiplicative semigroup S equipped with a mapping $^*: S \rightarrow S$ such that for all $x, y \in S$, $(xy)^* = y^*x^*$ and $x^{**} =$

$(x^*)^* = x$. An element $e \in S$ such that $e = e^2 = e^*$ is called a *projection*, and we agree to let $P = P(S)$ denote the partially ordered set of all projections of S, the partial ordering being given by $e \leq f$ if $e = ef$ (or equivalently, if $e = fe$).

A *Baer* **-semigroup* is an involution semigroup S with 0 having the property that for each $x \in S$, $\{y \in S : xy = 0\} = x'S$, where x' is a (necessarily unique) projection of S. For a Baer *-semigroup S, $P' = P'(S)$ is defined by $P' = \{x' : x \in S\}$. A subset S_1 of S is called a *Baer* **-subsemigroup of* S provided (i) S_1 is a subsemigroup of S; (ii) $S_1 = S_1^*$; (iii) $x \in S_1 \Rightarrow x' \in S_1$. Evidently, every Baer *-subsemigroup S_1 is a Baer *-semigroup in its own right with $P'(S_1) = P'(S) \cap S_1$.

A lattice L with 0 and 1 is said to be *orthocomplemented* if there is a mapping $' : L \to L$ such that (i) e' is a complement of e for all $e \in L$; (ii) $e = e''$ for all e; (iii) $e \leq f \Rightarrow f' \leq e'$ for all $e, f \in L$. If in addition, (iv)

$$e \leq f \Rightarrow f = e \vee (f \wedge e'),$$

then L is called an *orthomodular lattice*. Condition (iv) is frequently referred to as the *orthomodular identity*.

THEOREM 5.1. *Let S be a Baer *-semigroup. Then:*

(i) *For $e, f \in P(S)$, $e \leq f \Rightarrow f' \leq e'$.*

(ii) *For $e \in P(S)$, $e \leq e'' = (e')'$.*

(iii) *If $e \in P'(S)$ and if $a \in S$, then $ae = a$ if and only if $a'' \leq e$.*

(iv) *For $a \in S$, $a' = a'''$.*

(v) *$e \in P'(S)$ if and only if $e = e''$.*

(vi) *$0'$ (which we henceforth write as 1) is a unit for S.*

(vii) *$P'(S)$ is an orthomodular lattice under the partial order inherited from $P(S)$, with $e \to e'$ as its orthocomplementation.*

(viii) *For $a, b \in S$, $(ab)'' = (a''b)''$.*

(ix) *For $a \in S$, $(a^*a)'' = a''$.*

(x) *For $a, b \in S$, $(ab)'' \leq b''$.*

(xi) *For $a, b \in S$, if $b = b^*$, and if $ab = ba$, then $ab' = b'a$.*

Proof. See [4, Theorem 1, pp. 66–67].

LEMMA 5.2 [4, Lemma 4, p. 68]. *Let S be a Baer *-semigroup. If M is a non-empty subset of S, define $Z(M)$ to be $\{s \in S : sm = ms$ for all $m \in M\}$. Then:*

(i) *$M \subset N \Rightarrow Z(N) \subset Z(M)$.*

(ii) *$M \subset Z(Z(M))$.*

(iii) *$Z(M) = Z(Z(Z(M)))$.*

(iv) *If $M = M^*$ then $Z(M)$ is a Baer *-subsemigroup of S and*

$$P'(Z(M)) = P'(S) \cap Z(M)$$

is closed under the formation of arbitrary suprema and infima in $P'(S)$ provided only that they exist in $P'(S)$.

Starting with an orthomodular lattice L, let $M(L)$ denote the set of monotone mapping $\varphi : L \rightarrow L$. Two elements $\varphi, \psi \in M(L)$ are said to be *mutually adjoint* if $(e\varphi)'\psi \leq e'$ and $(e\psi)'\varphi \leq e'$ for all $e \in L$.

It is easily seen that for each $\varphi \in M(L)$ there can be at most one $\psi \in M(L)$ such that φ and ψ are mutually adjoint. If φ^* denotes this uniquely determined ψ (when it exists), then $S(L)$ = the subset of mappings in $M(L)$ possessing adjoints is clearly an involution semigroup. For each $e \in L$, the mapping $\varphi_e : L \rightarrow L$ is defined by the formula $f\varphi_e = (f \vee e') \wedge e$ for all $f \in L$. The situation is summarized in the next theorem.

THEOREM 5.3. *Let L be an orthomodular lattice. Then:*

(i) *$S(L)$ is a Baer *-semigroup.*

(ii) *For each $e \in L$, φ_e is a projection in $S(L)$.*

(iii) *$f\varphi_e = f$ if and only if $f \leq e$.*

(iv) *$f\varphi_e = 0$ if and only if $f \leq e'$.*

(v) *The correspondence $e \leftrightarrow \varphi_e$ between L and $P'(S(L))$ is an orthocomplementation-preserving lattice isomorphism.*

Proof. See [2, Theorem 6, p. 652].

A Baer *-semigroup S is said to *coordinatize* an orthomodular lattice L if there exists an orthocomplementation-preserving lattice isomorphism between L and $P'(S)$. Although the coordinatization is highly non-unique, we do see that every orthomodular lattice L can be coordinatized by the Baer *-semigroup $S(L)$. In the next section we shall concern ourselves with certain coordinatizing Baer *-subsemigroups of $S(L)$.

6. Center valued quantifiers on an orthomodular lattice

In §4, the study of quantifiers on an orthomodular lattice was reduced to the study of center valued quantifiers. For this reason, and in order to develop certain results that will be needed in our treatment of dimension lattices, we devote this section to an examination of center valued quantifiers on an orthomodular lattice. In the next two lemmas, which, incidentally, are due to D. J. Foulis, it will be assumed that S is a Baer *-semigroup and $L = P'(S)$. Borrowing some notation from Kaplansky [11, Definition 3, p. 5] for each $x \in S$, let $x\Gamma = \wedge\{h \in L \cap Z(S) : xh = x\}$, provided of course that such an infimum exists.

LEMMA 6.1. *If $x\Gamma$ exists for every $x \in L$, then the restriction of Γ to L is a center valued quantifier.*

Proof. By Theorem 5.1 (xi), $Z(S) \cap L$ is closed under the formation of orthocomplements. It follows easily that $\Gamma|_L$ is a center valued symmetric closure operator (see [7, p. 1244]), and therefore a center valued quantifier.

LEMMA 6.2. *Let $x \in S$. If $f = \vee_{a \in S}(xa)''$ exists, then $x\Gamma$ exists and equals f.*

Proof. By [4, Theorem 1 (xvii)] and our Theorem 5.1 (viii),

$$(fb)'' = \vee_{a \in S}[(xa)''b]'' = \vee_{a \in S}[x(ab)]'' \leq f,$$

where $b \in S$ is arbitrary. By Theorem 5.1 (iii), $fb = fbf$. Similarly, $fb^* = fb^*f$, so $fb = fbf = (fb^*f)^* = (fb^*)^* = bf$. This shows that $f \in Z(S) \cap L$. If $h \in Z(S) \cap L$ and $xh = x$, then for any $a \in S$, $(xa)h = xha = xa$, so by Theorem 5.1 (iii), $(xa)'' \leq h$. Thus $f = \bigvee_{a \in S}(xa)''$ is a lower bound for $\{h \in Z(S) \cap L : xh = x\}$. Since $x'' = (x1)'' \leq f$, we have $xf = x$ and $f \in \{h \in Z(S) \cap L : xh = x\}$. It follows that $f = x\Gamma$.

LEMMA 6.3. *Let φ be a center valued quantifier on an orthomodular lattice L. An element $e \in L$ is invariant if and only if $\varphi\varphi_e = \varphi_e \varphi$.*

Proof. If e is to be invariant, it must also be central. For arbitrary $a \in L$, we then have $a\varphi_e \varphi = (a \wedge e)\varphi = a\varphi \wedge e = a\varphi\varphi_e$. Hence $\varphi\varphi_e = \varphi_e \varphi$. Conversely, if $\varphi\varphi_e = \varphi_e \varphi$, then $e = 1\varphi\varphi_e = 1\varphi_e \varphi = e\varphi$.

THEOREM 6.4. *Let φ be a center valued quantifier on an orthomodular lattice L. There exists a Baer *-semigroup S such that $L = P'(S)$ and $\varphi = \Gamma|_L$.*

Proof. It clearly suffices to show that there is a coordinatizing Baer *-subsemigroup S^φ of $S(L)$ having the property that $a = a\varphi \Leftrightarrow \varphi_a \in Z(S^\varphi)$. Identification of L with $P'(S(L))$ will then produce the desired result. Accordingly, let

$$M = \{\varphi_a \in S(L) : a = a\varphi \in L\} \quad \text{and} \quad S^\varphi = Z(M).$$

Since $M = M^*$, we know that S^φ is a Baer *-subsemigroup of $S(L)$, and using the fact that φ is center valued, we have from [4, Theorem 2 (ii), p. 67] that $\varphi_e \in S^\varphi$ for each $e \in L$. Hence S^φ coordinatizes L. By the definition of M,

$$a = a\varphi \Rightarrow \varphi_a \in M \subset Z(Z(M)) = Z(S^\varphi).$$

We see from Lemma 6.3 that $\varphi \in S^\varphi$, and consequently, $\varphi_a \in Z(S^\varphi) \Rightarrow a = a\varphi$. Thus $a = a\varphi \Leftrightarrow \varphi_a \in Z(S^\varphi)$.

COROLLARY. *If L is a complete orthomodular lattice, then φ is a center valued quantifier on L if and only if there exists a Baer *-semigroup S such that $L = P'(S)$ and $\varphi = \Gamma|_L$.*

The above results together with Lemma 2.2 give considerable insight into the nature of center valued quantifiers on an orthomodular lattice. We turn now to some auxiliary results that will prove useful in connection with the next section.

LEMMA 6.5. *Let φ be a center valued quantifier on the orthomodular lattice L. For arbitrary $e \in L$, $(\varphi_e \varphi)'' = \varphi_{e\varphi}$ and $\varphi\varphi_e \varphi = \varphi\varphi_{e\varphi}$.*

Proof. It follows from [2, Theorem 8, p. 654] that $(\varphi_e \varphi)'' = \varphi_{1\varphi_e\varphi} = \varphi_{e\varphi}$. Since φ is center valued, we have $a\varphi\varphi_e \varphi = (a\varphi \wedge e)\varphi = a\varphi \wedge e\varphi = a\varphi\varphi_{e\varphi}$. Hence $\varphi\varphi_e \varphi = \varphi\varphi_{e\varphi}$.

LEMMA 6.6. *Let φ be a center valued quantifier on the orthomodular lattice L. For an element $e \in L$, the following conditions are equivalent:*

(i) *e is simple.*

(ii) $\varphi_e = \varphi_e \varphi \varphi_e$.

(iii) *$\varphi\varphi_e$ restricted to $L(0, e)$ is the identity map.*

Proof. (i) $\Rightarrow$ (ii). Given $g \in L$, $g\varphi_e \leq e$, so that $g\varphi_e = (g\varphi_e)\varphi \wedge e = g\varphi_e \varphi\varphi_e$. (ii) $\Rightarrow$ (iii). If $a \leq e$, then $a = a\varphi_e$, and $a\varphi\varphi_e = a\varphi_e \varphi\varphi_e = a\varphi_e = a$. (iii) $\Rightarrow$ (i). For $a \leq e$, $a = a\varphi\varphi_e = a\varphi \wedge e$.

THEOREM 6.7. *Let L be an orthomodular lattice and φ a center valued quantifier in L. There exists a Baer *-semigroup S and an element $q \in S$ such that*

(i) $L = P'(S)$;

(ii) $(L)\varphi = Z(S) \cap L$;

(iii) $q = q^2 = q^*$;

(iv) *for each $e \in L$, $(eq)'' = e\varphi$ and $qeq = q(eq)''$;*

(v) *e is simple $\Leftrightarrow e = eqe$;*

(vi) *e is invariant $\Leftrightarrow eq = qe$;*

(vii) *e is faithful $\Leftrightarrow q = qeq$.*

Proof. Conditions (i), (ii) and (iii) follow from Theorem 6.4; (iv) follows from Lemma 6.5; (v) from Lemma 6.6; and (vi) from Lemma 6.3.

(vii) If e is faithful, then $qeq = q(eq)'' = q(e\varphi) = q$. Conversely, if $q = qeq$, then

$$1 = 1\varphi = (1q)'' = (1qeq)'' = [(1q)''eq]'' = (eq)'' = e\varphi.$$

7. Loomis dimension lattices

A Loomis dimension lattice [12, p. 4] is a complete orthomodular lattice equipped with an equivalence relation ($\sim$) that satisfies:

(A) If $a \sim 0$ then $a = 0$.

(B) If $a_1 \perp a_2$ and $b \sim a_1 \vee a_2$ then there exists an orthogonal decomposition of b, $b = b_1 \vee b_2$, such that $a_1 \sim b_1$ and $a_2 \sim b_2$.

(C) If (a_α) and (b_α) are two families of pairwise orthogonal elements over the same indexing set such that $a_\alpha \sim b_\alpha$ for every α, then $\bigvee_\alpha a_\alpha \sim \bigvee_\alpha b_\alpha$.

(D) If a and b are not orthogonal, then there exist non-zero elements $a_1 \leq a$, $b_1 \leq b$ such that $a_1 \sim b_1$.

It will at times prove convenient to replace (B) by the following stronger condition:

(B′) If (a_α) is a family of pairwise orthogonal elements and $b \sim \bigvee_\alpha a_\alpha$, then there is a family (b_α) of pairwise orthogonal elements such that $b = \bigvee_\alpha b_\alpha$ and $a_\alpha \sim b_\alpha$ for every α.

Our goal in this section will be to show that every type I Loomis dimension lattice satisfying (B′) can be coordinatized by a Baer *-semigroup in such a way that the abstract dimension relation ($\sim$) is induced in a natural way by *-equivalence. The first step in such a program is a discussion of *-equivalence. *Until further notice it is assumed that S is a Baer *-semigroup and $L = P'(S)$.*

DEFINITION 7.1. Given $e, f \in L$, we shall say that e is *-equivalent to f, and write $e \sim^* f$, if there exists an element $x \in S$ such that $xx^* = e$ and $x^*x = f$. The element x is said to *implement* the *-equivalence of e and f; this will be indicated symbolically by writing $x : e \sim^* f$.

LEMMA 7.2. *Let $x : e \sim^* f$. Then $x'' = f$, $(x^*)'' = e$ and $x = exf$.*

Proof. By hypothesis, $xx^* = e$ and $x^*x = f$. It follows from Theorem 5.1 (iii) that $x = xx''$, so that $x^* = (xx'')^* = x''x^*$. Thus $e = xx^* = (xx^*)'' = (x''x^*)'' = (x^*)''$. Similarly, $f = x''$, and $x = (x^*)''xx'' = exf$.

LEMMA 7.3. *The relation ($\sim^*$) is an equivalence relation.*

Proof. The relation in question is clearly reflexive and symmetric. It is easily shown that if $x : e \sim^* f$ and $y : f \sim^* g$, then $xy : e \sim^* g$.

LEMMA 7.4. *Let $x : e \sim^* f$. If $a \leq e$ and $a \in L$, then $b = x^*ax \in L$; furthermore, $b \leq f$ and $a \sim^* b$.*

Proof. Clearly $b = b^*$, and $b^2 = (x^*ax)(x^*ax) = x^*aeax = x^*aax = x^*ax = b$; hence $b \in P(S)$. The fact that $bf = x^*axf = x^*ax = b$ shows that $b \leq f$. To prove that $b \in L$, note first that since $b \leq f$, we have $b'' \leq f'' = f = x''$. Thus $b = x''b = bx''$ and $b'' = x''b'' = b''x''$. Repeated application of Theorem 5.1 (viii) now produces

$$a = eae = xx^*axx^* = xbx^* = (xbx^*)'' = (x''bx^*)''$$
$$= (bx^*)'' = (b''x^*)'' = (x''b''x^*)'' = (xb''x^*)''.$$

This shows that $(xb''x^*)'' = xbx^*$, so that

$$xb''x^* = (xb''x^*)(xb''x^*)'' = (xb''x^*)(xbx^*) = xb''fbx^* = xb''bx^* = xbx^*.$$

We now have $b'' = fb''f = x^*xb''x^*x = x^*xbx^*x = fbf = b$, so that $b \in L = P'(S)$. To prove that $a \sim^* b$, set $y = ax$, and note that $yy^* = axx^*a = aea = a$, while $y^*y = x^*aax = x^*ax = b$.

THEOREM 7.5. *Let $x : e \sim^* f$. Define mappings*

$$E : L(0, e) \to L(0, f), \qquad F : L(0, f) \to L(0, e)$$

by the formulas $aE = x^*ax$, $bF = xbx^*$ for $a \in L(0, e)$, $b \in L(0, f)$. Then E is an orthocomplementation-preserving lattice isomorphism from $L(0, e)$ onto $L(0, f)$ whose inverse is F.

Proof. Lemma 7.4 shows that E and F are well defined. The proof that they are mutually inverse lattice isomorphisms is routine, and will therefore be omitted. We shall, however, prove that E preserves orthocomplements.

Let $a \in L(0, e)$, $b = aE$, $a^+ =$ the orthocomplement of a in $L(0, e)$, and $b^+ =$ the orthocomplement of b in $L(0, f)$. It is easily shown that $a^+ = a' \wedge e = a'e = ea'$, and $b^+ = b' \wedge f = b'f = fb'$. Since $bb^+ = 0$, we have $a(b^+F) = (bF)(b^+F) = xbx^*xb^+x^* = xbb^+x^* = 0$, and $b^+F \leq a'$. Then

$$b^+F \leq a' \wedge e = a^+ \quad \text{and} \quad b^+ = b^+FE \leq a^+E.$$

Similarly, one shows that $aa^+ = 0$ implies $b(a^+E) = 0$, whence $a^+E \leq b^+$. Therefore $a^+E = b^+$.

This concludes the material on *-equivalence in an arbitrary Baer *-semigroup. Suppose now that φ is a center valued quantifier on an orthomodular lattice L. Let S and q be as described in Theorem 6.7, and note that by Lemma 6.2, $e\varphi = \bigvee_{x \in S}(ex)''$ for every $e \in L$. In what follows we shall be referring to *-equivalence implemented by elements of S.

Lemma 7.6. *Let a, b be simple elements of L with $a\varphi = b\varphi$. Then $a \sim^* b$.*

Proof. Set $x = aqb$. Clearly $x \in S$, and $xx^* = (aqb)(bqa) = a(qbq)a = aq(bq)''a = aq(b\varphi)a = aq(a\varphi)a = aqa = a$. Similarly, $x^*x = b$.

Lemma 7.7. *Let $x : e \sim^* f$. Then $e\varphi = f\varphi$.*

Proof. By Lemma 7.2, $x'' = f$ and $(x^*)'' = e$. Hence $e\varphi = (eq)'' = (xx^*q)'' = (x''x^*q)'' = (fx^*q)'' \leq f\varphi$. Similarly, $f\varphi \leq e\varphi$.

Lemma 7.8. *Let $x : e \sim^* f$. If $a \leq e$ is simple and $b = x^*ax$, then b is simple.*

Proof. Lemma 7.4 assures us that $a \sim^* b$, so that $a\varphi = b\varphi$. Let $d \leq b$ and suppose $d\varphi = b\varphi$. Then $x\,dx^* \leq a$, and $(x\,dx^*)\varphi = d\varphi = b\varphi = a\varphi$. Since a is simple, we have from Theorem 3.4 that $x\,dx^* = a$. It follows that $d = b$, and consequently that b is simple.

We are finally ready to apply all of this to dimension lattices, *so we now assume that L is a type I Loomis dimension lattice.* This means that every element of L can be expressed as the supremum of a family of simple elements. Given $e \in L$, let $e\varphi$ denote the *hull* of e (see [12, p. 13]). Then φ is a center valued quantifier and L can be coordinatized in the manner described in Theorem 6.7.

Definition 7.9. Given $e, f \in L$, write $e \sim^+ f$ in case there exist two families (e_α), (f_α) of pairwise orthogonal elements over the same indexing set such that $e = \bigvee_\alpha e_\alpha$, $f = \bigvee_\alpha f_\alpha$ and $e_\alpha \sim^* f_\alpha$ for every α. Clearly $e \sim^* f \Rightarrow e \sim^+ f$. It will be our goal to show that $e \sim f \Leftrightarrow e \sim^+ f$, at least in the presence of axiom (B′).

Lemma 7.10. *Every non-zero element e of L is the supremum of a family of non-zero mutually orthogonal simple elements.*

Proof. This is [12, Lemma 29, p. 17].

LEMMA 7.11. *Let $a, b \in L$ be simple elements. Then $a \sim b \Leftrightarrow a \sim^* b$.*

Proof. By [12, Lemma 30, p. 17] $a \sim b$ is equivalent to $a\varphi = b\varphi$. Our Lemma 7.6 shows that $a\varphi = b\varphi \Rightarrow a \sim^* b$, while Lemma 7.7 gives the reverse implication.

LEMMA 7.12. *For $e, f \in L$, $e \sim^* f \Rightarrow e \sim f$.*

Proof. If $e = 0$, the result is clear, so we may assume $e \neq 0$. Let $x : e \sim^* f$, and by virtue of Lemma 7.10 we write $e = \bigvee_\alpha e_\alpha$ where (e_α) is a family of non-zero mutually orthogonal simple elements. We then use Theorem 7.5 and Lemma 7.8 to write $f = \bigvee_\alpha f_\alpha$ where (f_α) is a family of non-zero mutually orthogonal simple elements with $e_\alpha \sim^* f_\alpha$ for every α. By Lemma 7.11, $e_\alpha \sim f_\alpha$ for every α; hence $e \sim f$.

THEOREM 7.13. *If L satisfies (B′), then $e \sim f \Leftrightarrow e \sim^+ f$.*

Proof. The preceding lemma shows that $e \sim^* f \Rightarrow e \sim f$. It is immediate that $e \sim^+ f \Rightarrow e \sim f$. Suppose conversely that $e \sim f$. Writing $e = \bigvee_\alpha e_\alpha$ where (e_α) is a family of mutually orthogonal simple elements, we can apply axiom (B′) to write $f = \bigvee_\alpha f_\alpha$ with (f_α) a family of mutually orthogonal elements and $e_\alpha \sim f_\alpha$ for every α. By [12, Lemma 10, p. 7] each f_α is simple. We then have each $e_\alpha \sim^* f_\alpha$, and consequently $e \sim^+ f$.

The net result of all this is the result mentioned at the beginning of this section: Every type I Loomis dimension lattice satisfying (B′) can be coordinatized by a Baer *-semigroup in such a way that the abstract dimension relation ($\sim$) is induced naturally by *-equivalence.

In conclusion, the author wishes to acknowledge that none of this would be possible without the inspiration and patient guidance provided by Professor D. J. Foulis.

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